

ON THE HODGE METRIC OF THE UNIVERSAL DEFORMATION SPACE OF CALABI-YAU THREEFOLDS

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1. INTRODUCTIONS

A polarized Calabi-Yau manifold is a pair (X, ω) of a compact algebraic manifold X with zero first Chern class and a Kähler form $\omega \in H^2(X, \mathbb{Z})$. The form ω is called a polarization. Let \mathcal{M} be the universal deformation space of (X, ω) . \mathcal{M} is smooth by a theorem of Tian [5]. By [8], we may assume that each $X' \in \mathcal{M}$ is a Kähler-Einstein manifold. i.e. the associated Kähler metric $(g'_{\alpha\bar{\beta}})$ is Ricci flat. The tangent space $T_{X'}\mathcal{M}$ of \mathcal{M} at X' can be identified with $H^1(X', T_{X'})_\omega$ where

$$H^1(X', T_{X'})_\omega = \{\phi \in H^1(X', T_{X'}) \mid \phi \lrcorner \omega = 0\}$$

The Weil-Petersson metric G_{WP} on \mathcal{M} is defined by

$$G_{WP}(\phi, \psi) = \int_{X'} g'^{\alpha\bar{\beta}} g'_{\gamma\bar{\delta}} \phi_{\bar{\beta}}^\gamma \bar{\psi}_\alpha^\delta dV_{g'}$$

where $\phi = \phi_{\bar{\beta}}^\gamma \frac{\partial}{\partial z^\gamma} d\bar{z}^\beta$, $\psi = \psi_{\bar{\alpha}}^\delta \frac{\partial}{\partial \bar{z}^\delta} d\bar{z}^\alpha$ are in $H^1(X', T_{X'})_\omega$, $g' = g'_{\alpha\bar{\beta}} dz^\alpha d\bar{z}^\beta$ is the Kähler-Einstein metric on X associated with the polarization ω .

In this paper, we consider the universal deformation space \mathcal{M} of a simply connected Calabi-Yau threefold. Let ω_{WP} be the Kähler form of the Weil-Petersson metric and set $n = \dim H^1(X, T_X)$ for some $X \in \mathcal{M}$. We proved

Theorem 1.1. *Let $\omega_H = (n + 3)\omega_{WP} + Ric(\omega_{WP})$. Then*

1. ω_H is a Kähler metric on \mathcal{M} ;
2. The holomorphic bisectional curvature of ω_H is nonpositive.

Furthermore, Let $\alpha = ((\sqrt{n} + 1)^2 + 1)^{-1} > 0$. Then the Ricci

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curvature $\text{Ric}(\omega_H) \leq -\alpha\omega_H$ and the holomorphic sectional curvature is also less than or equal to $-\alpha$.

3. If $\text{Ric}(\omega_H)$ is bounded, then the Riemannian sectional curvature of ω_H is also bounded.

Because of the following theorem, we call ω_H the Hodge metric of the universal deformation space. For the definitions, see Section 2 and Section 3.

Theorem 1.2. *Let U be an open neighborhood of \mathcal{M} , and let $U \rightarrow D$ be the period map to the classifying space D . Then up to a constant, ω_H is the pull back of the invariant Hermitian metric of the classifying space D .*

Remark 1.1. In fact, we have proved more. The theorems are also true on the normal horizontal slices. A normal horizontal slice is a horizontal slice such that the Weil-Petersson metric can be defined. See Section 3 for details.

The proof of the first theorem is a straightforward computation using the Strominger's formula [4]. Using this method, we can find the optimal upper bound of the Ricci curvature and the holomorphic sectional curvature. The combination of the first and the second theorem is somewhat unexpected: let's explain this a little bit more in detail. By a theorem of Griffiths, we know that the holomorphic sectional curvature on the horizontal directions of the classifying space is negative away from zero. Using the same method, we know that the holomorphic bisectional curvature are nonpositive on certain directions. If D is a homogeneous Kähler manifold, then by the Gauss theorem, we should be able to prove that the holomorphic sectional curvature and the holomorphic bisectional curvature of the horizontal slice are smaller than the corresponding curvatures on the classifying space. However, D is not a homogeneous Kähler manifold in general. Nevertheless, the theorems tell us that we still have the negativity of the curvatures.

In order to prove the second theorem, we make use of the fact that D is the dual homogeneous manifold of a Kähler C-space. Write $D = G/V$ where G is a noncompact semi-simple Lie group without compact factors and V is its compact subgroup. Let K be the maximal compact

subgroup containing V . We write out explicitly the projection $G/V \rightarrow G/K$ via local coordinate. Then the metric $(n+3)\omega_{WP} + Ric(\omega_{WP})$ and the restriction of the invariant Hermitian metric of D on U can be identified.

In the last section, we gave an asymptotic estimate of the Weil-Petersson metric to the degeneration of Calabi-Yau threefolds. Such an estimate was obtained by Tian [6] in the case that the degenerated Calabi-Yau threefold has only ordinary double singular points. C-L. Wang also got such a result using a completely different method.

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2. THE CLASSIFYING SPACE AND THE HORIZONTAL SLICES

The concepts of the classifying space and the horizontal slice were introduced by Griffiths [2]. We recall his definitions and notations in this section.

Suppose X is a simply connected algebraic Calabi-Yau three-fold. The Hodge decomposition of the cohomology group $H = H^3(X, C)$ is

$$H^3(X, C) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$$

where

$$H^{p,q} = H^q(X, \Omega^p)$$

and Ω^p is the sheaf of the holomorphic p -forms. The quadratic form Q on X is defined by

$$Q(\xi, \eta) = - \int_X \xi \wedge \eta$$

By the Serre duality and the fact that the canonical bundle is trivial, $\dim H^{2,1} = \dim H^{1,2} = \dim H^1(X, T_X) = n$, and $\dim H^{3,0} = \dim H^{0,3} = 1$. Thus $H^3(X, C) = C^{2n+2}$ is a $(2n+2)$ -dimensional complex vector space.

It is easy to check that Q is skew-symmetric. Furthermore, we have the following two Hodge-Riemannian relations:

1. $Q(H^{p,q}, H^{p',q'}) = 0$ unless $p' = 3 - p$ and $q' = 3 - q$;
2. $(\sqrt{-1})^{p-q} Q(\psi, \bar{\psi}) > 0$ for any nonzero element $\psi \in H^{p,q}$.

