ON THE CURVATURE TENSOR OF THE HODGE METRIC OF MODULI SPACE OF POLARIZED CALABI-YAU THREEFOLDS

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Contents

1. Introduction

This paper is the continuation of the paper [6] of our study of the Moduli space of polarized Calabi-Yau threefold.

A polarized Calabi-Yau manifold is a pair (X, ω) of a compact algebraic manifold X with zero first Chern class and a Kähler form $\omega \in H^2(X, Z)$. The form ω is called a polarization. Let U be the universal deformation space of (X, ω). U is smooth by a theorem of Tian [12]. By [15], we may assume that each $X' \in U$ is a Kähler-Einstein manifold. i.e. the associated Kähler metric $(g'_{\alpha\overline{\beta}})$ is Ricci flat. The tangent space $T_{X'}U$ of U at X' can be identified with $H^1(X', T_{X'})_\omega$ where

$$H^1(X', T_{X'})_\omega = \{\phi \in H^1(X', T_{X'}) | \phi \lrcorner \omega = 0\}$$

The Weil-Petersson metric G_{PW} on U is defined by

$$G_{WP}(\phi, \psi) = \int_{X'} g'^{\alpha\overline{\beta}} g'_{\gamma\overline{\delta}} \phi^\gamma_{\overline{\beta}} \overline{\psi^\delta_{\overline{\alpha}}} dV_{g'}$$

where $\phi = \phi^\gamma_{\overline{\beta}} \frac{\partial}{\partial z^\gamma} d\overline{z}^\beta$, $\psi = \psi^\delta_{\overline{\alpha}} \frac{\partial}{\partial z^\delta} d\overline{z}^\alpha \in H^1(X', T_{X'})_\omega$, and $g' = g'_{\alpha\overline{\beta}} dz^\alpha d\overline{z}^\beta$ is the Kähler-Einstein metric on X associated to the polarization ω.

Let $n = \dim U$. As showed in [6], we defined the Hodge metric ω_H by

$$\omega_H = (n+3)\omega_{WP} + Ric(\omega_{WP})$$

where ω_{WP} is the Kähler form of the Weil-Petersson metric.

The main result of [6] is the following

Theorem 1.1. *Let* $\omega_H = (n+3)\omega_{WP} + Ric(\omega_{WP})$. *Then*

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1. ω_H *is a Kähler metric on* U*;*
2. *The holomorphic bisectional curvature of* ω_H *is nonpositive. Furthermore, Let* $\alpha = ((\sqrt{n}+1)^2+1)^{-1} > 0$*. Then the Ricci curvature* $Ric(\omega_H) \leq -\alpha\omega_H$ *and the holomorphic sectional curvature is also less than or equal to* $-\alpha$*.*
3. *If* $Ric(\omega_H)$ *is bounded, then the Riemannian sectional curvature of* ω_H *is also bounded.*

The main result of this paper builds on the above theorem:

Theorem 1.2. *Let* X *be a Calabi-Yau threefold. Let* $\varphi_1, \cdots, \varphi_n$ *be the orthonormal harmonic basis of* $H^1(X, T_X)_\omega$*. Then there is a constant* C*, depending only on* n*, such that the* L^∞ *norm of the sectional curvature* $|R|$ *satisfies*

$$|R| \leq C \sum_{i=1}^{n} ||\varphi_i||_{L^4}^8$$

Remark 1.1. The crucial part of this theorem is that the curvature has an upper bound which only depends on the L^4 norm of the harmonic basis, rather than depends on the derivative of the harmonic basis. Upper bound of the sectional curvature of the Hodge metric is very important in the compactification of the moduli space (cf. [5]).

In order to prove the theorem, we need to estimate the covariant derivative of the Yukawa coupling with respect to the Weil-Petersson metric. As a by-product, we proved the following theorem (for definitions, see §2):

Theorem 1.3. *Let* $F = (F_{ijk})$ *be the Yukawa coupling. Let*

$$F_{ijk,l} = \partial_l F_{ijk} - \Gamma_{il}^m F_{mjk} - \Gamma_{jl}^m F_{imk} - \Gamma_{kl}^m F_{ijm} + 2K_l F_{ijk}$$

Then $F_{ijk,l} = F_{ijl,k}$*.*

Remark 1.2. The moduli space of a Calabi-Yau threefold is a *projective* special Kähler manifold in the sense of D. Freed [2]. In [4], the Yakuwa coupling of special Kähler manifolds is discussed.

The motivation behind this paper and the paper [6] is that we want to give a differential geometric proof of the theorem of Viehweg [13] in the case of the moduli space of Calabi-Yau threefolds. Viehweg's theorem states that moduli spaces of polarized algebraic varieties are quasi-projective. The boundedness of the curvature of the Hodge metric is very important because of the work of Mok [7], Mok-Zhong [8] and Yeung [16]. By their theorems, if a complete Kähler manifold of finite volume has negative Ricci curvature and bounded sectional curvature, then it must be quasi-projective. In the case of the moduli space of Calabi-Yau threefolds, the Ricci curvature is negative (Theorem 1.1), and the condition on the boundedness of the sectional curvatures can be weakened, thus it is very important to get various upper bound estimates of the sectional curvatures.

In the last section, we give an extra restriction on the limit of Hodge structures for a one dimensional degeneration of a family of Calabi-Yau threefolds.

