WEIL-PETERSSON GEOMETRY ON CALABI-YAU MODULI

MICHAEL DOUGLAS AND ZHIQIN LU

Contents

1. Introduction

(Put the introduction here in the future) By Freed, the moduli space of polarized Calabi-Yau manifolds is called a projective special Kähler manifold. The results in the previous sections apply to projective special Kähler manifolds. Thus we expect that there are parallel results in special Kähler manifolds. In this section, we study these results.

2. Variation of Hodge structures

Let Z be a compact Kähler manifold of dimension s. A $(1,1)$ form ω is called a polarization if $[\omega]$ is the first Chern class of an ample line bundle over Z. The pair (Z,ω) is called a polarized algebraic manifold.

Let $H^k(X,\mathbb{C})$ be the k-th cohomology group of Z with $\mathbb{C}$ as the coefficient. Using ω, one can define

$$L : H^k(Z,\mathbb{C}) \to H^{k+2}(Z,\mathbb{C}), \quad [\alpha] \mapsto [\alpha \wedge \omega]$$

Date: September 19, 2005.

2000 *Mathematics Subject Classification.* Primary: 53A30; Secondary: 32C16.

Key words and phrases. Chern classes, Calabi-Yau moduli, Weil-Petersson metric, Hodge metric, incompleteness of the Weil-Petersson metric.

The first author is supported by DOE grant DE-FG02-96ER40959; the second author is supported by the NSF grant DMS 0347033 and the Alfred P. Sloan Research Fellowship.

to be the multiplication by ω for $k = 0, \cdots, 2s-2$. Let $P^k(Z,\mathbb{C}) = \ker L^{s-t+1}$ for $k \leq s$. The groups $P^k(Z,\mathbb{C})$ are called the k-th primitive cohomology groups.

By the Hodge decompositon theorem and the two Lefschetz theorems (cf. [10]), we make the following definitions: let $H_Z = P^k(Z,\mathbb{C}) \cap H^k(Z,\mathbb{Z})$ and $H^{p,q} = P^k(Z,\mathbb{C}) \cap H^{p,q}(Z)$ for $p+q=k$. Let $H_R = H_Z \otimes \mathbb{R}$ and $H = H_Z \otimes \mathbb{C}$. Suppose that S is the quadratic form on H_Z induced by the cup product of the cohomology group $H^k(X,\mathbb{C})$. S can be represented by

$$S(\phi,\psi) = (-1)^{k(k-1)/2} \int_X \phi \wedge \psi \wedge \omega^{s-k} \tag{2.1}$$

for $\phi, \psi \in H$. Then by the following definition, the triple $\{H_Z, H^{p,q}, S\}$ defines the so-called polarized Hodge structure.

Definition 2.1. *A polarized Hodge structure of weight k, which is denoted by $\{H_Z, H^{p,q}, S\}$, or $\{H_Z, F^p, S\}$, is given by a lattice H_Z, a decomposition*

$$H = \oplus H^{p,q}$$

with $p+q=k$ and

$$H^{p,q} = \overline{H^{q,p}}.$$

together with a bilinear form

$$S : H_Z \otimes H_Z \to \mathbb{Z}, \tag{2.2}$$

which is skew-symmetric if k is odd and symmetric if k is even such that it satisfies the two Hodge-Riemann relations:

(1) $S(H^{p,q}, H^{p',q'}) = 0$ *unless* $p' = k-p, q' = k-q$;
(2) $(\sqrt{-1})^{p-q} S(\phi,\overline{\phi}) > 0$ *for any nonzero element* $\phi \in H^{p,q}$.

Alternatively, the decomposition (2.2) *can be described as the filtration $\{F^p\}$ of H:*

$$0 \subset F^k \subset F^{k-1} \subset \cdots \subset F^0 \subset H,$$

such that

$$H = F^p \oplus \overline{F^{k-p+1}}, \quad H^{p,q} = F^p \cap \overline{F^q}.$$

In this case, the Hodge-Riemann relations can be written as

(3) $S(F^p, F^{k-p+1}) = 0$ *for* $p = 1, \cdots k$;
(4) $(\sqrt{-1})^{p-q} S(\phi,\overline{\phi}) > 0$ *for any nonzero element* $\phi \in H^{p,q}$.

Definition 2.2. *The dual classifying space $\hat{D}$ for the polarized Hodge structure of weight k is the set of all filtrations*

$$0 \subset F^k \subset \cdots \subset F^1 \subset H, \qquad F^p \oplus \overline{F^{k-p+1}} = H,$$

or the set of all the decompositions

$$\sum H^{p,q} = H, \qquad H^{p,q} = \overline{H^{q,p}} \qquad (p+q=k)$$

on which S satisfies the Hodge-Riemann relation 1 or 3 above. The classifying space D is an open set of $\hat{D}$ defined by the Hodge-Riemann relations 2 or 4 above.

D and $\hat{D}$ are dual in the following Lie group theoretic sense: let

$$G_R = \{\xi \in \mathrm{Hom}(H_R, H_R) \mid S(\xi\phi, \xi\psi) = S(\phi, \psi)\}. \tag{2.3}$$

Then D can also be written as the homogeneous space

$$D = G/V, \tag{2.4}$$

where V is the compact subgroup of G which leaves a fixed Hodge decomposition $\{H^{p,q}\}$ invariant. G_R is a semisimple real Lie group of noncompact type without compact factors.

Let M be the compact dual of G_R, and let G_C be the complexification of G_R. Then $V \subset M \cap G_R$ and

$$\hat{D} = M/V = G_C/B,$$

where B is some parabolic subgroup of G_C. For details, see [11].

Over the classifying space D we have the holomorphic vector bundles $\underline{F}^k, \cdots, \underline{F}^1, \underline{H}$ whose fibers at each point are $F^k, \cdots, F^1, H$, respectively. These bundles are called Hodge bundles.

We identify the holomorphic tangent bundle $T^{1,0}(D)$ as a subbundle of $Hom(\underline{H}, \underline{H})$:

$$T^{1,0}(D) \subset \oplus Hom(\underline{F}^p, \underline{H}/\underline{F}^p) = \bigoplus_{r>0} Hom(\underline{H}^{p,q}, \underline{H}^{p-r,q+r}), \tag{2.5}$$

such that the following compatible condition holds

$$\begin{array}{ccc} F^p & \longrightarrow & F^{p-1} \\ \downarrow & & \downarrow \\ H/F^p & \longleftarrow & H/F^{p-1} \end{array}.$$

Definition 2.3. *A subbundle $T_h(D)$ is called the horizontal distribution of D, if*

$$T_h(D) = \{\xi \in T^{1,0}(D) \mid \xi F^p \subset F^{p-1}, p = 1, \cdots, k\}.$$

Definition 2.4. *A horizontal slice $\mathcal{M}$ of D is a complex integral submanifold of the distribution $T_h(D)$. A map $\phi : \mathcal{M} \to D$ is called horizontal, if its image is a horizontal slice at the regular values.*

Let U be an open neighborhood of the universal deformation space (Kuranishi space) of Z with fixed polarization. Assume that U is smooth. Then for each Z' near Z, we have the isomorphism $H^*(Z', \mathbb{C}) = H^*(Z, \mathbb{C})$. Under this isomorphism, $\{H^{p,q}(Z') \cap P^k(Z', \mathbb{C})\}_{p+q=k}$ can be considered as a point of D. The map

$$\phi : U \to D, \qquad Z' \mapsto \{H^{p,q}(Z') \cap P^k(Z', \mathbb{C})\}_{p+q=k}$$

is called the period map.

In general, the universal deformation space allows some singularities. Thus in general, U is not simply connected and the monodromy group is not trivial. Let Γ is the monodromy group of the variation of Hodge structures. Then the period map is a map from U to $\Gamma\backslash D$. Let $\tilde{U}$ be the universal covering space of U. Then ϕ lifts to $\tilde{\phi}: \tilde{U} \to D$. We will call both ϕ and $\tilde{\phi}$ period maps.

The most important property of the period map is the following [9]:

Theorem 2.1 (Griffiths)**.** *The period map $\phi: U \to \Gamma\backslash D$ or $\tilde{\phi}: \tilde{U} \to D$ is holomorphic. Furthermore, it is an immersion and $\tilde{\phi}(\tilde{U})$ is a horizontal slice of the classifying space at regular values of $\tilde{\phi}$.*

3. Asymptotic behavior of the period map

We don't know much about the degeneracy of a general given family of compact Kähler manifolds. However, the asymptotic behaviors of the corresponding Hodge structures were known to Schmid [19] and Cattani-Kaplan-Schmid [2] by their beautiful results of nilpotent and SL_2 orbit theorems of one and several variables. These theorems are not only deep, but also very long. For the full version of the theorems, we refer to the above papers. In this section, we will only define and discuss what we need for the rest of the paper. Most of the materials of this section can be found in [19, 1, 2].