We define the Weil operator $C : H \rightarrow H$ by

$$C|_{H^{p,q}} = (\sqrt{-1})^{p-q}$$

For any collection of $\{H^{p,q}\}$'s, set

$$\begin{aligned} F^3 &= H^{3,0} \\ F^2 &= H^{3,0} \oplus H^{2,1} \\ F^1 &= H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \end{aligned}$$

Then F^1, F^2, F^3 defines a filtration of H

$$0 \subset F^3 \subset F^2 \subset F^1 \subset H$$

Under this terminology, the Hodge-Riemannian relations can be re-written as

3. $Q(F^3, F^1) = 0, Q(F^2, F^2) = 0;$
4. $Q(C\psi, \overline{\psi}) > 0$ if $\psi \neq 0$

Now we suppose that $\{h^{p,q}\}$ is a collection of integers such that $p+q = 3$ and $\sum h^{p,q} = 2n + 2$.

Definition 2.1. *With the notations as above, the classifying space D of the Calabi-Yau three-fold is the set of all collection of subspaces $\{H^{p,q}\}$ of H such that*

$$H = \bigoplus_{p+q=3} H^{p,q} \quad H^{p,q} = \overline{H^{q,p}}, \quad \dim H^{p,q} = h^{p,q}$$

and on which Q satisfies the two Hodge-Riemannian relations 1,2.

Set $f^p = h^{n,0} + \dots + h^{p,n-p}$. Then D is also the set of all filtrations

$$0 \subset F^3 \subset F^2 \subset F^1 \subset H, \quad F^p \oplus \overline{F^{4-p}} = H$$

with $\dim F^p = f^p$ on which Q satisfies the bilinear relations 3,4.

D is a homogeneous complex manifold. The horizontal distribution $T_h(D)$ is defined as

$$T_h(D) = \{X \in T(D) \mid XF^3 \subset F^2, XF^2 \subset F^1\}$$

where $T(D)$ is the holomorphic tangent bundle which can be identified as a subbundle of the (locally trivial) bundle $Hom(H^3(X, C), H^3(X, C))$. So X naturally acts on F^p .

Definition 2.2. *A complex integral submanifold of the horizontal distribution $T_h(D)$ is called a horizontal slice.*

Suppose $U \subset \mathcal{M}$ is a neighborhood of \mathcal{M} at the point X . Then there is a natural map $p : U \rightarrow D$, called the period map, which sends a Calabi-Yau threefold to its ‘‘Hodge Structure’’. To be precise, Let $X' \in U$. Then there is a natural identification of $H^3(X', C)$ to $H^3(X, C) = H$. So $\{H^{p,q}(X')\}_{p+q=3}$ are the subspaces of H satisfying the Hodge-Riemannian Relations. We define $p(X') = \{H^{p,q}(X')\} \in D$.

3. THE WEIL-PETERSSON METRIC AND THE HODGE METRIC

On the classifying space D , we can define the so called Hodge holomorphic bundles $\underline{F}^3, \underline{F}^2, \underline{F}^1$, which are the subbundles of the locally trivial bundle \underline{C}^{2n+2} . The fiber of the bundle \underline{C}^{2n+2} at $X \in \mathcal{M}$ is $H^3(X, C)$. The fibers of $\underline{F}^3, \underline{F}^2, \underline{F}^1$ at X are $H^{3,0}(X), H^{3,0}(X) \oplus H^{2,1}(X), H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X)$, respectively. Note that \underline{F}^3 is in fact a line bundle. Let Ω be a (nonzero) local holomorphic section of \underline{F}^3 . The curvature form of the bundle \underline{F}^3 is then $\sigma = -\frac{\sqrt{-1}}{2} \partial \bar{\partial} \log Q(\Omega, \bar{\Omega})$.

Let U be a horizontal slice, define

$$\omega = \sigma|_U$$

Proposition 3.1. *Let $\omega = \frac{\sqrt{-1}}{2} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ in local coordinate. Then $g_{\alpha\bar{\beta}} \geq 0$ is semi-positive definite.*

Proof: Let $K = -\log Q(\Omega, \bar{\Omega})$. Then

$$g_{\alpha\bar{\beta}} = -\frac{Q(\partial_\alpha \Omega + K_\alpha \Omega, \overline{\partial_\beta \Omega + K_\beta \Omega})}{Q(\Omega, \bar{\Omega})}$$

But $\partial_\alpha \Omega + K_\alpha \Omega \in H^{2,1}$. The proposition follows from the second Hodge-Riemannian Relation. \square

Definition 3.1. *The horizontal slice is called normal if the form ω is positive definite at any point. In that case, $\omega = \omega_{WP}$ is called the Weil-Petersson metric on the normal horizontal slice.*

Remark 3.1. By the theorem of Tian [5], we know that if \mathcal{M} is a universal deformation space, then $\sigma|_{\mathcal{M}}$ is the Weil-Petersson metric defined in the introduction. Thus the universal deformation space is a normal horizontal slice.

Definition 3.2. *The cubic form $F = F_{ijk}$ is a (local) section of the bundle $\text{Sym}^3(T^*\mathcal{M}) \otimes (F^3)^{\otimes 2}$ defined by*

$$F_{ijk} = Q(\Omega, \partial_i \partial_j \partial_k \Omega)$$

in local coordinates (z^1, \dots, z^n) .

Definition 3.3. *Let U be a normal horizontal slice. Suppose ω_D is the Kähler form of the invariant Hermitian metric on D , then we call $\omega_D|_U$ the Hodge metric on U .*

We are going to prove

Theorem 3.1. *Suppose ω_{WP} is the Kähler form of the Weil-Petersson metric. Let*

$$\omega_1 = (n+3)\omega_{WP} + \text{Ric}(\omega_{WP})$$

then ω_1 is a constant multiple of the Hodge metric.