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2. The Covariant Derivatives of the Yukawa Coupling

Suppose $\pi : \mathfrak{X} \to U$ is the total space over the (local) universal deformation space U of a Calabi-Yau threefold X. Thus for any point $X' \in U$, $\pi^{-1}(X')$ is a Calabi-Yau threefold. The Hodge bundle $\underline{F}^3 = \pi_* \omega_{\mathfrak{X}/U}$ is the holomorphic line bundle over U where $\omega_{\mathfrak{X}/U}$ is the relative canonical bundle of $\mathfrak{X}$.

There is a natural Hermitian metric on $\underline{F}^3$ defined by the Ricci flat metric on each fiber of π. Such a metric can be written out explicitly as follows: since for any $X' \in U$, $\pi^{-1}(X')$ is differmorphic to $\pi^{-1}(0) = X$, there is a natural identification of $H^3(X', C) \to H^3(X, C)$. Suppose $\varphi, \psi \in H^3(X, C)$. Define the cup product

$$Q(\varphi, \psi) = -\int_X \varphi \wedge \psi$$

Let Ω be a local holomorphic section of $\underline{F}^3$. Thus for each $X' \in U$, Ω at X' is a holomorphic $(3,0)$ form on $H^{3,0}(X') \subset H^3(X', C)$ and under the identification $H^3(X', C) \to H^3(X, C)$, $\Omega(X') \in H^3(X, C)$.

The Hermitian metric on $\underline{F}^3$ is defined by setting $||\Omega||^2 = \sqrt{-1} Q(\Omega, \overline{\Omega})$.

The technical heart of this paper is to compute the covariant derivative of the Yukawa coupling with respect to the Weil-Petersson metric and the Hermitian metric on the bundle $\underline{F}^3$. Recall that by definition(see[1], for example), the Yukawa coupling is the (local) section F of the bundle $Sym^3((R^1_*(T_{\mathfrak{X}/U}))^*) \otimes (\underline{F}^3)^{\otimes 2}$ over U such that for any $\varphi_1, \varphi_2, \varphi_3 \in H^1(X', T_{X'})$ and $\Omega \in H^{3,0}(X')$,

$$F(\varphi_1, \varphi_2, \varphi_3) = \int_{X'} (\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \lrcorner \Omega) \wedge \Omega$$

Here $T_{\mathfrak{X}/U}$ is the relative tangent sheaf of $\mathfrak{X} \to U$.

The basic property of the Yukawa coupling is that it is a holomorphic section. In fact, Let $t^1, \cdots, t^n$ be the local holomorphic coordinate system of U. Let Ω be a local nonzero section of the holomorphic bundle $\underline{F}^3$, We have

$$F_{ijk} = F(\rho(\frac{\partial}{\partial t^i}), \rho(\frac{\partial}{\partial t^j}), \rho(\frac{\partial}{\partial t^k})) = Q(\Omega, \partial_i \partial_j \partial_k \Omega), \qquad 1 \leq i, j, k \leq n$$

where $\rho : T_X U \to H^1(X, T_X)$ is the Kodaira-Spencer map.

Let Γ^k_{ij} be the Christoffel symbols of the Kähler metric $g_{i\bar{j}}$ and let $K_l = -\partial_l \log ||\Omega||^2$ be the connection of the Hermitian bundle $\underline{F}^3$ with respect to the local section Ω. We make the following definition:

Definition 2.1. *For $1 \leq i, j, k \leq n$, the covariant derivative of F_{ijk} is defined as*

$$F_{ijk,l} = \partial_l F_{ijk} - \Gamma^s_{li} F_{sjk} - \Gamma^s_{lj} F_{isk} - \Gamma^s_{lk} F_{ijs} + 2F_{ijk} K_l$$

In this section, we are going to compute $F_{ijk,l}$ at a point $X \in U$ in terms of the information of the fixed Calabi-Yau threefold X.

We use the method developed by Siu[11], Nannicini[9] and Schumacher[10]. By Kähler geometry, there is a holomorphic coordinate $(t^1, \cdots, t^n)$ of U such that at X, $\Gamma^k_{ij} = 0$, $1 \leq i, j, k \leq n$. Furthermore, if the local section Ω of $\underline{F}^3$ is carefully chosen, then $K_l = 0$, $1 \leq l \leq n$ at X.

Consider the Kodaira-Spencer map $\rho : T_{X'}U \to H^1(X', T_{X'})$. Let $\varphi_j = \rho(\frac{\partial}{\partial t^j}), 1 \leq j \leq n$. Suppose φ_j's are harmonic $T_{X'}$-valued $(0, 1)$ forms. These φ_j's can be realized by the canonical lift in the sense of Siu [11] (See also Nannicini [9] and Schumacher [10]): suppose (z^1, z^2, z^3) is the holomorphic coordinate on X. Then for each $\frac{\partial}{\partial t^j}$, there is a vector v_j, called the canonical lift of $\frac{\partial}{\partial t^j}$, on $\mathfrak{X}$ which locally can be represented as $v_j = \frac{\partial}{\partial t^j} + v^\alpha_j \frac{\partial}{\partial z^\alpha}$ such that $\varphi_j = \overline{\partial} v^\alpha_j \frac{\partial}{\partial z^\alpha}$ is a harmonic $T_{X'}$-valued (0,1) form.