We first introduce the nilpotent orbit theorem of several variables. Let $f: \mathscr{X} \to \mathscr{U}$ be a family of compact polarized Kähler manifolds. In order to study the degeneration of the variation of the Hodge structure, we let $\mathscr{U} = \Delta^{*n} \times \Delta^m$, where Δ, Δ^* are the unit disk and the punctured unit disk in the complex plane, respectively. Let $(z_1, \cdots, z_n, w_1, \cdots, w_m)$ be the standard coordinate system of $\mathscr{U}$. Consider the period map

$$\phi: \Delta^{*n} \times \Delta^m \to \Gamma\backslash D,$$

where Γ is the monodromy group. By going to the universal covering $U^n \times \Delta^m$, one can lift ϕ to a map

$$\tilde{\phi}: U^n \times \Delta^m \to D,$$

where U is the upper half plane. Corresponding to each of the first n variables, we choose a monodromy transformation $T_j \in \Gamma$, so that

$$\begin{aligned}&\tilde{\phi}(z_1, \cdots, z_{j+1}, \cdots, z_n, w_1, \cdots, w_m)\\ &\quad = T_j \circ \tilde{\phi}(z_1, \cdots, z_n, w_1, \cdots, w_m),\end{aligned}$$

holds identically in all variables. T_j's commute with each other. By a theorem of Borel, after passing a finite cover, we may assume that the eigenvalues of T_j are all 1 (we call such T_j's *unipotent*), so that we can define $N_j = \log T_j$. These N_j's are nilpotent operators. All N_j's are commute with each other.

Let $\mathbf{z} = (z_1, \cdots, z_n)$, and $\mathbf{w} = (w_1, \cdots, w_m)$. The map

$$\tilde{\psi}(\mathbf{z}, \mathbf{w}) = \exp(-\sum_{j=1}^n z_j N_j) \circ \tilde{\phi}(\mathbf{z}, \mathbf{w}) \tag{3.6}$$

remains invariant under the translation $z_i \mapsto z_i + 1, 1 \le j \le n$. It follows that $\tilde{\psi}$ drops to a map

$$\psi: \Delta^{*n} \times \Delta^m \to D \hookrightarrow \hat{D}.$$

Theorem 3.1 (Nilpotent Orbit Theorem [19]). *The map ψ extends holomorphically to Δ^{n+m}. For $\mathbf{w} \in \Delta^m$, the point*

$$F(\mathbf{w}) = \psi(0, \mathbf{w}) \in \hat{D}$$

is left fixed by $T_j, 1 \leq j \leq n$. For any given number η with $0 < \eta < 1$, there exist constants $\alpha, \beta \geq 0$, such that under the restrictions

$$\mathrm{Im}\, z_i \geq \alpha, 1 \leq j \leq n \quad \text{and} \quad |w_j| \leq \eta, 1 \leq j \leq m,$$

the point $\exp(\sum_{j=1}^n z_j N_j) \cdot F(\mathbf{w})$ lies in D and satisfies the inequality [1]

$$d(\exp(\sum_{i=1}^n z_i N_i) \cdot F(\mathbf{w}), \tilde{\phi}(\mathbf{z}, \mathbf{w})) \leq C \sum_{j=1}^n (\mathrm{Im}\, z_j)^\beta \exp(-2\pi \mathrm{Im}\, z_j),$$

where C is a constant and d is the G_R invariant Riemannian distance function on D. Finally, the mapping

$$(\mathbf{z}, \mathbf{w}) \mapsto \exp(\sum_{j=1}^n z_j N_j) \cdot F(\mathbf{w})$$

is horizontal.

□

Using the above theorem, we make the following definition

Definition 3.1. *A nilpotent orbit is a map $\theta : \mathbb{C}^n \to \hat{D}$ of the form*

$$\theta(\mathbf{z}) = \exp(\sum z_j N_j) \cdot F, \tag{3.7}$$

where
i.) $F \in \hat{D}$;
ii.) $\{N_j\}_{j=1}^n$ is a commuting set of nilpotent elements of $\mathfrak{g}_R$, the Lie algebra of G_R;
iii.) θ is horizontal, that is $N_j(F^p) \subset F^{p-1}$;
iv.) There exists $\alpha \in \mathbb{R}$ such that $\theta(\mathbf{z}) \in D$ for $\mathrm{Im}\,(z_j) > \alpha$.

Given a nilpotent endomorphism N of a finite dimensional complex vector space H, we can consider the monodromy weight filtration $W = W(N)$. This can be defined as the unique increasing filtration $(W_n)_{n \in Z}$ satisfying
i.) $N(W_l) \subset W_{l-2}$;
ii.) For every $l \geq 0$, $N^l : Gr_l^W \to Gr_{-l}^W$ is an isomorphism.
Assume that $N^{k+1} = 0$. Unless otherwise stated, we will use a shifting

$$W_l(N, k) = W_{l+k}(N).$$

Note that $W_l(N, k) = \{0\}$ for $l < 0$ and $W_l(N, k) = H$ for $l \geq 2k$.

[1] The inequality is a refinement of [19, (4.12)], which was observed by Deligne. See [2, page 465] for details.

We have the following abstract Lefschetz decomposition:

$$Gr_l^W = \bigoplus_{j\geq 0} N^j(P_{l+2j}),$$

where the *primitive* subspaces $P_l \subset Gr_l^W$ are defined by

$$\begin{array}{ll} P_l = \ker\{N^{l+1} \mid Gr_l^W \to Gr_{-l-2}^W\}, & \text{if } l \geq 0, \\ P_l = \{0\}, & \text{if } l < 0. \end{array}$$

Let $F \in \hat{D}$, $N \in \mathfrak{g}_0$ a nilpotent element such that $N^{k+1} = 0$ and W an increasing filtration on H. We shall say that (W, F, N) is a *polarized mixed Hodge structure*, if

i.) W is the monodromy weight filtration of (N, k);

ii.) (W, F) is a mixed Hodge structure; that is, for $l \geq 0$, the filtration induced by F on Gr_l^W is a Hodge structure of weight l;

iii.) $N(F^p) \subset F^{p-1}$ for $0 \leq p \leq k$;

iv.) For $l \geq k$, the Hodge structure induced by F on the primitive subspace $P_l \subset Gr_l^W$ is polarized by the bilinear form S_l, where $S_l(\cdot,\cdot) = S(\cdot, N^l\cdot)$, and S is a nondegenerated bilinear form on H which is symmetric if k is even and skewsymmetric of k is odd.

A splitting of a mixed Hodge structure (W, F) is a bigrading $H = \oplus J^{p,q}$ such that

$$W_l = \bigoplus_{p+q\leq l} J^{p,q}, \qquad F^p = \bigoplus_{r\geq p} J^{r,s}.$$

We have

$$J^{p,q} \equiv \overline{J^{q,p}} \,(\mathrm{mod}\, W_{p+q-2})$$

for any splitting of (W, F). A mixed Hodge structure is called to split over $\mathbb{R}$, if $J^{p,q} = \overline{J^{q,p}}$. An (r, r)-morphism X of (W, F) is compatible with $\{J^{p,q}\}$ if $X(J^{p,q}) \subset J^{p+r,q+r}$. To any mixed Hodge structure (W, F) on $H = H_R \otimes_R \mathbb{C}$ we associated the nilpotent algebra

$$L_R^{-1,-1} = L_R^{-1,-1}(W, F) = \{X \in \mathfrak{gl}(H) \mid X(J^{p,q}) \subset \bigoplus_{r\leq p-1, s\leq q-1} J^{r,s}\}.$$

The following result is from [2, Proposition 2.20]:

Proposition 3.1. *Given a mixed Hodge structure (W, F), there exists a unique $\delta \in L_R^{-1,-1}(W, F)$ such that $(W, e^{-i\delta} \cdot F)$ is a mixed Hodge structure which splits over $\mathbb{R}$. Every morphism of (W, F) commutes with δ; thus, the morphisms of (W, F) are precisely those morphisms of $(W, e^{-i\delta} \cdot F)$ which commute with this element.*

Now we turn to the several variable cases. Let $N_1, \cdots, N_n$ be a commutative set of nilpotent operators coming from the unipotent monodromy operators $T_1, \cdots, T_n$. Let

$$C = \{\lambda_1 N_1 + \cdots + \lambda_n N_n \mid \lambda_j > 0, 1 \leq j \leq n\} \tag{3.8}$$

be the cone of the monodromy operators. The basic fact about the above setting is the following [19, 1, 2]:

Theorem 3.2. *Any $N \in C$ defines the same monodromy weight filtration. Moreover, let F be in* (3.7)*. Then (W, F) defines a mixed Hodge structure, polarized by each $N \in C$.*

In the above situation, we also call that the mixed Hodge structure is polarized by $(N_1, \cdots, N_n)$.

