Special Geometric Structures in String Theory
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# GRADIENT ESTIMATES OF THE YUKAWA COUPLING

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## Contents



## 1. Introduction

A polarized Calabi-Yau manifold is a pair $(X,\omega)$ of a compact algebraic manifold $X$ with zero first Chern class and a Kähler form $\omega \in H^2(X,Z)$. The form $\omega$ is called a polarization. Let $\mathcal{M}$ be the universal deformation space of $(X,\omega)$. $\mathcal{M}$ is smooth by the theorem of Tian [8]. By [9], we may assume that each $X' \in \mathcal{M}$ is a Kähler-Einstein manifold. i.e. the associated Kähler metric $(g'_{\alpha\overline{\beta}})$ is Ricci flat. The tangent space $T_{X'}\mathcal{M}$ of $\mathcal{M}$ at $X'$ can be identified with $H^1(X',T_{X'})_\omega$ where

$$H^1(X',T_{X'})_\omega = \{\varphi \in H^1(X',T_{X'}) | \varphi \lrcorner \omega = 0\}.$$

The Weil-Petersson metric $G_{PW}$ on $\mathcal{M}$ is defined by

$$G_{WP}(\varphi,\psi) = \int_{X'} g'^{\alpha\overline{\beta}} g'_{\gamma\overline{\delta}} \varphi^{\gamma}_{\overline{\beta}} \overline{\psi^{\delta}_{\overline{\alpha}}} dV_{g'},$$

where $\varphi = \varphi^{\gamma}_{\overline{\beta}} \frac{\partial}{\partial z^\gamma} d\overline{z}^\beta$, $\psi = \psi^{\delta}_{\overline{\alpha}} \frac{\partial}{\partial z^\delta} d\overline{z}^\alpha$ are in $H^1(X',T_{X'})_\omega$, $g' = g'_{\alpha\overline{\beta}} dz^\alpha d\overline{z}^\beta$ is the Kähler-Einstein metric on $X'$ associated with the polarization $\omega$.

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A natural question on $\mathcal{M}$ is that whether the Weil-Petersson metric is complete. In [4], the author proved that there are no non-trivial complete special Kähler manifolds. If $\dim M = 3$, then $\mathcal{M}$ is a projective special Kähler manifold [2]. The corresponding conjecture in this case would be:

**Conjecture.** If the moduli space $\mathcal{M}$ of a Calabi-Yau threefold is complete with respect to the Weil-Petersson metric, then it is locally symmetric.

*Remark* 1. We don't have examples of Calabi-Yau threefolds whose moduli spaces are locally symmetric . But locally symmetric moduli space is the best we can expect. Our results in this paper can easily be extended to horizontal slices. In the last section, we will construct a horizontal slice which is locally symmetric (and complete).

In the rest of this paper, we will concentrate on moduli space of Calabi-Yau threefolds. General moduli space will be considered elsewhere [5].

In the three dimensional case, associated to the Weil-Petersson metric is the Yukawa coupling. It can be defined as

$$F(\varphi, \psi, \xi) = \int_{X'} \varphi \wedge \psi \wedge \xi \lrcorner \Omega,$$

where $\Omega$ is a $(3,0)$ form on $X'$. Since $X'$ is Calabi-Yau, $(3,0)$ forms on $X'$ differ by constants.

Note that the Yukawa coupling depends not only on $\varphi, \psi, \xi$ but depends on $\Omega$ as well. In fact, one can prove [3] that $F$ is a holomorphic section of the bundle $\mathrm{Sym}^3(T^*M) \otimes (\underline{F}^3)^{\otimes 2}$, where $\underline{F}^3$ is the first Hodge bundle on $\mathcal{M}$(cf. [3]).