Before proving the theorem, we first prove

Proposition 3.2. *There is a basis e_1, \dots, e_{2n+2} of H under which Q can be represented as*

$$Q = \sqrt{-1} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

And if we let

$$f^3 = \text{span}\{e_1 - \sqrt{-1}e_{n+2}\}$$

$$f^2 = \text{span}\{e_1 - \sqrt{-1}e_{n+2}, e_2 + \sqrt{-1}e_{n+3}, \dots, e_{n+1} + \sqrt{-1}e_{2n+2}\}$$

and f^1 is the hyperplane perpendicular to f^3 with respect to Q , then

$$\{0 \subset f^3 \subset f^2 \subset f^1 \subset H\} \in D$$

□

The point $\{f^3, f^2, f^1\} \in D$ is called the original point of D . Sometimes we write it as eV if $D = G/V$.

By the curvature formula of Strominger [4], the Ricci curvature of the Weil-Petersson metric

$$R_{i\bar{j}} = -(n+1)g_{i\bar{j}} + e^{2K} F_{ipq} \overline{F_{jmn}} g^{p\bar{m}} g^{q\bar{n}}$$

where we set $\omega_{WP} = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ in the local coordinates (z^1, \dots, z^n) and K is the local function: $K = -\log Q(\Omega, \bar{\Omega})$.

Let $\omega_1 = \frac{\sqrt{-1}}{2} h_{i\bar{j}} dz^i \wedge d\bar{z}^j$. Then

$$h_{i\bar{j}} = 2g_{i\bar{j}} + e^{2K} F_{ipq} \overline{F_{jmn}} g^{p\bar{m}} g^{q\bar{n}}$$

Clearly $(h_{i\bar{j}}) > 0$.

Suppose

$$\pi' : D \rightarrow CP^{2n+1}$$

is the projection of D to CP^{2n+1} by sending (F^3, F^2, F^1) to F^3 . Let U be a normal horizontal slice. Let Ω be a (nonzero) local section of $\underline{F^3}$. Then $\partial_i \Omega + K_i \Omega$ is not zero because ω_{WP} is positive. Thus

$$\pi' : U \rightarrow D \rightarrow CP^{2n+1}$$

is an immersion.

Now we consider the result of Bryant and Griffiths [1]. Their results can be briefly written as follows:

We assume that $eV \in U$. i.e. the normal horizontal slice passes the original point of D , where the original point is defined as $\{f^3, f^2, f^1\} \in D$ in the Proposition 3.2. Then according to Bryant and Griffiths, there is a holomorphic function u defined on a neighborhood of the original point of \mathbb{C}^n such if (z^1, \dots, z^n) is the local holomorphic coordinate of U at eV , the original point, then

$$(3.1) \quad \Omega = (1, \frac{1}{\sqrt{2}}z^1, \dots, \frac{1}{\sqrt{2}}z^n, u - \sum_i \frac{1}{2}z^i u_i, \frac{1}{\sqrt{2}}u_1, \dots, \frac{1}{\sqrt{2}}u_n)$$

with $F^1 = \text{Span}\{(\Omega)\}$, $F^2 = \text{span}\{\nabla\Omega\}$, $F^1 \perp F^3$ via Q . In particular, $u(0) = -\sqrt{-1}$, $|\nabla u(0)| = 0$, $\nabla^2 u(0) = \sqrt{-1}I$, where I is the unit matrix.

In order to prove Theorem 3.1, we need only to prove it at the original point, because any point of the homogeneous space D can be taken as the original point.

Under these notations, at eV , $D_i \Omega = \partial_i \Omega + K_i \Omega = \frac{1}{\sqrt{2}}(e_{i+1} + \sqrt{-1}e_{n+i+2})$. Thus

$$\sqrt{-1}Q(\Omega, \bar{\Omega}) = -2$$

and

$$g_{i\bar{j}} = -\frac{Q(D_i\Omega, \overline{D_j\Omega})}{Q(\Omega, \overline{\Omega})} = \frac{1}{2}\delta_{ij}$$

Furthermore, the cubic form F_{ijk} at eV is

$$F_{ijk} = -\frac{1}{2}\frac{\partial^3 u}{\partial z^i \partial z^j \partial z^k}(0) = -\frac{1}{2}u_{ijk}(0)$$

Thus

$$h_{i\bar{j}} = 2g_{i\bar{j}} + e^{2K} F_{imn} \overline{F_{jpq}} g^{m\bar{p}} g^{n\bar{q}} = \delta_{ij} + \frac{1}{4}u_{imn}(0)\overline{u_{jmn}}(0)$$

Now we are going to prove that $(h_{i\bar{j}})$ is a constant multiple of the Hodge metric. Consider the projection

$$\pi : D = G/V \rightarrow G/K$$

where K is the maximal connected compact subgroup of G containing V . We have

Lemma 3.1. *Let U be a horizontal slice, then π is an isometry between the Riemannian submanifold U of D and the Riemannian submanifold $\pi(U)$ of G/K .*

Proof: Note that U is a horizontal slice of D . The lemma follows from the definition of the invariant Hermitian metric on both manifold. \square

From the above lemma, we know that in order to compute $\omega_D|_U$, we need only computed the metric of U as a submanifold of G/K , even the map π is not holomorphic (Recall that D is not homogeneous Kähler, so the map will not be holomorphic in general). In order to do this, we write out the projection

$$\pi : G/V \rightarrow G/K$$

explicitly now.

It is easy to prove from linear algebra that the projection π send

$$0 \subset F^3 \subset F^2 \subset F^1 \subset H$$

to

$$F^3 \oplus H^{1,2}$$

We have known that $G/K = Sp(n+1, R)/U(n+1)$ is the Hermitian symmetric space. G/K can be realized as the set of $(n+1)$ planes P in the C^{2n+2} space such that $-\sqrt{-1}Q(P, \overline{P}) > 0$. Thus G/K can be

represented as the set of all the symmetric $(n+1) \times (n+1)$ matrix Z satisfying $Im Z > 0$ where $Im Z > 0$ means $Im Z$ is a positive definite Hermitian matrix.

We write the entries of the matrix Z as functions of D .