To consider the boundary of the monodromy cone C in (3.8), we need the notation of *relative weight filtration*. We use the settings in [2, pp. 505]. Let W^0 be an increasing filtration of H and let N be a nilpotent endomorphism of H which preserves W^0. Delinge [4, 1.6.13] has shown that there exists at most one filtration $W = W(N, W^0)$ of H such that

i) $N(W_l) \subset W_{l-2}$;
ii) For each $j, l \geq 0$

$$N^l : Gr^W_{l+j} Gr^{W^0}_j \to Gr^W_{-l+j} Gr^{W^0}_j$$

is an isomorphism.

For each set of indices $I = \{i_1, \cdots, i_r\} \subset \{1, \cdots, n\}$ let C_I denote the cone spanned by $N_{i_1}, \cdots, N_{i_r}$. All elements of C_I define the same weight filtration $W(C_I)$ and any $T \in C_J$ preserves $W(C_I)$.

One can prove (cf. [2, Proposition 4.72]) that given $I, J \subset \{1, \cdots, n\}$, $W(C_{I \cup J})$ is the weight filtration of any $T \in C_J$ relative to $W(C_I)$.

In order to establish the results in the next sections, we need the result in [2] on the asymptotical behavior of the Hodge length. We begin with the following setting: by (3.6), we have

$$\tilde{\psi}(\mathbf{z}, \mathbf{w}) = \exp(\sum z_j N_j) \cdot \psi(\mathbf{s}, \mathbf{w}),$$

where $s_j = e^{2\pi i z_j}$, and $\mathbf{s} = (s_1, \cdots, s_n)$. Let $(W, \psi(0, 0))$ be the mixed Hodge structure, polarized by $(N_1, \cdots, N_n)$. By Proposition 3.1, we let $F = \exp(-i\delta) \cdot \psi(0, 0)$ be the $\mathbb{R}$-split mixed Hodge structure associated to $(W, \psi(0, 0))$. Then $\xi = \exp(-i\delta) \cdot \psi : \Delta^{n+m} \to \hat{D}$ is holomorphic and $\xi(0) = F \in \hat{D}$. Since the complex orthgonal group G_C of the flat form S acts transitively and holomorphically on $\hat{D}$, we can write $\xi(\mathbf{s}, \mathbf{w}) = u(\mathbf{s}, \mathbf{w}) \cdot F$ for a G_C-valued function $u(\mathbf{s}, \mathbf{w})$, holomorphic on Δ^{n+m} and such that $u(0, 0) = 1$. We choose a specific lifting of $u(\mathbf{s}, \mathbf{w})$ as follows. The $\mathbb{R}$-split mixed Hodge structure (W, F) determines the bigrading $H = \oplus I^{p,q}$, and $I^{p,q} = F^p \cap \overline{F}^q \cap W_{p+q}$, and the corresponding bigrading of the Lie algebra $\mathfrak{g} = \oplus \mathfrak{g}^{p,q}$. Because $F^p = \oplus_{r \geq p} I^{r,s}$, the Lie algebra of the stabilizer of F in G_C is $\mathfrak{g}_F = \oplus_{p \geq 0} \mathfrak{g}^{p,q}$ and has the nilpotent subalgebra $\mathfrak{v} = \oplus_{p<0} \mathfrak{g}^{p,q}$ as a linear complement. A standard argument shows that the map $V \mapsto \exp V \cdot F$ is a holomorphic diffeomorphism from $\mathfrak{v}$ onto a neighborhood of F in $\hat{D}$. We can write

$$\xi(\mathbf{s}, \mathbf{w}) = \exp V(\mathbf{s}, \mathbf{w}) \cdot F$$

for a unique holomorphic $V : \Delta^{n+m} \to \mathfrak{v}$. The function $\gamma(\mathbf{z},\mathbf{w}) = \exp i\delta \cdot \exp\sum z_j N_j \cdot \exp V(\mathbf{s},\mathbf{w})$ takes values in the unipotent subgroup $\exp \mathfrak{v}$ and we have

$$\tilde{\psi}(\mathbf{z},\mathbf{w}) = \gamma(\mathbf{z},\mathbf{w}) \cdot F, \quad \mathbf{z} \in U^n, \mathbf{w} \in \mathscr{C}, \tag{3.9}$$

where $\mathscr{C}$ is a small neighborhood of Δ^m at the origin. Let C_ϕ be the Weil operator. That is, $C_\phi = (\sqrt{-1})^{p-q}$ on $H^{p,q}$. Let $W^{(j)} = W(N_1, \cdots, N_j)$ $(1 \leq j \leq n)$ be the weight filtration of $(N_1, \cdots, N_j)$. Define

$$H \cong \oplus Gr_1^W(H), \qquad Gr_1^W(H) \overset{def}{=} Gr_{l_n}^{W^{(n)}} \cdots Gr_{l_1}^{W^{(1)}}(H).$$

We recall the following theorem in [2, Theorem 5.21]:

Theorem 3.3. *Let $(N_1, \cdots, N_n)$ be the monodromy logarithms of a variation of polarized Hodge structure over $\Delta^{*n} \times \Delta^m$, given with a specific ordering. If $v \in \bigcap_j W_{l_j}^{(j)}$, and $Gr_1^W(v) \neq 0$, then (and only then)*

$$||v|| \sim \left(\frac{\log|s_1|}{\log|s_2|}\right)^{l_1/2} \left(\frac{\log|s_2|}{\log|s_3|}\right)^{l_2/2} \cdots (-\log|s_n|)^{l_n/2}$$

on any region of the form

$$\left\{(\mathbf{s},\mathbf{w}) \in (\Delta^*)^n \times \Delta^m \;\middle|\; \frac{\log|s_1|}{\log|s_2|} > \varepsilon, \cdots, -\log|s_n| > \varepsilon, \mathbf{w} \in \mathscr{C}\right\}$$

for any $\varepsilon > 0$ and $\mathscr{C} \subset \Delta^m$ compact, where

$$||v||^2 = S(C_\phi \gamma(\mathbf{z},\mathbf{w}) \cdot v, \gamma(\mathbf{z},\mathbf{w}) \cdot \overline{v}).$$

□

4. The rationality of the Chern-Weil forms on moduli space

Let $\mathcal{M}$ be the moduli space of a polarized Kähler manifold. By the period map ϕ defined in Theorem 2.1, the Hodge bundles $\underline{F}^k, \underline{H}$, and $\underline{H}^{p,q}$ can be pulled back to holomorphic bundles $\underline{\mathcal{F}}^k, \underline{\mathcal{H}}$ and $\underline{\mathcal{H}}^{p,q}$ on $\mathcal{M}$, respectively. These bundles are Hermitian vector bundles with respect to the polarization $S(\cdot,\cdot)$ and the Weil operator C_ϕ. For the sake of convenience, we still call them (and the bundles $\underline{\mathcal{F}}^{p,q}$ defined below) Hodge bundles.

Let $p < q$. Consider the holomorphic bundle

$$\underline{\mathcal{F}}^{p,q} = \underline{\mathcal{F}}^p / \underline{\mathcal{F}}^q.$$

Let $\pi_{p,q}$ be the orthogonal projection operator on $\underline{\mathcal{F}}^{p,q}$. That is, let $\Omega \in \underline{\mathcal{F}}^p$. Then $\pi_{p,q}\Omega \in \underline{\mathcal{F}}^p$ and $S(\pi_{p,q}\Omega, \overline{\Omega}') = 0$ for any $\Omega' \in \underline{\mathcal{F}}^q$. $\pi_{p,q}\Omega$ defines a holomorphic section of the bundle $\underline{\mathcal{F}}^p/\underline{\mathcal{F}}^q$, but in general, it is not a holomorphic section of the bundle $\underline{\mathcal{F}}^p$. By abusing the notations, we will use $\pi_{p,q}\Omega$ for the section of both $\underline{\mathcal{F}}^p$ and $\underline{\mathcal{F}}^p/\underline{\mathcal{F}}^q$.

Now let Ω, Ω' be local holomorphic sections of $\underline{\mathcal{F}}^p$. Then they define holomorphic sections of $\underline{\mathcal{F}}^{p,q}$. The Hodge metric [2] of $\underline{\mathcal{F}}^{p,q}$ is defined as

$$\langle \Omega, \overline{\Omega}' \rangle = (\sqrt{-1})^{2p-k} S(\pi_{p,q} C_\phi \Omega, \overline{\Omega'}). \tag{4.10}$$

Let M be a quasi-projective submanifold of $\mathcal{M}$. For the sake of simplicity, we use $\underline{\mathcal{F}}^k, \underline{\mathcal{H}}, \underline{\mathcal{H}}^{p,q}$, and $\underline{\mathcal{F}}^{p,q}$ for both the bundles on $\mathcal{M}$ and their restrictions on M.