One of the fundamental properties of the Weil-Petersson metric and the Yukawa coupling is that they can be defined "extrinsically" in the sense that they can be defined only using the fact that the moduli space is a horizontal slice. In fact, let $Q(\Omega, \overline{\Omega})$ be defined in (3). We have

$$\omega_{WP} = -\partial\overline{\partial} \log Q(\Omega, \overline{\Omega}),$$

(cf. [8]), where $\omega_{WP} = \frac{\sqrt{-1}}{2\pi} h_{i\overline{j}} dz_i \wedge d\overline{z}_j$ is the Kähler form of the Weil-Petersson metric, and

$$F_{ijk} = Q(\partial_i \partial_j \partial_k \Omega, \Omega).$$

One can contract the Yukawa coupling to get the following $(1,1)$ tensor

$$P = \frac{\sqrt{-1}}{2\pi} P_{i\overline{j}} dz^i \wedge d\overline{z}^j = \frac{\sqrt{-1}}{2\pi} h^{p\overline{q}} h^{r\overline{s}} F_{ipr} \overline{F_{jqs}} dz^i \wedge d\overline{z}^j. \tag{1}$$

This tensor is important because of the following theorem [6]:

**Theorem 1.** *Let* $\omega_H = 2\omega_{WP} + P$, *and let* $n = \dim \mathcal{M}$. *Then*

(1) $\omega_H$ *is a Kähler metric on* $\mathcal{M}$*;*

(2) *The holomorphic bisectional curvature of* $\omega_H$ *is nonpositive. Furthermore, Let* $\alpha = ((\sqrt{n}+1)^2+1)^{-1} > 0$. *Then the Ricci curvature* $\mathrm{Ric}(\omega_H) \leq -\alpha\omega_H$ *and the holomorphic sectional curvature is also less than or equal to* $-\alpha$.

(3) *If* $\mathrm{Ric}(\omega_H)$ *is bounded, then the Riemannian sectional curvature of* $\omega_H$ *is also bounded.*

We call $\omega_H$ the Hodge metric on $\mathcal{M}$.

In this paper, we give a necessary and sufficient condition for the Weil-Petersson metric to be complete. It is easy to prove one side of the theorem. That is, if the Yukawa coupling is bounded, then the Weil-Petersson metric and the Hodge metric are equivalent. The other side of the theorem is the main result of this paper:

**Theorem 2.** *Assume that the Weil-Petersson metric is complete. Then there is a constant* $C_1(m,n)$, *depending only on* $m, n$, *such that*

$$|\nabla^m F|^2 \leq C_1(m,n),$$

*for any nonnegative integer* $m$, *where* $\nabla$ *is the Hermitian connection of the bundle* $\mathrm{Sym}^3(T^*M) \otimes (\underline{F}^3)^{\otimes 2}$ *and* $n$ *is the complex dimension of the moduli space* $\mathcal{M}$.

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## 2. On the Complete Weil-Petersson Metric

In this section, we use the method of gradient estimate to prove Theorem 2, the main result of this paper. We first prove the following weak version of Theorem 2.

**Theorem 3.** *Suppose* $\mathcal{M}$ *is the moduli space of a Calabi-Yau threefold. If the Weil Petersson metric on* $\mathcal{M}$ *is complete, then the normal of the Yukawa coupling with respect to the Weil-Petersson metric is bounded. On the other hand, if the Yukawa coupling with respect to the Weil-Petersson metric is bounded, then the Weil-Petersson metric is equivalent to the Hodge metric.*

**Proof:** Let $\omega_H$ be the Hodge metric, then by definition

$$\omega_H = 2\omega_{WP} + P, \tag{2}$$

where $P$ is defined in (1) and $\omega_{WP} = \frac{\sqrt{-1}}{2\pi} h_{i\bar{j}} dz_i \wedge d\bar{z}_j$ is the Kähler form of the Weil-Petersson metric. The trace of the tensor $P$ is the normal of the Yukawa coupling. Thus the Hodge metric and the Weil-Petersson metric are mutually equivalent if the Yukawa coupling is bounded.