Suppose now near the original point, F^3 is spanned by

$$(1, z^t, a, \alpha^t)$$

where $z, \alpha \in \mathbb{C}^n, a \in \mathbb{C}$. And suppose F^2 is spanned by the row vectors of the matrix

$$\begin{pmatrix} 1 & z^t & a & \alpha^t \\ 0 & 1 & \beta & A \end{pmatrix}$$

for $\beta \in \mathbb{C}^n, A \in \mathfrak{gl}(n, \mathbb{C})$. Then by the first Hodge-Riemannian relation

$$Q(F^2, F^2) = 0$$

we know that

$$\beta = \alpha - Az, \quad A^t = A$$

So locally, we can represented F^2 by the matrix

$$(3.2) \quad \begin{pmatrix} 1 & z^t & a & \alpha^t \\ 0 & 1 & \alpha - Az & A \end{pmatrix} \quad A^t = A$$

Let $\Omega = (1, z^t, a, \alpha^t)$, and let $\begin{pmatrix} \Omega \\ \Theta \end{pmatrix}$ be a local section of F^2 with $\Theta = (0, 1, \alpha - Az, A)$. Set

$$\begin{aligned} m &= Q(\Omega, \bar{\Omega}) = -a + \bar{a} - \alpha^t \bar{z} + \bar{\alpha}^t z \\ \xi &= Q(\Omega, \bar{\Omega}) = -\alpha + \bar{\alpha} - \bar{A}(\bar{z} - z) \end{aligned}$$

where $m \in \mathbb{C}, \xi \in \mathbb{C}^n$. It is easily checked that

$$Q(\Omega, \bar{\Theta} - \frac{\xi}{m} \bar{\Omega}) = 0$$

So $\bar{\Theta} - \frac{\xi}{m} \bar{\Omega} \in F^1$ and since Ω and Θ are in $H^{2,1}$, $\bar{\Theta} - \frac{\xi}{m} \bar{\Omega} \in H^{1,2}$.

The projection π can be locally written as

$$\begin{aligned} \begin{pmatrix} 1 & z^t & a & \alpha^t \\ 0 & 1 & \alpha - Az & A \end{pmatrix} &\longrightarrow \begin{pmatrix} \Omega \\ \bar{\Theta} - \frac{\xi}{m} \bar{\Omega} \end{pmatrix} = \\ \begin{pmatrix} 1 & z^t & a & \alpha^t \\ -\frac{\xi}{m} & 1 - \frac{\xi \bar{z}^t}{m} & \frac{a}{\alpha - Az} - \frac{\bar{a} \xi}{m} & \bar{A} - \frac{\xi}{m} \bar{\alpha}^t \end{pmatrix} \end{aligned}$$

where the right hand side of the above represents an $(n+1)$ -plane in H .

The symmetric $(n+1) \times (n+1)$ matrix Z can be obtained as follows: let

$$\mu = \frac{1}{m - (\bar{z}^t - z^t)\xi}$$

Then as a matrix

$$1 + \mu\xi(\bar{z}^t - z^t) = \left(1 - \frac{\xi}{m}(\bar{z}^t - z^t)\right)^{-1}$$

We have

$$\begin{pmatrix} 1 & z^t \\ -\frac{\xi}{m} & 1 - \frac{\xi\bar{z}^t}{m} \end{pmatrix}^{-1} = \begin{pmatrix} 1 - z^t B \frac{\xi}{m} & -z^t B \\ B \frac{\xi}{m} & B \end{pmatrix}$$

where $B = 1 + \mu\xi(\bar{z}^t - z^t)$. Let

$$\begin{pmatrix} 1 - z^t B \frac{\xi}{m} & -z^t B \\ B \frac{\xi}{m} & B \end{pmatrix} \begin{pmatrix} \frac{a}{\alpha - Az} - \frac{\bar{a}\xi}{m} & \frac{\alpha^t}{\bar{A} - \frac{\xi}{m}\bar{\alpha}^t} \end{pmatrix} = \begin{pmatrix} D_1 & D_2^t \\ D_3 & D_4 \end{pmatrix}$$

for $D_1 \in \mathbb{C}, D_2, D_3 \in \mathbb{C}^n, D_4 \in \mathfrak{gl}(n, \mathbb{C})$. Then it can be computed

$$(3.3) \quad \begin{cases} D_1 = a - z^t(\alpha - \bar{A}z) + \mu(z^t\xi)^2 \\ D_2 = D_3 = (a - \bar{a})\mu\xi + B(\alpha - Az) \\ D_4 = \bar{A} + \mu\xi\xi^t \end{cases}$$

Then the matrix Z is obtained:

Proposition 3.3. *Under the notation as above, the map*

$$\pi : G/V \rightarrow G/K$$

under the local coordinate described as above is

$$\begin{pmatrix} 1 & z^t & a & \alpha^t \\ & 1 & \alpha - Az & A \end{pmatrix} \longrightarrow \begin{pmatrix} D_1 & D_2^t \\ D_3 & D_4 \end{pmatrix} = Z$$

where the D_i 's are defined as in equation (3.3).

The Hermitian metric on G/K is $-\sqrt{-1}\partial\bar{\partial} \log \det \operatorname{Im} Z$. In particular, at the original point, it is $\sum_{ij} dZ^{ij} \wedge d\bar{Z}^{ij}$, where we set $Z^{ij} = Z^{ji}$ if $i > j$. By Equation 3.1 we see that in order to get

the map $U \rightarrow G/K$, z, a, α, A in Equation 3.3 should be replaced by $\frac{1}{\sqrt{2}}z, u - \sum \frac{1}{2}z^i u_i, \frac{1}{\sqrt{2}}\nabla u, \frac{1}{\sqrt{2}}u_{ij}$, respectively. Thus we have

$$(3.4) \quad \begin{cases} \frac{\partial D_1}{\partial t_k} = 0, \frac{\partial D_1}{\partial \bar{t}_k} = 0 \\ \frac{\partial (D_3)_r}{\partial t_k} = -\sqrt{2}i\delta_{rk}, \frac{\partial (D_3)_r}{\partial \bar{t}_k} = 0 \\ \frac{\partial D_4}{\partial t_k} = 0, \frac{\partial (D_4)_{rs}}{\partial \bar{t}_k} = \bar{u}_{rsk} \end{cases}$$

By a straightforward computation, we know the restriction of the metric on G/K on U at the original point is a constant multiple of

$$h_{i\bar{j}} = \delta_{ij} + \frac{1}{4}u_{imn}(0)\overline{u_{jmn}}(0)$$

Thus completes the proof.