It should be noted that v_j is a vector field on $\mathfrak{X}$ but neither is $\frac{\partial}{\partial t^j}$ nor $v^\alpha_j \frac{\partial}{\partial z^\alpha}$ alone. In fact, the component $\frac{\partial}{\partial t^j}$ in the expression $v_j = \frac{\partial}{\partial t^j} + v^\alpha_j \frac{\partial}{\partial z^\alpha}$ is different from it is as the vector field on U. It is also easy to check that the real part of the vector field v_j defines differmorphisms between the fibers. Using these differmorphisms, tensor fields of the nearby fibers can be identified as tensor fields on X. By Nannicini [9] or Siu [11], the Lie derivative $L_{v_l}\cdot$ is defined as the usual $\frac{\partial}{\partial t^l}$ after pulling back via the differmorphisms.

Now let's analyze the conditions $K_l = 0$ and $\Gamma^i_{jk} = 0$ for $1 \leq i, j, k, l \leq n$ at X. We have

Proposition 2.1. *We use the notations as above. In particular, suppose $(\frac{\partial}{\partial t^1}, \cdots, \frac{\partial}{\partial t^n})$ and Ω are chosen such that $\Gamma^i_{jk} = 0$ and $K_l = 0$ at $X \in U$. Then we have*

1. $(L_{v_l}\Omega)^{3,0} = 0$*;*
2. $(L_{v_k}\varphi_i)$ *is a* $\overline{\partial}^*$*-boundary.*

Proof: The key point is to identify the derivatives with respect to cohomological classes and the derivatives with respect to forms. Suppose for fixed l, $v_l = \tau_1 + \sqrt{-1}\tau_2$ where τ_1 and τ_2 are real vector fields. Let $\sigma_1(s)$ and $\sigma_2(s)$ be the flows defined by τ_1 and τ_2, respectively. Consider the 3-forms $\sigma_i(s)^*\overline{\Omega}$, $i = 1, 2$. Suppose

$$\sigma_i(s)^*\overline{\Omega} = p_i(s) + dq_i(s)$$

be the Hodge decomposition of $\sigma_i(s)^*\overline{\Omega}$ in $H^3(X, C)$. Then we have

$$0 = \partial_l \overline{\Omega} = \frac{d}{ds}|_{s=0}(p_1(s) + \sqrt{-1}p_2(s))$$

by the definition of $\partial_l\overline{\Omega}$. This is equivalent to

$$\frac{d}{ds}|_{s=0}(\sigma_1(s)^*\overline{\Omega}+\sqrt{-1}\sigma_2(s)^*\overline{\Omega})-d\frac{d}{ds}|_{s=0}(q_1(s)+\sqrt{-1}q_2(s))=0$$

Or in other word

$$L_{v_l}\overline{\Omega}-d\sigma=0$$

for $\sigma=\frac{d}{ds}|_{s=0}(q_1+\sqrt{-1}q_2)$.

Using this, we have

$$\int_X\Omega\wedge L_{v_l}\overline{\Omega}=0$$

On the other hand, $K_l=0$ implies

$$0=\partial_l Q(\Omega,\overline{\Omega})=\int_X L_{v_l}\Omega\wedge\overline{\Omega}+\int_X\Omega\wedge L_{v_l}\overline{\Omega}$$

So the first part of the proposition follows from the following:

Claim. $\overline{\partial}L_{v_l}\Omega=0$.

Proof of the Claim: This follows from a straightforward computation. Let Ω be represented as

$$\Omega=a\,dz^1\wedge dz^2\wedge dz^3$$

where the functions a,z^1,z^2,z^3 are holomorphic on each fiber and have parameter t. Suppose $\rho(\frac{\partial}{\partial t})=\varphi=\varphi^{\alpha}_{\overline{\beta}}\partial_\alpha d\overline{z^\beta}$ is a harmonic T_X-valued $(0,1)$ form. We have

$$\begin{aligned}
&\overline{\partial}(\frac{\partial}{\partial t}(dz^1\wedge dz^2\wedge dz^3))\\
&=\overline{\partial}d\frac{\partial z^1}{\partial t}\wedge dz^2\wedge dz^3+\overline{\partial}(dz^1\wedge d\frac{\partial z^2}{\partial t}\wedge dz^3)+\overline{\partial}(dz^1\wedge dz^2\wedge d\frac{\partial z^3}{\partial t})\\
&=-\partial\overline{\partial}\frac{\partial z^1}{\partial t}\wedge dz^2\wedge dz^3+dz^1\wedge\partial\overline{\partial}\frac{\partial z^2}{\partial t}\wedge dz^3-dz^1\wedge dz^2\wedge\partial\overline{\partial}\frac{\partial z^3}{\partial t}\\
&=(-\partial_1\varphi^1_{\overline{\beta}}-\partial_2\varphi^2_{\overline{\beta}}-\partial_3\varphi^3_{\overline{\beta}})dz^1\wedge dz^2\wedge dz^3\wedge d\overline{z^\beta}
\end{aligned}\tag{2.1}$$

By the harmonicity of φ, we have

$$\partial_\alpha\varphi^\alpha_{\overline{\beta}}+\Gamma^\alpha_{\alpha\gamma}\varphi^\gamma_{\overline{\beta}}=0\tag{2.2}$$

where the notation $\Gamma^\alpha_{\beta\gamma}$ is the connection of X which is different from the connection Γ^i_{jk} on the universal deformation space U.