The main result of this section is:

Theorem 4.1. *Let M be a quasi-projective subvariety of the moduli space $\mathcal{M}$ of a polarized Kähler manifold. Let $R_{p,q}$ be the curvature tensor of $\underline{\mathcal{F}}^{p,q}$, with respect to the metric* (4.10). *Let $f_{p,q}$ be an invariant polynomial of* $\mathrm{Hom}(\underline{\mathcal{F}}^{p,q}, \underline{\mathcal{F}}^{p,q})$ *with rational coefficients. Then for any sequence*

$$(p_1, q_1), \cdots, (p_r, q_r),$$

of nonnegative integers, we have

$$\int_M f_{p_1,q_1}(\frac{\sqrt{-1}}{2\pi} R_{p_1,q_1}) \wedge \cdots \wedge f_{p_r,q_r}(\frac{\sqrt{-1}}{2\pi} R_{p_r,q_r}) \in \mathbb{Q}. \tag{4.11}$$

Remark 4.1. The curvatures of the Hodge bundles blow up at infinity of the moduli space. Thus the precise meaning of (4.11), *at this time*, is

$$\lim_{\varepsilon \to 0} \int_M \rho_\varepsilon f_{p_1,q_1}(\frac{\sqrt{-1}}{2\pi} R_{p_1,q_1}) \wedge \cdots \wedge f_{p_r,q_r}(\frac{\sqrt{-1}}{2\pi} R_{p_r,q_r}) \in \mathbb{Q},$$

where $\rho = \rho_\varepsilon$ is the cut-off function with compact support in M such that $0 \le \rho \le 1$, and $\rho \equiv 1$ outside the ε-neighborhood of the infinity. However, after next section, we shall know that the integration converges absolutely so that the result is independent of the choice of the cut-off functions and the expression of (4.11) makes sense.

If M *were* compact, the above result follows trivially because the integrand defines Chern classes. In general, the expression of (4.11) is not even topological (for example, see the result of Hartman [12]). Unless $\dim M = 1$, the Hodge metrics are not "good" in the sense of Mumford [18, Section 1][3]. By using the tricks in [17, 5], we proved Theorem 4.1 when the polynomials $f_{p,q}$ are linear, or $\dim M \le 2$. However, in its full generality, we must use the full power of the SL_2-orbit theorem of several variables.

Let M_{reg} be the smooth part of M. Then there is a projective manifold $\overline{M}$, called the *compactification* of M, such that $\overline{M} \backslash M$ is a divisor of $\overline{M}$ of normal crossings[4]. In order to study the asymptotic behavior of the curvatures $R_{p,q}$, we first need to extend the Hodge bundles to the compactification of M.

[2]The Hodge metric defined here are different from those defined in the next section. But since both of them are from the variation of the Hodge structures, we use the same name for convenience.

[3]See the footnote of page 515 of [2].

[4]It may be more accurate to say that, up to a finite covering, $\overline{M}$ can be chosen to be a manifold.

The following lemma is due to Kawamata [13]. For the sake of completeness, we sketch the proof here in our notations. Note that in [17], we proved it under the notations of Weil-Petersson geometry.

Lemma 4.1. *Let $Y = \overline{M}\backslash M$. Then there is a finite branched covering $\tilde{M}$ of $\overline{M}$ with branched locus Y such that the monodromy operators along Y in $\tilde{M}$ are all unipotent.*

Proof. Let T be a monodromy operator along certain irreducible component of the divisor which is not unipotent. Let $T = \gamma_s\gamma_u$ be the decomposition of T into its semi-simple part and its unipotent part. By a theorem of Borel, there is an integer m such that $\gamma_s^m = 1$. Let L be an ample line bundle of $\overline{M}$. Let $Y = \sum D_j$ be the decomposition of the divisor Y into irreducible components. We assume that the monodromy operator T is generated by $U\backslash D_1$, where U is a neighborhood of D_1. Assume that t is large enough such that the bundle $L^t(-D_1)$ is very ample. By taking the m-th root of the sections of $L^t(-D_1)$ we get a variety M_1 such that outside a possible divisor, it is a finite covering space of M. M_1 may have some singularities. However, we can always remove those divisors containing singularities to get a smooth manifold.

Let M' be an Zariski open set of M_1 that is a covering space of M minus of some divisors. The Hodge bundles can be pulled back to the manifold M'. At any neighborhood of $\overline{M}\backslash M$, the transform of (M_1, M') to $(\overline{M}, M)$ is the m-branched covering defined by $z_1 \mapsto \sqrt[m]{z_1}$, where $z_1 = 0$ is corresponding to the divisor D_1. Evidently, the monodromy operator T is transformed to T^m, which becomes a unipotent operator.

One can observe that if T' is a unipotent monodromy operator, then under the transform (M_1, M') to $(\overline{M}, M)$, T' is still unipotent. Since there are only finitely many irreducible components of Y, there are only finitely many monodromy operators which are not unipotent.[5] Thus by finitely many transforms, we can get a compact complex manifold $\tilde{M}$ on which all monodromy operators are unipotent.

□

For the rest of the paper, we let D be the divisor of $\tilde{M}$ of normal crossings such that $\tilde{M}\backslash D$ is a finite covering of M. We also assume that $\tilde{M}$ is covered by locally finite open coordinate neighborhoods $\{U_\alpha\}_{\alpha=1,\cdots,t}$, and $\{\psi_\alpha\}$ is the partition of unity subordinating to the covering.

Using Lemma 4.1 and the nilpotent orbit theorem (Theorem 3.1), the Hodge bundles $\underline{\mathcal{F}}^p$, $\underline{\mathcal{H}}^{p,q}$, $\underline{\mathcal{H}}$, and $\underline{\mathcal{F}}^{p,q}$ extend to vector bundles over $\tilde{M}$. We use the same notations to denote these (extended) bundles.

The following definition is standard to experts. However, we find the notation quite convenient to use:

[5] During the process, it is possible that some divisors are added. However, along these divisors, the monodromy operators are the identity operator.

Definition 4.1. *Let Δ, Δ^* be the unit disk and the punctured unit disk of $\mathbb{C}$, respectively. Let $U = \Delta^{*n} \times \Delta^m$ and let the standard coordinate system of U be $(s_1, \cdots, s_n, w_1, \cdots, w_m)$. A differential form on U is called of Poincaré bounded, if it is bounded under the coframe*

$$\frac{ds_i}{s_i \log |s_i|}, \quad \frac{d\bar{s}_i}{\bar{s}_i \log |s_i|}, dw_j, d\overline{w}_j$$

for $i = 1, \cdots, n, j = 1, \cdots, m$.

Apparently, if a form is bounded, then it is Poincaré bounded. If η_1, η_2 are Poincaré bounded, so is $\eta_1 \wedge \eta_2$. Moreover, the notation of Poincaré boundedness is independent of the choice of coordinates:

Lemma 4.2. *Let M be a quasi-projective manifold and let $\overline{M}$ be its compactification such that $\overline{M} \backslash M$ is a divisor of normal crossings. Let U, U' be two neighborhoods of the divisor such that $U \cap U' \neq \emptyset$. Then a smooth form η is Poincaré bounded on $U \cap M$ if and only if it is Poincaré bounded on $U' \cap M$.*

Proof. We assume that $U \approx \Delta^{*n} \times \Delta^m$ and $U' \approx \Delta^{*n'} \times \Delta^{m'}$. Let the coordinates of the two neighborhoods be $(s_1, \cdots, s_n, w_1, \cdots, w_m)$ and $(s_1', \cdots, s_{n'}', w_1', \cdots, w_{m'}')$, respectively. We further assume that the divisor is the zero locus of either $\{s_1 \cdots s_n = 0\}$ or $\{s_1' \cdots s_{n'}' = 0\}$. Assuming $n \leq n'$, then on $U \cap U'$, we can rearrange the order of s_j' such that

$$s_j = \xi_j s_j'$$

for $j = 1, \cdots, n$, where ξ_j are smooth nonzero functions. Since

$$d \log s_j = d \log s_j' + d \log \xi_j,$$

we concluded that $\frac{ds_j}{s_j \log |s_j|}, \quad \frac{d\bar{s}_j}{\bar{s}_j \log |s_j|}$ are bounded under the coframe $\frac{ds_j'}{s_j' \log |s_j'|}$, $\frac{d\bar{s}_j'}{\bar{s}_j' \log |s_j'|}$ for $j = 1, \cdots, n$, and vice vesa. On the other hand, s_j' for $j > n$ are bounded [6] on $U \cap U'$. The lemma is proved.