4. THE CURVATURE COMPUTATION

In this section we give an optimal estimate of the upper bound of the holomorphic sectional curvature, bisectional curvature and the Ricci curvature of a normal horizontal slice.

Let U be a normal horizontal slice. Suppose $(g_{i\bar{j}})$ is the Weil-Petersson metric, (F_{ijk}) is the cubic form, and $K = -\log Q(\Omega, \bar{\Omega})$. The Hodge metric $(h_{i\bar{j}})$ is:

$$h_{i\bar{j}} = 2g_{i\bar{j}} + \sum_{rspq} e^{2K} F_{irs} \overline{F_{jpq}} g^{r\bar{p}} g^{s\bar{q}}$$

As we have proved, $(h_{i\bar{j}})$ is a Kähler metric. So the curvature tensor $\tilde{R}_{i\bar{j}k\bar{l}}$ of $(h_{i\bar{j}})$ is

$$\tilde{R}_{i\bar{j}k\bar{l}} = \frac{\partial^2 h_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} - h^{n\bar{m}} \frac{\partial h_{i\bar{m}}}{\partial z^k} \frac{\partial h_{n\bar{j}}}{\partial \bar{z}^l}$$

Now we suppose that at point p , the local coordinate for the Weil-Petersson metric is normal. i.e., at point p , $g_{i\bar{j}} = \delta_{ij}$ and $dg_{i\bar{j}} = 0$. Furthermore, assume $K(p) = 0$. The curvature tensor $R_{i\bar{j}k\bar{l}}$ of $(g_{i\bar{j}})$ then is

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l}$$

also we have

$$(4.1) \quad \frac{\partial h_{i\bar{m}}}{\partial z^k} = \sum_{rs} F_{irs,k} \overline{F_{mrs}}$$

where

$$F_{irs,k} = \partial_k F_{irs} + 2K_k F_{irs}$$

is the covariant derivative of the cubic form with respect to the Weil-Petersson metric. By using the Strominger formula at p ,

$$R_{i\bar{j}k\bar{l}} = \delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} - F_{ikm}\overline{F_{jlm}}$$

We get

$$(4.2) \quad \begin{aligned} \frac{\partial^2 h_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} &= 2R_{i\bar{j}k\bar{l}} - 2 \sum_{sqr} R_{q\bar{s}k\bar{l}} F_{irs} \overline{F_{jrq}} \\ &+ 2\delta_{kl} \sum_{rs} F_{irs} \overline{F_{jrs}} + \sum_{rs} F_{irs,k} \overline{F_{jrs,l}} \end{aligned}$$

Combining Equation 4.1 and Equation 4.2, we have

Proposition 4.1. *If $K = 0$ at the point p ,*

$$(4.3) \quad \begin{aligned} \tilde{R}_{i\bar{j}k\bar{l}} &= 2R_{i\bar{j}k\bar{l}} + 2\delta_{kl} \sum_{rs} F_{irs} \overline{F_{jrs}} - 2 \sum_{sqr} R_{q\bar{s}k\bar{l}} F_{irs} \overline{F_{jrq}} \\ &+ \sum_{rs} F_{irs,k} \overline{F_{jrs,l}} - \sum_{mn} \left(\sum_{rs} F_{irs,k} \overline{F_{mrs}} \right) \overline{\left(\sum_{rs} F_{jrs,l} \overline{F_{nrs}} \right)} h^{n\bar{m}} \end{aligned}$$

Based on the above proposition, we get

Theorem 4.1. *Let $c(n) = ((\sqrt{n} + 1)^2 + 1)$, then*

$$\begin{aligned} Ric(\omega_H) &\leq -\frac{1}{c(n)}\omega_H \\ R &\leq -\frac{1}{c(n)} \end{aligned}$$

where R is the superium of the holomorphic sectional curvature. The constant here is optimal. Furthermore, the bisectional curvature is non-positive.

Proof: We consider the point p and the normal coordinate at p with respect to the Weil-Petersson metric. Fixing i , let

$$A_m = \sum_{rsk} F_{irs,k} a_k \overline{F_{mrs}}$$

for a vector $a = (a_1, \dots, a_n)$. Then it is easy to see that

$$(4.4) \quad \begin{aligned} &\sum_{pq} \left| \sum_k F_{ipq,k} a_k \right|^2 - \sum_{mn} \left(\sum_{rsk} F_{irs,k} a_k \overline{F_{mrs}} \right) \overline{\left(\sum_{rsk} F_{irs,k} a_k \overline{F_{nrs}} \right)} h^{n\bar{m}} \\ &= \sum_{pq} \left| \sum_k F_{ipq,k} a_k - \sum_{mn} h^{n\bar{m}} A_m F_{npq} \right|^2 + 2 \sum_a \left| \sum_m h^{a\bar{m}} A_m \right|^2 \end{aligned}$$

where we use the fact that $h_{i\bar{j}} = 2\delta_{ij} + F_{imn}\overline{F_{jmn}}$ at p .

Define a generic vector $a^k = \delta_{ik}, k = 1, \dots, n$. Using Equation 4.4, we have

$$\sum_{pq} |F_{ipq,i}|^2 - \sum_{mn} h^{n\bar{m}} \left(\sum_{rs} F_{irs,i} \overline{F_{mrs}} \right) \overline{\sum_{rs} F_{irs,i} \overline{F_{nrs}}} \geq 0$$

Now using Proposition 4.1, we get

$$\begin{aligned} \tilde{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} a^\alpha \bar{a}^\beta a^\gamma \bar{a}^\delta &= \tilde{R}_{i\bar{i}i\bar{i}} \geq 2R_{i\bar{i}i\bar{i}} + 2 \sum_{rs} |F_{irs}|^2 - 2 \sum_{sqr} R_{q\bar{s}i\bar{i}} F_{irs} \overline{F_{irq}} \\ &\geq 4 - 4 \sum_r |F_{iir}|^2 + 2 \sum_{rp} \left| \sum_q F_{qip} \overline{F_{irq}} \right|^2 \end{aligned}$$