From the theory of deformation of complex structures, we know that $\overline{\partial}-t\varphi$ defines the $\overline{\partial}$-operator on the nearby fibers. Thus we have

$$\overline{\partial}\frac{\partial a}{\partial t}=\varphi^\alpha_{\overline{\beta}}\partial_\alpha a\,d\overline{z^\beta}\tag{2.3}$$

Using Equation (2.1), (2.2) and (2.3), we have

$$\overline{\partial}\frac{\partial}{\partial t}\Omega = \overline{\partial}\frac{\partial a}{\partial t}dz^1\wedge dz^2\wedge dz^3 + a\overline{\partial}\frac{\partial}{\partial t}(dz^1\wedge dz^2\wedge dz^3)$$
$$= (\varphi_{\overline{\beta}}^{\alpha}\partial_\alpha a - a\Gamma^{\alpha}_{\alpha\gamma}\varphi^{\gamma}_{\overline{\beta}})d\overline{z}^{\beta}\wedge dz^1\wedge dz^2\wedge dz^3$$

So $\overline{\partial}\frac{\partial}{\partial t}\Omega = 0$ follows from the fact

$$a\Gamma^{\alpha}_{\alpha\gamma} = a\partial_\gamma \log|a|^2 = a\partial_\gamma\log a = \partial_\gamma a$$

and the claim is proved.

The second part of the proposition is implied in [9]. We prove it for the sake of completeness. We assume at X, (z^1, z^2, z^3) are normal coordinates. By definition,

$$L_{v_k}\varphi_i = (\partial_k(\varphi_i)^{\alpha}_{\overline{\beta}} - (\varphi_i)^{\gamma}_{\overline{\beta}}\partial_\gamma v_k^\alpha)\partial_\alpha d\overline{z}^{\beta}$$

Thus

$$\overline{\partial}^* L_{v_k}\varphi_i = (\partial_\beta\partial_k(\varphi_i)^{\alpha}_{\overline{\beta}} - (\varphi_i)^{\gamma}_{\overline{\beta}}\partial_\beta\partial_\gamma v_k^\alpha)\partial_\alpha \tag{2.4}$$

We are going to prove $\overline{\partial}^* L_{v_k}\varphi_i = 0$. By the harmonicity of φ_i, we have

$$g^{\beta_1\overline{\beta}}(\varphi_i)^{\alpha}_{\overline{\beta},\beta_1} = 0$$

or

$$g^{\beta_1\overline{\beta}}(\partial_{\beta_1}(\varphi_i)^{\alpha}_{\overline{\beta}} + \Gamma^{\alpha}_{\beta_1\beta_2}(\varphi_i)^{\beta_2}_{\overline{\beta}}) = 0$$

Taking derivative with respect to ∂_k gives

$$g^{\beta_1\overline{\beta}}\partial_{\partial_k\beta_1}(\varphi_i)^{\alpha}_{\overline{\beta}} + \partial_k g^{\beta_1\overline{\beta}}\partial_{\beta_1}(\varphi_i)^{\alpha}_{\overline{\beta}} + \partial_k(g^{\beta_1\overline{\beta}}\Gamma^{\alpha}_{\beta_1\beta_2})(\varphi_i)^{\beta_2}_{\overline{\beta}} = 0 \tag{2.5}$$

Since $L_{v_k}\omega = 0$ (See Nannicini [9], for example), we have

$$0 = L_{v_k} g_{\beta\overline{\beta}_1} dz^\beta\wedge d\overline{z}^{\beta_1} = (\partial_k g_{\beta\overline{\beta}_1} + \partial_\beta v_k^{\beta_1})dz^\beta\wedge d\overline{z}^{\beta_1}$$

So we have

$$\partial_k g_{\beta\overline{\beta}_1} = -\partial_\beta v_k^{\beta_1} \tag{2.6}$$

We also have

$$\partial_k(g^{\beta_1\overline{\beta}}\Gamma^{\alpha}_{\beta_1\beta_2}) = \partial_k\Gamma^{\alpha}_{\beta\beta_2} = \partial_k\Gamma_{\beta\overline{\alpha}\beta_2} = \partial_k\partial_{\beta_2}g_{\beta\overline{\alpha}} = -\partial_{\beta_2}\partial_\beta v_k^\alpha \tag{2.7}$$

In addition, we have

$$\partial_k\partial_{\beta_1}(\varphi_i)^{\alpha}_{\overline{\beta}} = \partial_{\beta_1}\partial_k(\varphi_i)^{\alpha}_{\overline{\beta}} - \partial_{\beta_1}v_k^\gamma\partial_\gamma(\varphi_i)^{\alpha}_{\overline{\beta}} \tag{2.8}$$

Using Equation (2.6), (2.7) and (2.8), from Equation (2.4) and (2.5), we get $\overline{\partial}^* L_{v_k}\varphi_i = 0$.

By [9], we see that

$$0 = \Gamma_{i\overline{j}k} = \int_X < L_{v_k}\varphi_i, \varphi_j >$$

So the harmonic part of $L_{v_k}\varphi_i$ is zero. Thus $L_{v_k}\varphi_i$ is a $\overline{\partial}^*$-boundary by the Hodge decomposition.