□

As the first step of the proof of Theorem 4.1. We need to choose the cut-off function on $\tilde{M}$ carefully. The following definition of the cut-off function depends on the particular choice of the covering and the partition of unity. However, the main feature is that the complex Hessian of the function is of order $O(\frac{1}{\varepsilon^2 (\log \frac{1}{\varepsilon})^2})$, a little bit better than $O(\frac{1}{\varepsilon^2})$. More precisely, we have the following:

Lemma 4.3. *For any $\varepsilon > 0$, there is a smooth real function $\rho = \rho_\varepsilon$ on $\tilde{M}$ such that*

(1) $0 \leq \rho \leq 1$*;*

[6] In fact, they are bounded on any compact subset of $U \cap U'$. But that is enough for our applications.

(2) $\partial\rho, \bar\partial\rho$, *and* $\partial\bar\partial\rho$ *are Poincaré bounded;*
(3) The Euclidean measure of $\operatorname{supp}(\partial\rho)$ *goes to zero as* $\varepsilon \to 0$*;*
(4) In a neighborhood of D, $\rho \equiv 0$*; and* $\rho(p) = 1$ *if the distance of* $p \in M$ *to* D *is greater than* 2ε.

Proof. Let $U_1, \cdots, U_s$ be an open covering of the divisor D such that $(U_{s+1} \cup \cdots \cup U_t) \cap D = \emptyset$. We further assume that $U_\alpha \backslash D = (\Delta^*)^{n_\alpha} \times \Delta^{m_\alpha}$ and the coordinates of $U_\alpha \backslash D$ are $(s_1^\alpha, \cdots, s_{n_\alpha}^\alpha, w_1^\alpha, \cdots, w_{m_\alpha}^\alpha)$. Suppose D is the zero locus of

$$s_1^\alpha \cdots s_{n_\alpha}^\alpha = 0.$$

Let $\phi : \mathbb{R} \to \mathbb{R}$, $0 \le \phi \le 1$ be a smooth decreasing function defined as

$$\phi(t) = \begin{cases} 0 & t \ge 1 \\ 1 & t \le 0 \end{cases}.$$

Let

$$\phi_\varepsilon(z) = \phi\left(\frac{(\log\frac{1}{r})^{-1} - \varepsilon}{\varepsilon}\right),$$

and let

$$\phi_\varepsilon^\alpha(s_1^\alpha, \cdots, s_{n_\alpha}^\alpha) = \prod_{j=1}^{n_\alpha}(1 - \phi_\varepsilon(s_j^\alpha)).$$

Then the function $\rho = \rho_\varepsilon$ is defined as

$$\rho_\varepsilon = \sum_{\alpha=1}^{s} \psi_\alpha \phi_\varepsilon^\alpha + \sum_{\alpha=s+1}^{t} \psi_\alpha,$$

where $\{\psi_\alpha\}$ is the fixed partition of unity defined before. It is standard to check that ρ_ε satisfies all the assertions in the Lemma.

□

By pulling back the Hodge bundles to $\tilde M \backslash D$, R_{p_j,q_j} makes sense as the $\operatorname{Hom}(\underline{\mathcal{F}}^{p_j,q_j}, \underline{\mathcal{F}}^{p_j,q_j})$-valued $(1,1)$ form on $\tilde M \backslash D$. Since $\tilde M \backslash D$ is a finite covering of M, there is a positive integer μ such that (after the next section that the integrands are proved to be integrable)

$$\begin{aligned}&\int_{\tilde M} f_{p_1,q_1}(\frac{\sqrt{-1}}{2\pi} R_{p_1,q_1}) \wedge \cdots \wedge f_{p_r,q_r}(\frac{\sqrt{-1}}{2\pi} R_{p_r,q_r}) \\ &= \mu \int_M f_{p_1,q_1}(\frac{\sqrt{-1}}{2\pi} R_{p_1,q_1}) \wedge \cdots \wedge f_{p_r,q_r}(\frac{\sqrt{-1}}{2\pi} R_{p_r,q_r}).\end{aligned}$$

Let $c_1^j, \cdots, c_{d_j}^j$ be the elementary invariant polynomials on $\operatorname{Hom}(\mathbb{C}^{d_j}, \mathbb{C}^{d_j})$. Then there are polynomials g_j such that

$$f_{p_j,q_j} = g_j(c_1^j, \cdots, c_{d_j}^j)$$

for $j = 1, \cdots, r$.

The following theorem implies the main result of this section, Theorem 4.1:

Theorem 4.2. *Using the same assumptions and notations as in Theorem 4.1 and Lemma 4.3, we have*

$$\lim_{\varepsilon\to 0}\int_{\tilde M}\rho_\varepsilon f_{p_1,q_1}(\frac{\sqrt{-1}}{2\pi}R_{p_1,q_1})\wedge\cdots\wedge f_{p_r,q_r}(\frac{\sqrt{-1}}{2\pi}R_{p_r,q_r})\in\mathbb{Z}, \tag{4.12}$$

if the coefficients of the polynomials g_j $(j=1,\cdots,r)$ are integers.

Let $\tilde f_{p_j,q_j}$ be the polarization of f_{p_j,q_j}. That is,

$$\tilde f_{p_j,q_j}:(\mathbb{C}^{d_j\times d_j})^{n_j}\to\mathbb{C},$$

where n_j is the degree of f_{p_j,q_j} and d_j is the rank of the vector bundle $\underline{\mathcal{F}}^{p_j,q_j}$, and $\tilde f_{p_j,q_j}$ satisfies

(1) $\tilde f_{p_j,q_j}(A_1,\cdots,A_{n_j})$ is linear with each A_l $(1\le l\le n_j)$;
(2) $\tilde f_{p_j,q_j}(A,\cdots,A)=f_{p_j,q_j}(A)$.

We let

$$\mathcal{E}=\bigoplus_{j=1}^r\underline{\mathcal{F}}^{p_j,q_j}.$$

Then

$$R=\begin{pmatrix}R_{p_1,q_1}&&\\&\ddots&\\&&R_{p_r,q_r}\end{pmatrix}$$

is the curvature tensor of $\mathcal{E}$. Let f be an invariant polynomial such that

(1) $f(R,\cdots,R)=\tilde f_{p_1,q_1}(R_{p_1,q_1},\cdots,R_{p_1,q_1})\wedge\cdots\wedge\tilde f_{p_r,q_r}(R_{p_r,q_r},\cdots,R_{p_r,q_r})$;
(2) f is linear with each component.

Let Γ^0 be a smooth $(1,0)$-type connection of $\mathcal{E}$ over $\tilde M$ and let $R^0=\overline\partial\Gamma^0$ be the curvature tensor. Since the integrand below defines a integral cohomology class, we have

$$\int_{\tilde M}f(\frac{\sqrt{-1}}{2\pi}R^0,\cdots,\frac{\sqrt{-1}}{2\pi}R^0)\in\mathbb{Z}. \tag{4.13}$$

Using the notations as above, we have

Lemma 4.4. *If there is a smooth connection Γ^0 on $\tilde M$ such that for any j,*

$$\overline\partial\rho\wedge f(\underbrace{R,\cdots,R}_{j},\Gamma-\Gamma^0,R^0,\cdots,R^0)$$

is Poincaré bounded, then Theorem 4.2 is true. Here Γ is the connection operator of the Hodge bundles $\mathcal{E}$.

Proof. By the linearity of the function f and the obvious equation

$$f(R,\cdots,R)-f(R^0,\cdots,R^0)=\sum_{j=0}^{n+m-1}f(\underbrace{R,\cdots,R}_{j},\overline\partial(\Gamma-\Gamma^0),R^0,\cdots,R^0),$$

and noting that $\Gamma-\Gamma^0$ is globally defined, we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\tilde M}\rho_\varepsilon(f(R,\cdots,R)-f(R^0,\cdots,R^0))\\
&=-\lim_{\varepsilon\to 0}\int_{\tilde M}\overline{\partial}\rho_\varepsilon\wedge\sum_{j=0}^{n+m-1} f(\underbrace{R,\cdots,R}_{j},(\Gamma-\Gamma^0),R^0,\cdots,R^0).
\end{aligned}
\tag{4.14}
$$

By the assumption, the integrand of the above is Poincaré bounded. Let $\xi ds_1\wedge\cdots\wedge d\overline{s}_n\wedge dw_1\wedge\cdots\wedge d\overline{w}_m$ be the $(n+m,n+m)$ component of the integrand on a general neighborhood $U=U_\alpha$, where ξ is a smooth function on U. Then there is a constant C such that

$$
|\xi|\leq C\prod_{j=1}^n\frac{1}{|s_j|^2(\log\frac{1}{|s_j|})^2}.
$$

It is elementary to see that that above function is Euclidean integrable. Since the Euclidean measure of supp $(\partial\rho_\varepsilon)$ goes to zero, by the Lebesgue theorem, the right hand side of (4.14) is zero.