On the other side, let

$$f = |F|^2 = e^{2K} F_{ijk} \overline{F_{abc}} h^{i\bar{a}} h^{j\bar{b}} h^{k\bar{c}},$$

where $K = -log\sqrt{-1}Q(\Omega, \overline{\Omega})$, and

$$Q(\Omega, \overline{\Omega}) = \int_{X'} \Omega \wedge \overline{\Omega}. \tag{3}$$

We have

$$\begin{aligned} f_{\overline{\alpha}} &= 2e^{2K} K_{\overline{\alpha}} F_{ijk} \overline{F_{abc}} h^{i\bar{a}} h^{j\bar{b}} h^{k\bar{c}} \\ &+ e^{2K} F_{ijk} \overline{\partial_\alpha F_{abc}} h^{i\bar{a}} h^{j\bar{b}} h^{k\bar{c}} + e^{2K} F_{ijk} \overline{F_{abc}} \partial_{\overline{\alpha}} (h^{i\bar{a}} h^{j\bar{b}} h^{k\bar{c}}) \end{aligned}$$

for $\alpha = 1, \cdots, n$. Thus we have

$$\Delta f = e^{2K} |\partial_\alpha F_{ijk} + 2K_\alpha F_{ijk}|^2 + 2n|F|^2 + e^{2K} F_{ijk} \overline{F_{abc}} \partial_\alpha \overline{\partial_\alpha} (h^{i\bar{a}} h^{j\bar{b}} h^{k\bar{c}}),$$

where $\Delta$ is the complex Laplacian of the Weil-Petersson metric. Under the normal coordinates,

$$e^{2K} F_{ijk} \overline{F_{abc}} \partial_\alpha \overline{\partial_\alpha} (h^{i\bar{a}} h^{j\bar{b}} h^{k\bar{c}}) = 3e^{2K} F_{ijk} \overline{F_{ajk}} R_{a\bar{i}}$$

where $R_{a\bar{i}}$ is the Ricci curvature of the Weil-Petersson metric. Thus

$$\Delta f \geq 2n|F|^2 + 3e^{2K} F_{ijk} \overline{F_{ajk}} R_{a\bar{i}}.$$

It is known in [7] that

$$R_{a\bar{i}} = -(n+1)\delta_{a\bar{i}} + e^{2K} F_{amn} \overline{F_{imn}}. \tag{4}$$

Thus

$$\begin{aligned} \Delta f &\geq 2n|F|^2 - 3(n+1)|F|^2 + 3e^{4K} \sum_{a,i} |\sum_{j,k} F_{ijk} \overline{F_{ajk}}|^2 \\ &\geq -(n+3)|F|^2 + 3e^{4K} \sum_i (\sum_{j,k} |F_{ijk}|^2)^2 \\ &\geq -(n+3)|F|^2 + \frac{3}{n} e^{4K} (\sum_{i,j,k} |F_{ijk}|^2)^2 \\ &= \frac{3}{n} f^2 - (n+3) f. \end{aligned}$$

We now recall a version of the maximum principle from [8].

**Proposition 1.** *Suppose that $(M, g)$ is a complete Kähler manifold. If the Ricci curvature of $g$ is bounded from below and $\varphi$ is a nonnegative function satisfying*

$$\Delta\varphi \geq c_1\varphi^\alpha - c_2\varphi - c_3,$$

*where $\alpha > 1, c_1 > 0, c_2, c_3 \geq 0$ are constants. then*

$$\sup\varphi \leq Max(1, (\frac{c_2+c_3}{c_1})^{\frac{1}{\alpha}}).$$

By equation (4) we know that the Ricci curvature is bounded from below. Thus using Proposition 1, we have

$$f \leq \sqrt{\frac{n(n+3)}{3}}.$$

□

*Remark* 2. We can also get similar estimates on moduli spaces with incomplete Weil-Petersson metric. In that case, a different version of Maximum principle should be set up.