□

The condition that $(\frac{\partial}{\partial t^1}, \cdots, \frac{\partial}{\partial t^n})$ is an orthonormal basis implies that $\varphi_1, \cdots, \varphi_n \in H^1(X, T_X)$ is a set of orthnormal basis of harmonic T_X-valued forms. Let Ω be the local nonzero section of $\underline{F}^3$. We make the following definition:

Definition 2.2.

$$a_{jk} = (\varphi_j \wedge \varphi_k)^{\#} = \varphi_j \wedge \varphi_k \lrcorner \Omega$$

is an (1,2) form for $1 \leq j, k \leq n$. *Here*

$$\varphi_j \wedge \varphi_k = (\varphi_j)^{\alpha}_{\overline{\beta}} (\varphi_k)^{\gamma}_{\overline{\delta}} \frac{\partial}{\partial z^\alpha} \wedge \frac{\partial}{\partial z^\gamma} \otimes d\overline{z}^\beta \wedge d\overline{z}^\delta$$

If $\Omega = a dz^1 \wedge dz^2 \wedge dz^3$. *Then* a_{jk} *can be represented as*

$$a_{jk} = a(\varphi_j)^{\alpha}_{\overline{\beta}} (\varphi_k)^{\gamma}_{\overline{\delta}} sgn(\xi, \alpha, \gamma) dz^\xi \wedge d\overline{z}^\beta \wedge d\overline{z}^\delta \tag{2.9}$$

Lemma 2.1. *For* $1 \leq j, k \leq n$,

$$\partial^* a_{jk} = 0$$

Proof: Since X is a Kähler manifold, we have $\partial^* = \sqrt{-1}[\Lambda, \overline{\partial}]$. First we have $\overline{\partial} a_{jk} = 0$. Next, suppose (z^1, z^2, z^3) is a normal coordinate system. Then

$$\begin{aligned}\Lambda a_{jk} &= \Lambda(a(\varphi_j)^{\alpha}_{\overline{\beta}} (\varphi_k)^{\gamma}_{\overline{\delta}} sgn(\alpha, \gamma, \xi) dz^\xi \wedge d\overline{z}^\beta \wedge d\overline{z}^\delta) \\ &= a(\varphi_j)^{\alpha}_{\overline{\beta}} (\varphi_k)^{\gamma}_{\overline{\delta}} sgn(\alpha, \gamma, \beta) d\overline{z}^\delta - a(\varphi_j)^{\alpha}_{\overline{\beta}} (\varphi_k)^{\gamma}_{\overline{\delta}} sgn(\alpha, \gamma, \delta) d\overline{z}^\beta\end{aligned}$$

However, the fact that $\varphi_j \lrcorner \omega = 0$ and $\varphi_k \lrcorner \omega = 0$ implies $(\varphi_j)^{\alpha}_{\overline{\beta}} = (\varphi_j)^{\beta}_{\overline{\alpha}}$, and $(\varphi_k)^{\gamma}_{\overline{\delta}} = (\varphi_k)^{\delta}_{\overline{\gamma}}$. So $\Lambda a_{jk} = 0$ and the lemma is proved.

□

Definition 2.3. *The Hodge *-operator on* X *is defined as*

$$* : A^{p,q} \to A^{n-p,n-q}, \qquad (\varphi, \psi) dV = \varphi \wedge * \psi$$

for $\varphi, \psi \in A^{p,q}(X)$.

Lemma 2.2. *For* $1 \leq j, k \leq n$

$$* a_{jk} = \overline{a_{jk}}$$

Proof: By Equation (2.9),

$$a_{jk} = a(\varphi_j)^{\alpha}_{\overline{\beta}} (\varphi_k)^{\gamma}_{\overline{\delta}} sgn(\alpha, \gamma, \xi) dz^\xi \wedge d\overline{z}^\beta \wedge d\overline{z}^\delta$$

We have

$$\begin{aligned}* a_{jk} &= \sum_{m<n} \overline{a(\varphi_j)^{\alpha}_{\overline{\beta}} (\varphi_k)^{\gamma}_{\overline{\delta}}} sgn(\alpha, \gamma, \xi) sgn(\xi, m, n) sgn(\beta, \delta, \eta) d\overline{z}^m \wedge d\overline{z}^n \wedge dz^\eta \\ &= \overline{a(\varphi_j)^{\alpha}_{\overline{\beta}} (\varphi_k)^{\gamma}_{\overline{\delta}} d\overline{z}^\alpha \wedge d\overline{z}^\gamma sgn(\beta, \delta, \eta) d\overline{z}^\eta} = \overline{a_{jk}}\end{aligned}$$

Here we again use the fact $\varphi_j \lrcorner \omega = 0$ and $\varphi_k \lrcorner \omega = 0$.

□

Lemma 2.3. *For $1 \leq j, k \leq n$,*

$$\overline{\partial}(L_{v_j}\varphi_k \lrcorner \Omega) = \partial a_{jk}$$

Proof: In Nannicini [9], it is proved that

$$\overline{\partial} L_{v_j}\varphi_k = D_1^*(\varphi_j \wedge \varphi_k)$$

where

$$D_1^*(\varphi_j \wedge \varphi_k) = \partial_\alpha((\varphi_j)_{\overline{\beta}}^{\alpha}(\varphi_k)_{\overline{\delta}}^{\gamma})\partial_\gamma d\overline{z}^\beta \wedge d\overline{z}^\delta$$

The lemma follows from Equation (2.9).

Now we are going to prove the main theorem of this section.

Theorem 2.1. *Let G be the Green's operator on differential forms of X. Suppose $\varphi_1, \cdots, \varphi_n$ are the orthnormal basis of $H^1(X, T_X)$. Then*

$$F_{ijk,l} = \int_X (G\partial a_{li}, \overline{\partial a_{jk}})dV + \int_X (G\partial a_{lj}, \overline{\partial a_{ik}})dV + \int_X (G\partial a_{lk}, \overline{\partial a_{ij}})dV$$

where a_{ij}'s are defined in Definition 2.2.