Let

$$x = \sum_r |F_{iir}|^2$$

Then we have

$$\begin{aligned} \sum_{rp} \left| \sum_q F_{qip} \overline{F_{irq}} \right|^2 &\geq \left| \sum_q |F_{iiq}|^2 \right|^2 = x^2 \\ \sum_{rp} \left| \sum_q F_{qip} \overline{F_{irq}} \right|^2 &\geq \sum_r \left| \sum_q |F_{qir}|^2 \right|^2 \geq \frac{1}{n} (h_{i\bar{i}} - 2)^2 \end{aligned}$$

So for $a, b > 0, a + b = 1$, we have

$$\begin{aligned} \frac{1}{2} \tilde{R}_{i\bar{i}i\bar{i}} &\geq 2 - 2x + ax^2 + \frac{b}{n} (h_{i\bar{i}} - 2)^2 \\ &\geq 2 - \frac{1}{a} + \frac{b}{n} (h_{i\bar{i}} - 2)^2 \end{aligned}$$

On the other hand, we have

$$h_{\alpha\bar{\beta}} a^\alpha \bar{a}^\beta = h_{i\bar{i}}$$

Let $a = \frac{2+\sqrt{n}}{2+2\sqrt{n}}$ and $b = 1 - a$, we have

$$\tilde{R}_{i\bar{i}i\bar{i}} \geq \frac{1}{(\sqrt{n} + 1)^2 + 1} h_{i\bar{i}}^2 = \frac{1}{c(n)} |h_{i\bar{j}} a^i \bar{a}^j|^2$$

It is a straightforward computation that the constant here is optimal.

Thus we proved $\tilde{R}(a, \bar{a}, a, \bar{a}) \geq \|a\|^2$. Since a can be any vector by making a linear transformation of the normal coordinate. We have already proved the assertion of the theorem about the holomorphic sectional curvature.

Now we turn to the bisectonal curvature. For any (a^1, \dots, a^n) , using the same inequalities before, we have

$$\begin{aligned} \tilde{R}_{i\bar{i}k\bar{k}} a^k \bar{a}^l &\geq 2 \sum_k |a^k|^2 + 2|a^i|^2 - 4 \sum_r \left| \sum_k F_{irk} a^k \right|^2 \\ &+ 2 \sum_{mr} \left| \sum_{qk} F_{qkm} \overline{F_{irq}} a^k \right|^2 \geq 2|a^i|^2 + 2 \sum_r \left| \sum_{qk} F_{iqk} \overline{F_{iqr}} a^k - a^r \right|^2 \geq 0 \end{aligned}$$

This proves the nonpositivity of the bisectonal curvature.

Finally we consider the Ricci curvature. Suppose that ξ is a unit vector. Then by the definition of the Ricci curvature and above results, we have

$$-Ric(\xi, \bar{\xi}) \geq \tilde{R}(\xi, \bar{\xi}, \xi, \bar{\xi})$$

This completes the proof of the theorem.

5. THE BOUNDNESS OF THE SECTIONAL CURVATURE

In this section, we prove that the boundness of the Ricci curvature implies the boundness of the Riemannian sectional curvature.

Theorem 5.1. *Suppose U is a normal horizontal slice. Suppose $p \in U$ is a fixed point such that the Ricci curvature has a lower bound C_p at p . That is*

$$Ric(\omega_H)_p \geq -C_p(\omega_H)_p$$

Then the Riemannian sectional curvature has a bound

$$|\tilde{R}(X, Y, X, Y)| \leq (3 + C_p) \|X\|^2 \|Y\|^2$$

where $X, Y \in T_p U$ and $X \perp Y$.

We begin by restating Proposition 4.1 in the section 4.

Proposition 5.1. *Suppose we have the notations as in the proposition 4.1, then we have*

$$\tilde{R}_{i\bar{j}k\bar{l}} = A_{i\bar{j}k\bar{l}} + B_{i\bar{j}k\bar{l}}$$

where

$$\begin{aligned}
A_{i\bar{j}k\bar{l}} &= 2\delta_{ij}\delta_{kl} + 2\delta_{il}\delta_{kj} - 4 \sum_s F_{iks}\overline{F_{jls}} + 2 \sum_{mnpq} F_{qkm}\overline{F_{plm}}F_{inp}\overline{F_{jnq}} \\
B_{i\bar{j}k\bar{l}} &= \sum_{rs} (F_{irs,k} - \sum_{mn} A_{ik\bar{m}}F_{nrs}h^{n\bar{m}}) \overline{(F_{jrs,l} - \sum_{mn} A_{jl\bar{m}}F_{nrs}h^{n\bar{m}})} \\
&\quad + 2 \sum_{mm_1n} A_{ik\bar{m}}h^{n\bar{m}}\overline{A_{jl\bar{m}_1}}h^{m_1\bar{n}}
\end{aligned}$$

here we define

$$A_{ik\bar{m}} = \sum_{rs} F_{irs,k}\overline{F_{mrs}}$$

Proof: A straightforward computation. \square

Lemma 5.1. Suppose that $\xi, \eta \in T_p^{(1,0)}U$ and define $\|\xi\|^2 = h_{i\bar{j}}\xi^i\overline{\xi^j}$, then

$$|\tilde{R}_{i\bar{j}k\bar{l}}\xi^i\xi^k\overline{\eta^j\eta^l}| \leq (6 + C_p)\|\xi\|^2\|\eta\|^2$$

Proof: Note that the holomorphic bisectional curvature of U is non-positive. We know that the holomorphic sectional curvature is bounded by C_p , i.e.

$$|\tilde{R}_{i\bar{j}k\bar{l}}\xi^i\xi^k\overline{\xi^j\xi^l}| \leq C_p\|\xi\|^4$$