**Proof of Theorem 2.** We define

$$f_m = |\nabla^m F|^2 \tag{5}$$

for $m = 0, 1, 2, \cdots$. The inequality is true for $m = 0$ by Theorem 3. Assume that the inequality is also true for all $0 \leq i \leq m-1$. That is, we have a constant $\tilde{C}_1(m, n)$ such that

$$|\nabla^i F|^2 \leq \tilde{C}_1(m, n) \tag{6}$$

for any $0 \leq i \leq m-1$. We are going to prove that $|\nabla^m F|$ is bounded. First we have the following lemma:

**Lemma 1.** *With the above assumption, there is a constant $C_2$ depending only on $m, n$ such that*

$$\Delta f_m \geq f_{m+1} - C_2(m, n)(f_m + 1).$$

**Proof.** By (5), we have

$$\begin{aligned} \Delta f_m &= |\nabla^{m+1}F|^2 + |\overline{\partial}\nabla^m F|^2 \\ &+ < h^{i\bar{j}}\nabla_i\overline{\nabla}_j\nabla^m F, \overline{\nabla^m F} > + < \nabla^m F, \overline{h^{j\bar{i}}\overline{\nabla}_i\nabla_j\nabla^m F} > . \end{aligned} \tag{7}$$

Changing the order of the covariant derivative, we get

$$\overline{\nabla}_i\nabla_j\nabla^m F = R(\frac{\partial}{\partial\bar{z}^i}, \frac{\partial}{\partial z^j})\nabla^m F + \nabla_j\overline{\nabla}_i\nabla^m F. \tag{8}$$

By the Strominger's formula [7] of the curvature of the Weil-Petersson metric

$$R_{i\bar{j}k\bar{l}} = h_{i\bar{j}}h_{k\bar{l}} + h_{i\bar{l}}h_{k\bar{j}} - e^{2K}h^{p\bar{q}}F_{ikp}\overline{F_{jlq}}, \tag{9}$$

and by the assumption (6), we see that

$$|\nabla^m R_{i\bar{j}k\bar{l}}| \leq C_2|\nabla^m F| + C_3 \tag{10}$$

for some constants $C_2$ and $C_3$ depending only on $m, n$. Thus by (8)

$$|h^{i\bar{j}}\overline{\nabla}_i\nabla_j\nabla^m F| + |h^{i\bar{j}}\nabla_i\overline{\nabla}_j\nabla^m F| \leq C_4 f_m + C_5 \tag{11}$$

for constants $C_4$ and $C_5$ depending only on $m, n$. By (7) and (11), there is a constant $C_2(m,n)$ such that

$$\Delta f_m \geq f_{m+1} - C_2(m,n)(f_m + 1).$$

In particular, using (6), we have

$$\Delta f_i \geq f_{i+1} - C_4 \tag{12}$$

for $0 \leq i \leq m-1$, and the constant $C_4$ depending only on $m, n$.

□

**Continuation of the proof of Theorem 2.** It is not hard to see that

$$|\nabla f_m| \leq 2\sqrt{f_{m+1} f_m}. \tag{13}$$

Let

$$g_m = f_m(A + f_{m-1}),$$

where constant $A$ is to be determined. Then using Lemma 1 and (12), (13), we have

$$\Delta g_m \geq A f_{m+1} + f_m^2 - C_5(A) f_m - C_6(A) - C_7 f_m \sqrt{f_{m+1}}, \tag{14}$$

where $C_5(A)$, $C_6(A)$ are constants depending only on $m, n$ and $A$ and $C_7$ is the constant depending only on $m, n$. We choose that $A = C_7^2$. Then

$$A f_{m+1} + \frac{1}{4} f_m^2 \geq C_7 f_m \sqrt{f_{m+1}}.$$

Thus there are constants $\delta > 0$ and $C_8$, depending only on $m, n$, such that

$$\Delta g_m \geq \delta g_m^2 - C_8. \tag{15}$$

Using the maximal principal Proposition 1,

$$g_m \leq C_8/\delta + 1.$$

Since we may have chosen $A \geq 1$, we have

$$f_m \leq g_m \leq C_8/\delta + 1,$$

and the theorem is proved. □

## 3. An example

In this section, we give an example of locally symmetric horizontal slice.

We first introduce the notion of classifying space by recalling the definitions and notations in [3].