It should be noted that the notation of the inner product is defined as $(a, a) = ||a||^2$. So $(a, \overline{a})$ is *not* the norm of a.

Proof: Since we choose the local coordinate $(t_1, \cdots, t_n)$ and the local section Ω such that $\Gamma_{jk}^i = K_l = 0$, we have $F_{ijk,l} = \partial_l F_{ijk}$. By Proposition 2.1,

$$\partial_l F_{ijk} = \partial_l \int_X (\varphi_i \wedge \varphi_j \wedge \varphi_k)\lrcorner\Omega) \wedge \Omega = \int_X (L_{v_l}\varphi_i \wedge \varphi_j \wedge \varphi_k \lrcorner \Omega) \wedge \Omega$$
$$+ \int_X (\varphi_i \wedge L_{v_l}\varphi_j \wedge \varphi_k \lrcorner\Omega) \wedge \Omega + \int_X (\varphi_i \wedge \varphi_j \wedge L_{v_l}\varphi_k \lrcorner \Omega) \wedge \Omega$$

Thus we need only to prove that

$$\int_X ((L_{v_l}\varphi_i) \wedge \varphi_j \wedge \varphi_k \lrcorner\Omega) \wedge \Omega = \int_X (G\partial a_{li}, \overline{\partial a_{jk}})dV$$

Let $L_{v_l}\varphi_i = b_{li}$. Recall that $\Omega = a dz^1 \wedge dz^2 \wedge dz^3$. Then

$$(b_{li}\lrcorner\Omega) = (-1)^\alpha a (b_{li})_{\overline{\alpha}_1}^{\alpha} dz^1 \wedge \cdots d\hat{z}^\alpha \cdots \wedge dz^3 \wedge d\overline{z}^{\alpha_1} \tag{2.10}$$

By Equation (2.9) we see that

$$\begin{aligned}(b_{li}\lrcorner\Omega \wedge a_{jk}) &= (-1)^\alpha a^2 (b_{li})_{\overline{\alpha}_1}^{\alpha} (\varphi_j)_{\overline{\beta}_1}^{\beta} (\varphi_k)_{\overline{\gamma}_1}^{\gamma} dz^1 \wedge \cdots d\hat{z}^\alpha \cdots \wedge dz^3 \wedge d\overline{z}^{\alpha_1} \\ &\qquad sgn(\beta, \gamma, \xi) dz^\xi \wedge d\overline{z}^{\beta_1} \wedge d\overline{z}^{\gamma_1} \\ &= a^2 (b_{li})_{\overline{\alpha}_1}^{\alpha} (\varphi_j)_{\overline{\beta}_1}^{\beta} (\varphi_k)_{\overline{\gamma}_1}^{\gamma} sgn(\alpha, \beta, \gamma) sgn(\alpha_1, \beta_1, \gamma_1) \\ &\qquad dz^1 \wedge dz^2 \wedge dz^3 \wedge d\overline{z}^1 \wedge d\overline{z}^2 \wedge d\overline{z}^3 \\ &= -(b_{li} \wedge \varphi_j \wedge \varphi_k \lrcorner\Omega) \wedge \Omega\end{aligned}$$

By Lemma 2.2 and the above equation, we have

$$
\begin{aligned}
(b_{li} \wedge \varphi_j \wedge \varphi_k \lrcorner \Omega) \wedge \Omega &= -(b_{li} \lrcorner \Omega \wedge a_{jk}) \\
&= (b_{li} \lrcorner \Omega \wedge * \overline{a_{jk}}) = (b_{li} \lrcorner \Omega, \overline{a_{jk}}) dV
\end{aligned} \tag{2.11}
$$

By Lemma 2.3 and Proposition 2.1, we have

$$
\begin{aligned}
\Box(L_{v_l}\varphi_i \lrcorner \Omega) &= \overline{\partial}^* \overline{\partial}(L_{v_l}\varphi_i \lrcorner \Omega) + \overline{\partial}\overline{\partial}^*(L_{v_l}\varphi_i \lrcorner \Omega) \\
&= \overline{\partial}^* \partial a_{li} + \overline{\partial}((\overline{\partial}^* L_{v_l}\varphi_i) \lrcorner \Omega) = \overline{\partial}^* \partial a_{li}
\end{aligned}
$$

Since $L_{v_l}\varphi_i$ is a $\overline{\partial}^*$-boundary (Proposition 2.1), we know

$$
L_{v_l}\varphi_i \lrcorner \Omega = G \overline{\partial}^* \partial a_{li} \tag{2.12}
$$

where G is the Green operator of the Laplacian $\Box$. Thus by Equation (2.11) and (2.12),

$$
\begin{aligned}
\int_X ((L_{v_l}\varphi_i) \wedge \varphi_j \wedge \varphi_k \lrcorner \Omega) \wedge \Omega &= \int_X (b_{li} \lrcorner \Omega, \overline{a_{jk}}) dV \\
= \int_X (G \overline{\partial}^* \partial a_{li}, \overline{a_{jk}}) &= \int_X (G \partial a_{li}, \overline{\partial a_{jk}}) dV
\end{aligned}
$$

□

Corollary 2.1 (Theorem 1.3). *For $1 \leq i, j, k, l \leq n$,*

$$
F_{ijk,l} = F_{ijl,k}
$$

□

3. The Estimates

In this section, we give an upper bound of the curvature tensor of the Hodge metric. We use the same notations as in the previous section.