We have

$$\begin{aligned}
&\sum_{pq} \left| \sum_{nij} F_{inp}\overline{F_{jnq}}\xi^i\overline{\eta^j} \right|^2 \\
&= \sum_{pq} \left| \sum_n \left(\sum_i F_{inp}\xi^i \right) \overline{\left(\sum_j F_{jnq}\eta^j \right)} \right|^2 \\
&\leq \sum_{pq} \left(\sum_n \left| \sum_i F_{inp}\xi^i \right|^2 \sum_n \left| \sum_j F_{jnq}\eta^j \right|^2 \right) \\
&= \sum_{pn} \left| \sum_i F_{inp}\xi^i \right|^2 \sum_{qn} \left| \sum_j F_{jnq}\eta^j \right|^2 \\
&\leq \|\xi\|^2\|\eta\|^2
\end{aligned}$$

here we use the fact that

$$h_{i\bar{j}} = 2\delta_{ij} + \sum_{rs} F_{irs}\overline{F_{jrs}}$$

and

$$\sum_{pn} \left| \sum_i F_{inp} \xi^i \right|^2 = \sum_{i,j} \sum_{pn} F_{inp} \xi^i \overline{F_{jnp} \xi^j} \leq \|\xi\|^2$$

Thus

$$\begin{aligned} & \sum_{ijkl} \sum_{mnpq} F_{qkm} \overline{F_{plm}} F_{inp} \overline{F_{jnq}} \xi^i \xi^k \overline{\eta^j} \overline{\eta^l} \\ &= \sum_{pq} \left(\sum_{nij} F_{inp} \overline{F_{jnq}} \xi^i \overline{\eta^j} \right) \left(\sum_{mkl} F_{qkm} \overline{F_{plm}} \xi^k \overline{\eta^l} \right) \\ &\leq \sqrt{\sum_{pq} \left| \sum_{nij} F_{inp} \overline{F_{jnq}} \xi^i \overline{\eta^j} \right|^2} \sqrt{\sum_{pq} \left| \sum_{mkl} F_{qkm} \overline{F_{plm}} \xi^k \overline{\eta^l} \right|^2} \\ &\leq \|\xi\|^2 \|\eta\|^2 \end{aligned}$$

We also have

$$\sum_s \left| \sum_{ik} F_{iks} \xi^i \xi^k \right|^2 \leq \sum_s \left(\sum_k \left| \sum_i F_{iks} \xi^i \right|^2 \sum_k |\xi^k|^2 \right) \leq \|\xi\|^4$$

Thus by proposition 5.1, we have

$$\left| \sum_{ijkl} A_{i\bar{j}k\bar{l}} \xi^i \xi^k \overline{\eta^j} \overline{\eta^l} \right| \leq 6 \|\xi\|^2 \|\eta\|^2$$

We also have

$$\sum_{ijkl} B_{i\bar{j}k\bar{l}} \xi^i \xi^k \overline{\eta^j} \overline{\eta^l} \leq \sqrt{\sum_{ijkl} B_{i\bar{j}k\bar{l}} \xi^i \xi^k \overline{\xi^j} \overline{\xi^l}} \sqrt{\sum_{ijkl} B_{i\bar{j}k\bar{l}} \eta^i \eta^k \overline{\eta^j} \overline{\eta^l}}$$

Combining the above two inequalities we proved the lemma. \square

Proof of Theorem 5.1:

Let

$$\begin{aligned} \xi &= X - \sqrt{-1}JX \\ \eta &= Y - \sqrt{-1}JY \end{aligned}$$

Then

$$\tilde{R}(X, Y, X, Y) = \frac{1}{8} (\operatorname{Re} \tilde{R}(\xi, \bar{\eta}, \xi, \bar{\eta}) - \tilde{R}(\xi, \bar{\xi}, \eta, \bar{\eta}))$$

The bisectonal curvature is bounded by C_p

$$|\tilde{R}(\xi, \bar{\xi}, \eta, \bar{\eta})| \leq C_p \|\xi\|^2 \|\eta\|^2$$

Thus

$$|\tilde{R}(X, Y, X, Y)| \leq \frac{1}{4} (3 + C_p) \|\xi\|^2 \|\eta\|^2 = (3 + C_p) \|X\|^2 \|Y\|^2$$

6. AN ASYMPTOTIC ESTIMATE

In this section, we make use of the results in the previous sections to prove an asymptotic estimate of the Weil-Petersson metric of the degeneration of Calabi-Yau threefolds. The motivation for the estimate is from a result of G. Tian [6]. Although the argument can be generalized to study the Weil-Petersson metric of a normal horizontal slice near infinity, we restrict ourselves to the degeneration of Calabi-Yau threefolds.

We say $\pi : \mathfrak{X} \rightarrow \Delta$ is a degeneration of Calabi-Yau threefolds, if \mathfrak{X} , Δ are complex manifolds and π is holomorphic, and Δ is the unit disk in \mathbb{C} . $\forall t \in \Delta$, $t \neq 0$, $\pi^{-1}(t)$ is a smooth Calabi-Yau threefold while $\pi^{-1}(0)$ is a divisor of normal crossing. We also denote Δ^* to be the punctured unit disk.

Theorem 6.1. *Suppose $\mathfrak{X} \rightarrow \Delta$ is a degeneration of Calabi-Yau threefolds. Suppose ω be the Weil-Petersson metric on Δ^* . Then if*

$$\lim_{r \rightarrow 0} \frac{\log \omega}{\log \frac{1}{r}} = 0$$

Then

$$\omega \leq \sqrt{-1} C_1 \left(\log \frac{1}{r}\right)^{4c(n)} dz \wedge d\bar{z}$$

where $c(n) = ((\sqrt{n} + 1)^2 + 1)$, C_1 is a constant and z is the coordinate of Δ .

Remark 6.1. ω is a Kähler metric on Δ^* . So there is a function $\lambda(z) > 0$ on Δ^* such that

$$\omega = \sqrt{-1} \lambda(z) dz \wedge d\bar{z}$$

The assumption is understood as

$$\lim_{r \rightarrow 0} \frac{\log \lambda}{\log \frac{1}{r}} = 0$$

and the conclusion of the theorem is understood as

$$\lambda(z) \leq C_1 \left(\log \frac{1}{r}\right)^{4c(n)}$$

Proof: By a theorem of Tian, there are no obstructions towards the deformation of a Calabi-Yau three-fold. Suppose $M \in \pi^{-1}(\Delta^*)$ is a fiber. Let $n = \dim H^1(M, \Theta)$ and $\pi(M) = p$. Let \mathcal{M} be the universal deformation space at p . Then there is a neighborhood U of Δ^* at p such that $U \subset \mathcal{M}$. Suppose $z_1 = z$, and suppose U is defined by $z_2 = \cdots = z_n = 0$ near the point p . Using the notations as in the previous sections, by the Strominger's formula, we have

$$(6.1) \quad R_{1\bar{1}1\bar{1}} = 2g_{1\bar{1}}^2 - \frac{1}{(\Omega, \bar{\Omega})^2} F_{11\xi} \overline{F_{11\eta}} g^{\xi\bar{\eta}}$$