Suppose $X$ is a simply connected algebraic Calabi-Yau three-fold. The Hodge decomposition of the cohomology group $H = H^3(X, C)$ is

$$H^3(X, C) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3},$$

where

$$H^{p,q} = H^q(X, \Omega^p),$$

and $\Omega^p$ is the sheaf of the holomorphic $p$-forms. The quadratic form $Q$ on $X$ is defined by

$$Q(\xi, \eta) = -\int_X \xi \wedge \eta.$$

By the Serre duality and the fact that the canonical bundle is trivial, $\dim H^{2,1} = \dim H^{1,2} = \dim H^1(X, T_X) = n$, and $\dim H^{3,0} = \dim H^{0,3} = 1$. Thus $H^3(X, C) = C^{2n+2}$ is a (2n+2)-dimensional complex vector space.

It is easy to check that $Q$ is skew-symmetric. Furthermore, we have the following two Hodge-Riemannian relations:

1. $Q(H^{p,q}, H^{p',q'}) = 0$ unless $p' = 3 - p$ and $q' = 3 - q$;
2. $(\sqrt{-1})^{p-q} Q(\psi, \overline{\psi}) > 0$ for any nonzero element $\psi \in H^{p,q}$.

We define the Weil operator $C : H \to H$ by

$$C|_{H^{p,q}} = (\sqrt{-1})^{p-q}.$$

For any collection of $\{H^{p,q}\}$'s, set

$$F^3 = H^{3,0};$$
$$F^2 = H^{3,0} \oplus H^{2,1};$$
$$F^1 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2}.$$

Then $F^1, F^2, F^3$ defines a filtration of $H$

$$0 \subset F^3 \subset F^2 \subset F^1 \subset H.$$

Under this terminology, the Hodge-Riemannian relations can be re-written as

3. $Q(F^3, F^1) = 0, Q(F^2, F^2) = 0$;
4. $Q(C\psi, \overline{\psi}) > 0$ if $\psi \neq 0$.

Now we suppose that $\{h^{p,q}\}$ is a collection of integers such that $p+q=3$ and $\sum h^{p,q}=2n+2$.

**Definition 1.** *With the notations as above, the classifying space $D$ of the Calabi-Yau three-fold is the set of all collection of subspaces $\{H^{p,q}\}$ of $H$ such that*

$$H = \bigoplus_{p+q=3} H^{p,q} \qquad H^{p,q}=\overline{H^{q,p}}, \qquad \dim H^{p,q}=h^{p,q},$$

*and on which $Q$ satisfies the two Hodge-Riemannian relations 1,2.*

*Set $f^p = h^{n,0}+\cdots+h^{p,n-p}$. Then $D$ is also the set of all filtrations*

$$0\subset F^3\subset F^2\subset F^1\subset H, \qquad F^p\oplus\overline{F^{4-p}}=H$$

*with* $\dim F^p = f^p$ *on which $Q$ satisfies the bilinear relations 3,4.*

$D$ is a homogeneous complex manifold. The horizontal distribution $T_h(D)$ is defined as

$$T_h(D)=\{X\in T(D)|XF^3\subset F^2, XF^2\subset F^1\},$$

where $T(D)$ is the holomorphic tangent bundle which can be identified as a sub-bundle of the (locally trivial) bundle $Hom(H^3(X,C),H^3(X,C))$. So $X$ naturally acts on $F^p$.

**Definition 2.** *A complex integral submanifold of the horizontal distribution $T_h(D)$ is called a horizontal slice.*

Suppose $U\subset\mathcal{M}$ is a neighborhood of $\mathcal{M}$ at the point $X$. Then there is a natural map $p:U\to D$, called the period map, which sends a Calabi-Yau threefold to its "Hodge Structure". To be precise, Let $X'\in U$. Then there is a natural identification of $H^3(X',C)$ to $H^3(X,C)=H$. So $\{H^{p,q}(X')\}_{p+q=3}$ are the subspaces of $H$ satisfying the Hodge-Riemannian Relations. We define $p(X')=\{H^{p,q}(X')\}\in D$. It is proved in [3] that $p(U)$ is a horizontal slice.