Suppose $(z^1, \cdots, z^n)$ is the normal holomorphic coordinate system at $p \in U$ with respect to the Weil-Petersson metric $\omega_{WP} = \sqrt{-1} g_{i\overline{j}} dz^i \wedge d\overline{z}^j$.

We further assume that $Q(\Omega, \overline{\Omega}) = 1$. In [6] we have proved that

Theorem 3.1. *If $(z^1, \cdots, z^n)$ is the normal coordinate system of ω_{WP}, then the curvature tensor $R_{i\overline{j}k\overline{l}}$ of $\omega_H = \sqrt{-1} h_{i\overline{j}} dz^i \wedge d\overline{z}^j$ at p is*

$$
R_{i\overline{j}k\overline{l}} = A_{i\overline{j}k\overline{l}} + B_{i\overline{j}k\overline{l}}
$$

where

$$
A_{i\overline{j}k\overline{l}} = 2\delta_{ij}\delta_{kl} + 2\delta_{il}\delta_{kj} - 4\sum_s F_{iks}\overline{F_{jls}} + 2\sum_{mnpq} F_{qkm}\overline{F_{plm}} F_{inp}\overline{F_{jnq}}
$$

$$
B_{i\overline{j}k\overline{l}} = \sum_{rs} F_{irs,k}\overline{F_{jrs,l}} - \sum_{mn} (\sum_{rs} F_{irs,k}\overline{F_{mrs}}) \overline{(\sum_{rs} F_{jrs,l}\overline{F_{nrs}})} h^{n\overline{m}}
$$

Here F_{ijk} is the Yukawa coupling and $F_{ijk,l}$ is its covariant derivative with respect to the Weil-Petersson metric and the connection on $\underline{F}^3$.

It is proved in [6] that in order to bound the curvature tensor, we need only to bound the scalar curvature. By definition, the scalar curvature ρ is

$$\rho = -h^{i\bar{j}}h^{k\bar{l}}R_{i\bar{j}k\bar{l}} = -h^{i\bar{j}}h^{k\bar{l}}A_{i\bar{j}k\bar{l}} - h^{i\bar{j}}h^{k\bar{l}}B_{i\bar{j}k\bar{l}}$$

Lemma 3.1. *Suppose that the dimension of the universal deformation space U is n. Then*

$$|h^{i\bar{j}}h^{k\bar{l}}A_{i\bar{j}k\bar{l}}| \le 3n^6$$

Proof: Under the local coordinate $(z^1, \cdots, z^n)$, we proved in [6] that

$$h_{i\bar{j}} = 2\delta_{i\bar{j}} + \sum_{mn} F_{imn}\overline{F_{jmn}}$$

Suppose further that

$$\sum_{mn} F_{imn}\overline{F_{jmn}} = \lambda_i \delta_{ij}$$

Then

$$h_{i\bar{j}} = (2+\lambda_i)\delta_{ij}$$

In particular, for fixed i, m, n

$$|F_{imn}| \le \sqrt{\lambda_i}$$

So

$$\begin{aligned}
&\sum_{ijkl} h^{i\bar{j}}h^{k\bar{l}} \sum_{mnpq} F_{qkm}\overline{F_{plm}}F_{inp}\overline{F_{jnq}} \\
&= \sum_{ikmnpq} \frac{1}{2+\lambda_i}\cdot\frac{1}{2+\lambda_k} F_{qkm}\overline{F_{pkm}}F_{inp}\overline{F_{inq}} \\
&\le \sum_{ik} \frac{1}{2+\lambda_i}\cdot\frac{1}{2+\lambda_k}\lambda_i\lambda_k n^4
\end{aligned}$$

In [6], we have proved that $h^{i\bar{j}}h^{k\bar{l}}A_{i\bar{j}k\bar{l}} \ge 0$. Thus

$$|h^{i\bar{j}}h^{k\bar{l}}A_{i\bar{j}k\bar{l}}| \le \sum_{ik} \frac{4}{(2+\lambda_i)(2+\lambda_k)} + 2\sum_{ik} \frac{1}{2+\lambda_i}\cdot\frac{1}{2+\lambda_k}\lambda_i\lambda_k n^4 \le 3n^6$$

□

Now we consider $h^{i\bar{j}}h^{k\bar{l}}B_{i\bar{j}k\bar{l}}$. It is easy to see that

$$0 \le h^{i\bar{j}}h^{k\bar{l}}B_{i\bar{j}k\bar{l}} \le \sum_{ijklrs} h^{i\bar{j}}h^{k\bar{l}}F_{irs,k}\overline{F_{jrs,l}} \le \sum_{ijkl} |F_{ijk,l}|^2 \tag{3.1}$$

Lemma 3.2. *Using the notations in Theorem 2.1 , we have*

$$|F_{ijk,l}| \le 3(||\varphi_i||^4_{L^4} + ||\varphi_j||^4_{L^4} + ||\varphi_k||^4_{L^4} + ||\varphi_l||^4_{L^4})$$

Proof: Since the Green operator is a positive operator, we have

$$|\int_X (G\partial a_{li}, \overline{\partial a_{jk}})| \leq \sqrt{\int_X (G\partial a_{li}, \partial a_{li})}\sqrt{\int_X (G\partial a_{jk}, \partial a_{jk})}$$