□

Before giving the explicit construction of the connection Γ^0, we define the local frame on each $U=U_\alpha$ $(1\leq\alpha\leq s)$. Recall (3.9). Let F^p_∞ be the limiting Hodge filtration. Then for any basis $\{\tilde v_{p,j}\}$ of F^p_∞,

$$
\exp(\sqrt{-1}\delta)\exp V(s,w)\tilde v_{p,j}
$$

gives local frame of the bundle $\underline{\mathcal{F}}^p$. In fact, this is the local frame we use to define the extension of the Hodge bundles. We call such a local frame *defined by the nilpotent orbit theorem.* Likewise, if $v_{p,q,j}$ is a basis of the vector space

$$
\bigoplus_{j=1}^r F^{p_j}_\infty/F^{q_j}_\infty,
$$

then

$$
\exp(\sqrt{-1}\delta)\exp V(s,w)v_{p,q,j}
$$

gives the local frame of the bundle $\mathcal{E}$, which we also called it defined by the nilpotent orbit theorem.

Now we construct the connection Γ^0 explicitly: as before $U_\alpha\cap D\neq\emptyset$ if and only if $1\leq\alpha\leq t$. On each $\underline{\mathcal{F}}^{p_j,q_j}$, if $1\leq\alpha\leq t$, let $\Omega_{\alpha,p_j,q_j,a}$ where $a=1,\cdots,d_j$ be the local holomorphic frame of the bundle $\underline{\mathcal{F}}^{p_j,q_j}$ defined by the nilpotent orbit theorem; if $\alpha>t$, let $\Omega_{\alpha,p_j,q_j,a}$ be an arbitrary holomorphic local frame of $\mathcal{E}$. Let

$$
\Omega_\alpha=(\Omega_{\alpha,p_1,q_1,1},\cdots,\Omega_{\alpha,p_1,q_1,d_1},\cdots,\Omega_{\alpha,p_r,q_r,1},\cdots,\Omega_{\alpha,p_r,q_r,d_r}).
$$

Then the transition matrices of the vector bundles $A_{\alpha\beta}$ is a holomorphic matrix valued functions. Let

$$
\Omega_\alpha=\Omega_\beta\,A^t_{\alpha\beta}
$$

on $U_\alpha \cap U_\beta \neq \emptyset$, where $\{A_{\alpha\beta}\}$ are transition functions of the bundle $\mathcal{E}$. Then we define the connection matrix

$$\Gamma^0_\alpha = \sum_\gamma \psi_\gamma \partial A_{\alpha\gamma} A^{-1}_{\alpha\gamma}.$$

on U_α, where $\{\psi_\gamma\}$ is the partition of unity subordinating to the covering $\{U_\alpha\}$.

As a general fact, we have the following result: [7]

Lemma 4.5. *The collection of matrix valued* $(1,0)$ *forms* $\{\Gamma^0_\alpha\}$ *defines a smooth* $(1,0)$ *connection on the vector bundle* $\mathcal{E} \to \widetilde{M}$.

Proof. The compatibility conditions of the transition matrices are

$$A_{\alpha\gamma} = A_{\alpha\beta} A_{\beta\gamma}$$

on $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. Thus

$$\partial A_{\alpha\gamma} = \partial A_{\alpha\beta} A_{\beta\gamma} + A_{\alpha\beta} \partial A_{\beta\gamma}.$$

It follows that

$$\partial A_{\alpha\gamma} A^{-1}_{\alpha\gamma} = \partial A_{\alpha\beta} A^{-1}_{\alpha\beta} + A_{\alpha\beta} \partial A_{\beta\gamma} A^{-1}_{\beta\gamma} A^{-1}_{\alpha\beta}.$$

Thus we have

$$\Gamma^0_\alpha = \partial A_{\alpha\beta} A^{-1}_{\alpha\beta} + A_{\alpha\beta} \Gamma^0_\beta A^{-1}_{\alpha\beta}.$$

on $U_\alpha \cap U_\beta \neq \emptyset$. In particular, $\{\Gamma^0_\alpha\}$ defines a smooth connection of $\mathcal{E}$.

□

The problem to verify the assumption in Lemma 4.4 is purely local. So we will concentrate on a typical neighborhood $U = U_\alpha$ and suppress the subscript α for the sake of simplicity. We assume that $U \approx \Delta^{*n} \times \Delta^m$. Also, we will use R, R^0, Γ, Γ^0 to represent the curvature and the connection matrices, respectively, under the local frame defined by the nilpotent orbit theorem.

In order to study the properties of the smooth connection Γ^0, we use the following notations in [2, §5][8].

We assume that the local coordinate system of U is $(s_1, \cdots, s_n, w_1, \cdots, w_m)$, and $U \cap D$ is the zero locus of $s_1 \cdots s_n = 0$.

We define a cone $\mathcal{C}$ in Δ^{*n} by

$$\mathcal{C} = \{|s_1| \leq |s_2| \leq \cdots \leq |s_n| \mid (s_1, \cdots, s_n) \in \Delta^{*n}\}.$$

Let

$$I = \{i_\alpha\},\ 1 \leq i_1 < \cdots < i_r \leq i_{r+1} = n.$$

Define $|I| = r$, and

---

[7] The result doesn't depend on the particular choice of the frames.

[8] We will use the same notations as in [2] as much as possible since it is a large set of notations. Thus we have to sacrifice the uniqueness of the notations. For example, α is used in $U = U_\alpha$ and also as the subscripts of the set I, etc. It should be clear from the context.

$$
\begin{aligned}
& t_\alpha = y_{i_\alpha}/y_{i_{\alpha+1}} && \text{if } \alpha < r,\ t_r = y_n, \\
& u_\alpha^j = y_j/y_{i_\alpha} && \text{for } i_{\alpha-1} < j < i_\alpha, \\
& X_\alpha = X_\alpha(\mathbf{u}) = N_{i_\alpha} + \sum_{i_{\alpha-1}<j<i_\alpha} u_\alpha^j N_j, &&
\end{aligned}
\tag{4.15}
$$

where $N_1, \cdots, N_n$ are the nilpotent operators. Let $Y_\alpha(\mathbf{u})$ be the semisimple elements corresponding to X_α by the Jacobson-Morozov theorem. Define

$$
e(\mathbf{y}) = e(\mathbf{t},\mathbf{u}) = \exp(\sum_\alpha \frac{1}{2}\log y_{i_\alpha} Y_\alpha(\mathbf{u})) = \exp(\sum_\alpha \frac{1}{2}\log t_\alpha \mathbf{Y}_\alpha(\mathbf{u})),
$$

where $\mathbf{Y}_\alpha = Y_1 + \cdots + Y_\alpha$.

Moreover, let

- $\mathscr{A}$: analytic functions of $\mathbf{u} \in \mathbb{R}_+^{n-r}$;
- $\mathscr{L}$: Laurent polynomials in $\{t_\alpha^{1/2}\}$ (or $\{y_{i_\alpha}^{1/2}\}$ with coefficients in $\mathscr{A}$);
- $\mathcal{O}$: pull back to U^n of the ring of holomorphic germs at 0 in Δ^n, via $U^n \to \Delta^{*n} \to \Delta^n$;
- $\mathscr{L}^b$: polynomials in $\{t_\alpha^{-1/2}\}$ with coefficients in $\mathscr{A}$;
- $(\mathcal{O}\otimes\mathscr{L})^b$: subring of $(\mathcal{O}\otimes\mathscr{L})$ generated by $\mathcal{O}$, $\mathscr{L}^b$, and all polynomials of the form $s_j t_{\alpha_1}^{m_1/2}\cdots t_{\alpha_p}^{m_p/2}$ for $j \le i_{\alpha_l}$, $m_l \in \mathbb{Z}$, $l = 1, \cdots, p$;
- $(\mathcal{O}\otimes\overline{\mathcal{O}}\otimes\mathscr{L})^b$: ring generated by $(\mathcal{O}\otimes\mathscr{L})^b$, $(\overline{\mathcal{O}}\otimes\mathscr{L})^b$;
- $\mathscr{R}_{K,L}^b$: ring of rational expressions f/g, $f, g \in (\mathcal{O}\otimes\overline{\mathcal{O}}\otimes\mathscr{L})^b$, with g bounded away from zero on $(U^n)_{K,L}^I$ (defined below).