Now we introduce a result of Bryant and Griffiths [1]. Their results can be briefly written as follows:

We assume that $eV\in U$. i.e. the horizontal slice passes the original point of $D$, where the original point is defined as $\{f^3,f^2,f^1\}\in D$ as follows: there is a basis $e_1,\cdots,e_{2n+2}$ of $H$ under which $Q$ can be represented as

$$Q=\sqrt{-1}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}.$$

If we let

$$f^3 = span\{e_1 - \sqrt{-1}e_{n+2}\},$$
$$f^2 = span\{e_1 - \sqrt{-1}e_{n+2}, e_2 + \sqrt{-1}e_{n+3}, \cdots, e_{n+1} + \sqrt{-1}e_{2n+2}\},$$

and $f^1$ is the hyperplane perpendicular to $f^3$ with respect to $Q$, then

$$\{0 \subset f^3 \subset f^2 \subset f^1 \subset H\} \in D.$$

According to Bryant and Griffiths, there is a holomorphic function $u$ with $u(0) = -\sqrt{-1}$, $\nabla u(0) = 0$ and $\nabla^2 u(0) = \sqrt{-1}I$ ($I$ is the identity matrix) defined on a neighborhood of the original point of $\mathbb{C}^n$ such if $(z^1, \cdots, z^n)$ is the local holomorphic coordinate of $U$ at $eV$, the original point, then the horizontal slice passing through $eV$ can be represented by

$$F^3 = span(1, \frac{1}{\sqrt{2}}z_1, \cdots, \frac{1}{\sqrt{2}}z_n, u - \sum_i \frac{1}{2}z_i u_i, \frac{1}{\sqrt{2}}u_1, \cdots, \frac{1}{\sqrt{2}}u_n), \tag{16}$$

and $F^2 = \nabla F^3$, $F^1 \perp F^3$ via $Q$.

**Example 1.** *With the above notations, we choose*

$$u = -\sqrt{-1} + \frac{\sqrt{-1}}{2}\sum_{i=1}^n z_i^2.$$

*Then we can define the horizontal slice with $F^3$ being given by*

$$F^3 = (1, \frac{1}{\sqrt{2}}z_1, \cdots, \frac{1}{\sqrt{2}}z_n, -\sqrt{-1}, \frac{\sqrt{-1}}{\sqrt{2}}z_1, \cdots, \frac{\sqrt{-1}}{\sqrt{2}}z_n).$$

The Yukawa coupling of the above example is identically zero. Thus by (9), the curvature tensor is

$$R_{i\bar{j}k\bar{l}} = h_{i\bar{j}}h_{k\bar{l}} + h_{i\bar{l}}h_{k\bar{j}},$$

which is parallel. In order to see that the horizontal slice we defined is complete, we first observed that since the Yukawa coupling is zero, the Hodge metric is two times the Weil-Petersson metric because $P \equiv 0$ in (1). Using [6, Lemma 3.8], we can isometrically embed the horizontal slice to the Siegal manifold $H_{n+1}$, where $H_{n+1}$ is the set of all $(n+1) \times (n+1)$ matrices of the form $X + \sqrt{-1}Y$ with $X, Y$ symmetric and $Y$ positive. By [6], the embedding can be represented by the matrix

$$\begin{pmatrix} 1 & \frac{1}{\sqrt{2}}z_1 & \cdots & \frac{1}{\sqrt{2}}z_n & -\sqrt{-1} & \frac{\sqrt{-1}}{\sqrt{2}}z_1 & \cdots & \frac{\sqrt{-1}}{\sqrt{2}}z_n \\ 0 & \frac{1}{\sqrt{2}} & & & 0 & -\frac{\sqrt{-1}}{\sqrt{2}} & & \\ \vdots & & \ddots & & \vdots & & \ddots & \\ 0 & & \cdots & \frac{1}{\sqrt{2}} & 0 & & \cdots & -\frac{\sqrt{-1}}{\sqrt{2}} \end{pmatrix}.$$

Since the Siegal manifold is complete and since the above set is closed in $H_{n+1}$, the horizontal slice is complete with respect to the Weil-Petersson metric.

□

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