However, for fixed l, i, by Lemma 2.1, we have

$$G\partial^*\partial a_{li} = G\Box a_{li} = a_{li} - H(a_{li})$$

where $H(a_{li})$ is the harmonic part in the Hodge decomposition of a_{li}. Thus

$$|\int_X (G\partial a_{li}, \partial a_{li})| \leq |\int_X (G\partial^*\partial a_{li}, a_{li})| \leq \int_X |a_{li}|^2 \leq \int_X |\varphi_l|^4 + |\varphi_i|^4$$

So by Theorem 2.1,

$$|F_{ijk,l}| \leq 3(||\varphi_i||^4_{L^4} + ||\varphi_j||^4_{L^4} + ||\varphi_k||^4_{L^4} + ||\varphi_l||^4_{L^4})$$

□

Theorem 3.2 (Theorem 1.2)**.** *The scalar curvature ρ of the Hodge metric satisfies*

$$0 < -\rho \leq 3n^6 + 144n^3 \sum_i ||\varphi_i||^8_{L^4}$$

Proof: $\rho < 0$ follows from [6]. The upper bound is from Lemma 3.1 and Lemma 3.2. □

4. A Remark on the Theorem of C-L Wang

In his paper [14], C-L Wong gave a necessary and sufficient condition for the Weil-Petersson metric to be incomplete for a family of Calabi-Yau manifolds over a punctured disk. The main theorem of him is (for the precise definitions and notations, see Wong [14]):

Theorem 4.1 (C-L Wong [14])**.** *Let Δ^* be the parameter space of a family of Calabi-Yau manifolds. Let F^n_∞ be the limit of F^n in the sense of Hodge theory and N is the associated nilpotent operator. Then the necessary and sufficient condition for the Weil-Petersson metric to be incomplete is that $NF^n_\infty = 0$.*

In this section, we are going to prove, even if the Weil-Petersson metric is complete, we still have some restrictions on F^n_∞ for $n = 3$.

The classical Weil-Petersson metric is defined by giving a natural Hermitian metric on $H^1(X, T_X)$ induced by the Ricci flat Kähler metric for each Calabi-Yau manifold. However, by the theorem of Tian [12], We can look at the Weil-Petersson metric in a different way.

Recall that the Hodge bundle $\underline{F^n}$ over the classifying space D is the tautological bundle of the filtration

$$0 \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset H$$

The natural Hermitian metric on $\underline{F^n}$ is the polarization Q. Suppose ω is the curvature form of the Hermitian metric Q, then ω is an closed (1,1) form of D. Suppose M is a horizontal slice of D (see Griffiths [3], for example), then ω restricts to a semi-positive form on M. However, if M is the universal deformation space of a Calabi-Yau manifold, then by Tian's theorem [12], $\omega|_M$ must be positive definite and is the Weil-Petersson metric.

Thus there are some restrictions for a horizontal slice on which the ω is positive definite. The following theorem gives one of the restrictions on the limiting Hodge structure.

Theorem 4.2. *We use the notations in the above theorem and in* [14]*. If* $n = 3$*, then*

$$Q(F^3_\infty, N^3F^3_\infty - 3N^2F^3_\infty - 2NF^3_\infty) = 0$$

Proof: Let

$$\Omega = e^{\frac{1}{2\pi\sqrt{-1}}\log zN}A(z)$$

where $A(z)$ is a vector valued holomorphic function of $z \in \Delta^*$, the punctured unit disk. Let

$$F_{zzz} = (\Omega, \partial_z\partial_z\partial_z\Omega)$$

It is easy to check that

$$\lim_{z\to 0} z^3F_{zzz} = Q(F^3_\infty, N^3F^3_\infty - 3N^2F^3_\infty - 2NF^3_\infty)$$

So we need only to prove that

$$\lim_{z\to 0} z^3F_{zzz} = 0$$

Let $p \in \Delta^*$. Then since p represents a Calabi-Yau threefold, we have a map f from a neighborhood of p in Δ^* to the universal deformation space U. Suppose in local coordinates, the map f is $z \mapsto (z^1, \cdots, z^n)$. Let $Z^i = \frac{\partial z^i}{\partial z}$. Then from [6], we see that the Hodge metric on Δ^* can be written as

$$h = h_{i\bar{j}}Z^i\overline{Z}^j = (2g_{i\bar{j}} + g^{m\overline{n}}g^{p\overline{q}}F_{imp}\overline{F_{jnq}})Z^i\overline{Z}^j$$

where $g_{i\bar{j}}$ and $h_{i\bar{j}}$ are the Weil-Petersson metric and the Hodge metric, respectively. Since $g_{i\bar{j}} \leq h_{i\bar{j}}$, we have

$$h \geq h^{m\overline{n}}h^{p\overline{q}}F_{imp}\overline{F_{jnq}}Z^i\overline{Z}^j$$

By the Cauchy inequality, we see that

$$(h^{m\overline{n}}h^{p\overline{q}}F_{imp}\overline{F_{jnq}}Z^i\overline{Z}^j)h^2 \geq |F_{ijk}Z^iZ^jZ^k|^2$$

So we have

$$h^3 \geq |F_{ijk}Z^iZ^jZ^k|^2 = |F_{zzz}|^2$$

In [6], it is proved that the curvature of h is negative away from zero. So the Schwartz lemma gives,

$$h \leq \frac{C}{r^2(\log\frac{1}{r})^2}$$

Then

$$r^6|F_{zzz}|^2 \leq C\frac{r^6}{r^6(\log\frac{1}{r})^6} \to 0$$

The theorem is proved.

□

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