Here $(\Omega, \bar{\Omega}) = \sqrt{-1}Q(\Omega, \bar{\Omega})$

Lemma 6.1. *Let $\lambda = g_{1\bar{1}}$, then*

$$\lambda^{-1} \frac{1}{(\Omega, \bar{\Omega})^2} F_{11\xi} \overline{F_{11\eta}} g^{\xi\bar{\eta}} \leq \frac{1}{(\Omega, \bar{\Omega})^2} F_{1\alpha\xi} \overline{F_{1\beta\eta}} g^{\alpha\bar{\beta}} g^{\xi\bar{\eta}}$$

Proof: If $g_{1\bar{\beta}} = 0$ for $\beta \neq 1$, then the inequality is trivially true. Thus we would like to choose a coordinate such that $g_{1\bar{\beta}} = 0, \beta \neq 1$.

Let A be an $(n-1) \times (n-1)$ matrix. Let

$$w_i = \sum_{j=2}^n A_{ij} z_j$$

for $i = 2, \dots, n$. If A is a nonsingular matrix, then (z_1, w_2, \dots, w_n) will be local holomorphic coordinate of \mathcal{M} at p and Δ^* is again be defined by $w_2 = \cdots = w_n = 0$. Now we choose an A such that

$$\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial w^k} \right) = 0$$

for $k = 2, \dots, n$. Suppose $\tilde{g}_{\alpha\bar{\beta}}$ is the matrix under the coordinate (z_1, w_2, \dots, w_n) , $(\tilde{g}_{1\alpha}) = 0$ for $\alpha \neq 1$. □

Using the lemma, from Equation 6.1, we have

$$R_{1\bar{1}1\bar{1}} \geq 2\lambda^2 - \lambda(h_{1\bar{1}} - 2g_{1\bar{1}}) \geq -\lambda h_{1\bar{1}}$$

Suppose $\Delta = \frac{\partial^2}{\partial z_1 \partial \bar{z}_1}$, then the Gauss curvature of Δ^* with respect to λ is

$$K = -\frac{4}{\lambda} \Delta \log \lambda$$

On the other hand

$$\Delta \log \log \frac{1}{r} = -\frac{1}{4r^2 (\log \frac{1}{r})^2}$$

So we have

$$\Delta \log \lambda = -\frac{1}{4}\lambda K \geq \frac{1}{\lambda}R_{1\bar{1}1\bar{1}} \geq -h_{1\bar{1}}$$

where we use the Gauss formula $4R_{1\bar{1}1\bar{1}} \leq -K\lambda^2$. The holomorphic sectional curvature of $(h_{1\bar{1}})$ is less than $-\frac{1}{c(n)}$. Thus by the Schwartz lemma

$$h_{1\bar{1}} \leq \frac{c(n)}{(r \log \frac{1}{r})^2}$$

Thus

$$\Delta \log \frac{\lambda}{(\log \frac{1}{r})^{4c(n)}} \geq 0$$

The rest of the proof is quite elementary: let

$$f = \log \frac{\lambda}{(\log \frac{1}{r})^{4c(n)}}$$

Then

$$\lim_{r \rightarrow 0} \frac{f}{\log \frac{1}{r}} = 0$$

So for any ϵ , there is a δ such that $r < \delta$ implies $-f + \epsilon \log \frac{1}{r}$ large enough. Now

$$\Delta(-f + \epsilon \log \frac{1}{r}) \leq 0$$

So the minimum point must be obtained at $|r| = \frac{1}{2}$. So

$$f - \epsilon \log \frac{1}{r} \leq C_1 + \epsilon \log 2 \leq 2C_1$$

for any ϵ small. thus letting $\epsilon \rightarrow 0$, we have $f \leq 2C_1$, which completes the proof.

Remark 6.2. In Hayakawa [3], the author claimed a relation between the degeneration of the Calabi-Yau manifolds and the noncompleteness of the Weil-Petersson metric. But her proof was incomplete. C-L. Wang [7] gave a proof of this and studied the Weil-Petersson metric in great detail. In particular, he proved an asymptotic estimate for the degeneration of Calabi-Yau manifold which is slightly sharper than our estimate independently using a different method.

REFERENCES

- [1] Robert Bryant and Phillip Griffiths. Some Observations on the Infinitesimal Period Relations for Regular Threefolds with Trivial Canonical Bundle. In Michael Artin and John Tate, editors, *Arithmetic and Geometry*, pages 77–85. Boston, Birkhäuser, 1983.
- [2] Phillip Griffiths, editor. *Topics in Transcendental Algebraic Geometry*, volume 106 of *Ann. Math Studies*. Princeton University Press, 1984.
- [3] Yoshiko Hayakawa. Degeneration of Calabi-Yau Manifold with Weil-Petersson Metric. Technical Report alg-geom/9507016, Okolahoma State University, July 1995.
- [4] Andrew Strominger. Special Geometry. *Comm. Math. Phys.*, 133:163–180, 1990.
- [5] Gang Tian. Smoothness of the Universal Deformation Space of Compact Calabi-Yau Manifolds and its Peterson-Weil Metric. In Shing-Tung Yau, editor, *Mathematical aspects of string theory*, volume 1, pages 629–646. World Scientific, 1987.
- [6] Gang Tian. Smoothing 3-folds with Trivial Canonical Bundle and Ordinary Double Points. In Shing-Tung Yau, editor, *Essays in Mirror Symmetry*, pages 458–479. International Press, 1992.
- [7] Chin-Long Wang. On the Weil-Petersson Metrics and Degeneration of Calabi-Yau Manifolds. Technical report, Harvard University, Dec. 1996.
- [8] Shing-Tung Yau. On the Ricci Curvature of a Compact Kähler Manifold and the Complex Monge-Ampere Equation, I,. *Comm. Pure. Appl. Math.*, 31:339–411, 1978.

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