Via $\mathbf{y} \to (\mathbf{t},\mathbf{u})$, $\mathbb{R}_+^n$ is identified with $\mathbb{R}_+^r \times \mathbb{R}_+^{n-r}$. For any $K << L$, let

$$
\begin{aligned}
& (\mathbb{R}_+^n)_{K,L}^I = \left\{\mathbf{y} \in \mathbb{R}_+^n \mid t_\alpha > L(\alpha \neq r+1),\ 1 \le u_\alpha^j \le K\right\}, \\
& (U^n)_{K,L}^I = \left\{\mathbf{z} \in U^n \mid \mathbf{z} = \mathbf{x} + \sqrt{-1}\mathbf{y},\ \mathbf{y} \in (\mathbb{R}_+^n)_{K,L}^I\right\}, \\
& (\Delta^{*n})_{K,L}^I = \left\{\mathbf{s} \in (\Delta^*)^n \;\middle|\; \frac{\log s_j}{2\pi\sqrt{-1}} \in (U^n)_{K,L}^I\right\}.
\end{aligned}
\tag{4.16}
$$

Remark 4.2. The above definition is slightly different from the one in [2, page 509]. First, we use the notation $(\Delta^{*n})_{K,L}^I$ instead of $(\Delta^{*n})_K^I$ in that paper so that the C^∞ convergence when $L >> K$ in the asymptotic representation of Theorem 3.3 is easier to understand. Of course, we do have the C^∞ convergence on $(\Delta^{*n})_{K,K}^I = (\Delta^{*n})_K^I$. But the proof is well hidden in the §5 of [2] and more explanations are needed. Second, we introduced an extra $i_{r+1} = n$ in order to aviod one more level of math induction in our proof below.

With the above settings, we have the following combinatoric lemma:

Lemma 4.6. *Let*

$$
1 = K_{n+1} \le K_n < K_{n-1} < \cdots < K_0 = +\infty
$$

be a sequence such that

$$K_j > K_{j+1}^n$$

for $j = 1, \cdots, n$. *Let*

$$A_j = \bigcup_{|I|=j} (\Delta^{*n})^I_{K^n_{j+1}, K_j}$$

for $j = 1, \cdots, n$, *and let*

$$A_0 = \{\mathbf{s} \mid y_j < K_1^n, \text{for } 1 \le j \le n\}.$$

Then we have

$$\bigcup_{j=0}^{n} A_j \supset \mathcal{C}.$$

Proof. We let

$$\xi_1 = \frac{y_1}{y_2}, \ \ldots, \ \xi_{n-1} = \frac{y_{n-1}}{y_n}, \ \xi_n = y_n.$$

Consider $(n+1)$ open intervals

$$(K_{n+1}, K_n), \ \cdots, (K_2, K_1), \ (K_1, K_0(= +\infty)).$$

By the pigeonhole principle, there is an $1 \le l \le n+1$ such that

$$\xi_j \notin (K_l, K_{l-1})$$

for any $1 \le j \le n$. If $l = 1$, then $s \in A_0$. Otherwise, let

$$I = \{i_1 < \cdots < i_r \le n\}$$

such that

$$\begin{array}{ll} \xi_j \le K_l & \text{for } j \notin I; \\ \xi_j \ge K_{l-1} & \text{for } j \in I. \end{array}$$

Then we have

$$u_\alpha^j = \frac{y_j}{y_{i_\alpha}} = \frac{y_j}{y_{j+1}} \cdots \frac{y_{i_\alpha - 1}}{y_{i_\alpha}} \le K_l^n.$$

Thus we have $s \in A_{l-1}$ and the lemma is proved.

□

Let $\Omega = \Omega_\alpha$ be the local frame of $\mathcal{E}$ defined by the nilpotent orbit theorem. Let C be a fixed cone of U of the form $(\Delta^{*n})^I_{K,L} \times \mathscr{C}$, where $\mathscr{C}$ is a compact subset of Δ^m. In [2, pp. 514], for such a cone, there is a basis of $\{v_{C,j}\}$ of $\bigoplus_{j=1}^r F_\infty^p / F_\infty^q$ on the typical fiber H flagged according to the "limiting split" Hodge filtration F, which we call it *the basis* of the cone. We let Ω_C the frame of the cone defined by the above basis via the nilpotent orbit theorem, and let $\mathbf{e}$ be the matrix of $e = e(\mathbf{y})$ under the frame Ω_C. Then the following is true (cf. (5.19) of [2]):

Theorem 4.3. *Let* $\mathbf{h}$ *be the metric matrix of the Hodge bundle* $\mathcal{E}$ *under the basis* Ω_C, *and let*

$$\mathbf{h} = \mathbf{e}^t \mathbf{k} \bar{\mathbf{e}}.$$

Then the matrix $\mathbf{k}$ *and its inverse matrix are bounded on the cone* C.

□

By the definition of the basis of the cone, there is a *constant* matrix A_C such that

$$\Omega = \Omega_C A_C^t.$$

Let Γ_C, R_C be the connection and the curvature operators of the Hodge metric under the local frame Ω_C, respectively. Then in Proposition (5.22) of [2], the following was proved

Theorem 4.4. *The coefficients of the forms $Ad((\mathbf{e}^{-1})^t)\Gamma_C$ and $Ad((\mathbf{e}^{-1})^t)R_C$ are Poincaré bounded.*

□

The key technical lemma of this section is the following:

Lemma 4.7. *Let $U' = U_\gamma$ be an open set such that $U \cap U' \neq \emptyset$. Let $A = A_{\alpha\gamma}$. Then on $U \cap U' \neq \emptyset$,*

$$Ad((\mathbf{e}^{-1})^t) Ad(A_C^{-1})(\partial A A^{-1})$$

is Poincaré bounded.

Proof. Let $\Omega' = \Omega_\gamma$ be the local frame of U' defined by the nilpotent orbit theorem. Let C' be a fixed cone of U' and let $\Omega'_{C'}$ be the frame of the cone. We assume that $C \cap C' \neq \emptyset$. We just need to prove the assertion of the lemma on $C \cap C'$ because as C, C' are running over all the cones, the whole $U \cap U'$ will be covered.

Let $\mathbf{e}'$ be the matrix under the frame $\Omega'_{C'}$. Let $A_{C'}$ be the constant matrix defined as

$$\Omega' = \Omega'_{C'} A_{C'}^t.$$

Then we have

$$\Omega_C = \Omega'_{C'} A_{C'}^t A^t (A_C^{-1})^t.$$

We let $B = A_C^{-1} A A_{C'}$ and let $\mathbf{h}'$ be the metric matrix of $\Omega'_{C'}$. Then

$$\mathbf{h} = B \mathbf{h}' \overline{B}^t.$$

It follows that

$$\partial \mathbf{h} \mathbf{h}^{-1} = \partial B B^{-1} + Ad(B)(\partial \mathbf{h}' (\mathbf{h}')^{-1}). \tag{4.17}$$

By the definition of B, we have

$$\partial B B^{-1} = Ad(A_C^{-1})(\partial A A^{-1}).$$

By Theorem 4.4, from (4.17), we see that in order to prove the lemma, we need to prove that

$$Ad((\mathbf{e}^{-1})^t) Ad(B) \partial \mathbf{h}' (\mathbf{h}')^{-1}$$

is Poincaré bounded. But this follows from Theorem 4.4 and that

$$(\mathbf{e}^{-1})^t B (\mathbf{e}')^t$$

and its inverse are bounded. To prove the boundedness, we observe that if

$$\mathbf{h} = \mathbf{e}^t \mathbf{k} \overline{\mathbf{e}},$$

and if

$$\mathbf{h}' = (\mathbf{e}')^t \mathbf{k}' \overline{\mathbf{e}'},$$

then

$$\mathbf{k} = (\mathbf{e}^{-1})^t B(\mathbf{e}')^t \mathbf{k}' (\overline{(\mathbf{e}^{-1})^t B(\mathbf{e}')^t})^t,$$

and the lemma follows from the fact (Theorem 4.3) that $\mathbf{k}, (\mathbf{k}')$, and their inverse matrices are bounded.

□

Corollary 4.1. *Let $\Gamma_C^0 = Ad(A_C^{-1})\Gamma^0$ and $R_C^0 = Ad(A_C^{-1})R^0$ be the connection and the curvature matrices of the connection Γ^0 and the curvature R^0under the frame Ω_C. Then $Ad((\mathbf{e}^{-1})^t)(\Gamma_C^0)$ and $Ad((\mathbf{e}^{-1})^t)(R_C^0)$ are Poincaré bounded.*

Proof. We have

$$\Gamma_C^0 = Ad(A_C^{-1})(\sum_\gamma \psi_\gamma \partial A_{\alpha\gamma} A_{\alpha\gamma}^{-1}) = \sum_\gamma \psi_\gamma Ad(A_C^{-1})(\partial A_{\alpha\gamma} A_{\alpha\gamma}^{-1}).$$

By the above lemma, for each γ, $Ad((\mathbf{e}^{-1})^t)Ad(A_C^{-1})(\partial A_{\alpha\gamma} A_{\alpha\gamma}^{-1})$ is Poincaré bounded. Since $\{U_\alpha\}$ is a locally finite covering, the conclusion on Γ_C^0 follows. The result on R_C^0 follows from a similar formula:

$$Ad((\mathbf{e}^{-1})^t)R_C^0 = \sum_\gamma \overline{\partial}\psi_\gamma Ad((\mathbf{e}^{-1})^t)Ad(A_C^{-1})(\partial A_{\alpha\gamma} A_{\alpha\gamma}^{-1}).$$

□

Proof of Theorem 4.2. On any cone C, by the invariance of f,

$$\begin{aligned} &\overline{\partial}\rho \wedge f(R, \cdots, R, \Gamma - \Gamma^0, R^0, \cdots, R^0) \\ &= \overline{\partial}\rho \wedge f(R_C, \cdots, R_C, \Gamma_C - \Gamma_C^0, R_C^0, \cdots, R_C^0) \\ &= \overline{\partial}\rho \wedge f(Ad((\mathbf{e}^{-1})^t)(R_C), \cdots, Ad((\mathbf{e}^{-1})^t)(R_C), \\ &\quad Ad((\mathbf{e}^{-1})^t)(\Gamma_C - \Gamma_C^0), Ad((\mathbf{e}^{-1})^t)(R_C^0), \cdots, Ad((\mathbf{e}^{-1})^t)(R_C^0)). \end{aligned} \tag{4.18}$$

By Theorem 4.4, $Ad((\mathbf{e}^{-1})^t)(R_C)$ and $Ad((\mathbf{e}^{-1})^t)(\Gamma_C)$ are Poincaré bounded. On the other side, by Corollary 4.1, $Ad((\mathbf{e}^{-1})^t)(R_C^0)$ and $Ad((\mathbf{e}^{-1})^t)(\Gamma_C^0)$ are also Poincaré bounded. Thus the left hand side of (4.18) is Poincaré bounded. By Lemma 4.4, this implies Theorem 4.2 (hence Theorem 4.1).

□

For the moduli space $\tilde{M}$ itself, we have

Corollary 4.2. *Using the same notations as in Theorem 4.1, we have*

$$\int_{\mathcal{M}} f_{p_1,q_1}(\frac{\sqrt{-1}}{2\pi} R_{p_1,q_1}) \wedge \cdots \wedge f_{p_r,q_r}(\frac{\sqrt{-1}}{2\pi} R_{p_r,q_r}) \in \mathbb{Q}. \tag{4.19}$$

Proof. By the theorem of Viehweg [21], $\mathcal{M}$ is a quasi-projective variety.

□

5. The generalized Hodge metrics

In this section, we shall show that the form in (4.11) is absolutely integrable. The result can be proved using Proposition (5.22) in [2]. However, we provide a proof which is elementary (avoid using the SL_2-orbit theorem). More importantly, we give the intrinsic upper bound of the integrals, which can be regarded as Chern number inequalities on the moduli spaces.

In the first part of this section, we recall some notations and results from [6].

Let M be a complex manifold of dimensional s (in the last section, $s = n+m$). Suppose that M is the parameter space of a family of polarized compact Kähler manifolds $\pi : \mathcal{X} \to M$. By the functorial property, the Hodge bundles $\underline{\mathcal{H}}$, $\{\underline{\mathcal{F}}^p\}$, $\{\underline{\mathcal{H}}^{p,q}\}$, and $\underline{\mathcal{F}}^{p,q}$ on M can be defined as the pull-back of the Hodge bundles from the classifying space. The bundles can also be identified to the relative cohomology groups as follows

$$\underline{\mathcal{H}}^{p,q}|_M = PR^q\pi_*\Omega^p_{\mathcal{X}/M},\ \underline{\mathcal{F}}^p|_M = \underline{\mathcal{H}}^{p+q,0} \oplus \cdots \oplus \underline{\mathcal{H}}^{p,q}$$

for $p, q \geq 0$, where $\Omega^p_{\mathcal{X}/M}$ is the sheaf of relative holomorphic $(p,0)$ forms on $\mathcal{X}$. In particular, $\underline{\mathcal{H}} = PR^k\pi_*(\mathbb{C})$. For the sake of simplicity, we shall use $\underline{\mathcal{H}}^{p,q}, \underline{\mathcal{F}}^p$ to denote $\underline{\mathcal{H}}^{p,q}|_M, \underline{\mathcal{F}}^p|_M$, etc.

Let $Z_t = \pi^{-1}(t)$. The Kodaira-Spencer map $T_t(M) \to H^1(Z_t, \Theta_t)$ gives a bundle map

$$\frac{\partial}{\partial t_i} : \underline{\mathcal{H}}^{p,q} \to PR^k\pi_*(\mathbb{C})/\underline{\mathcal{H}}^{p,q}$$

for $k \leq s$ by differentiation, where $PR^k\pi_*(\mathbb{C})$ is the primitive part of $R^k\pi_*(\mathbb{C})$. In this way, we have a natural bundle map (compare to (2.5)):

$$T^{(1,0)}(M) \to \bigoplus_{p+q=k} \mathrm{Hom}\,(\underline{\mathcal{H}}^{p,q}, PR^k\pi_*(\mathbb{C})/\underline{\mathcal{H}}^{p,q}). \tag{5.20}$$

We make the following definition of the generalized Hodge metrics.

Definition 5.1. *Let h_{PH^k} be the pull back of the natural Hermitian metric on the bundle $\bigoplus_{p+q=k} \mathrm{Hom}(\underline{\mathcal{H}}^{p,q} \to PR^k\pi_*(\mathbb{C})/\underline{\mathcal{H}}^{p,q})$ to $T^{(1,0)}(M)$ for $k \leq s$. Then h_{PH^k} is semi-positive definite. We use ω_{PH^k} to denote the corresponding Kähler forms for $k \leq s$.*[9] *According to the Lefschetz decomposition theorem, we define*

$$\omega_{H^k} = \omega_{PH^k} + \omega_{PH^{k-2}} + \cdots . \tag{5.21}$$

We call both ω_{H^k} and ω_{PH^k} to be the generalized Hodge metrics.

Remark 5.1. The above construction is a generalization of the Hodge metric defined by the second author [15]. In fact, it is proved in [16] that

$$\omega_{PH^n} = \omega_H,$$

the latter being the Hodge metric.

[9] That is, if $h_{PH^k} = (h_{PH^k})_{i\bar{j}} dt_i \otimes d\bar{t}_j$, then $\omega_{PH^k} = \frac{\sqrt{-1}}{2\pi}(h_{PH^k})_{i\bar{j}} dt_i \wedge d\bar{t}_j$.

Remark 5.2. The Hodge metric on classifying space was studied as early as [11] (and later by Peters (**add reference here!!!**). The curvature properties (with respect to the Hermitian connection) of the Hodge metric were the key to the famous Nilpotent and SL_2 orbit theorems. The contribution of [15] and [6] is the observation that the generalized Hodge metrics are Kählerian. Thus the Levi-Civita connection and the Hodge connection are the same and the usages of many theorem in Kähler a geometry, including the Schwarz Lemma [22], become possible.

Because of the possible degeneration of the action (5.20), the generalized Hodge metric is only semi-positive definite; hence, it is a pseudo-metric. Nevertheless, it enjoys similar "curvature" properties of the Hodge metric [14]. We begin with the following result in [6, Proposition 2.8]:

Proposition 5.1. *Let $c_1(E)$ be the Ricci form of a vector bundle E. Then we have*

$$\omega_{PH^k} = \sum_{0\leq p\leq k} pc_1(\underline{\mathcal{H}}^{p,k-p}). \tag{5.22}$$

$$\omega_{H^k} = \sum_{l=0}^{[\frac{k}{2}]} \sum_{p\leq k-2[\frac{k}{2}]+2l} pc_1(\underline{\mathcal{H}}^{p,k-2[\frac{k}{2}]+2l-p}). \tag{5.23}$$

for $k\leq s$.

Proof. For the sake of completeness, we include the proof here. Fixing a $k\leq s$, we have

$$\mathcal{F}_k^p = \underline{\mathcal{H}}^{k,0}\oplus\cdots\oplus\underline{\mathcal{H}}^{p,k-p}$$

for $p=0,\cdots,k$. Thus for $q=k-p$,

$$\underline{\mathcal{H}}^{p,q} = \underline{\mathcal{F}}_k^p/\underline{\mathcal{F}}_k^{p+1}.$$

In terms of the curvatures, we have

$$c_1(\underline{\mathcal{H}}^{p,q}) = c_1(\underline{\mathcal{F}}_k^p) - c_1(\underline{\mathcal{F}}_k^{p+1}). \tag{5.24}$$

By the Abel summation formula, we have

$$\sum_{0\leq p\leq k} pc_1(\underline{\mathcal{H}}^{p,k-p}) = c_1(\underline{\mathcal{F}}_k^k)+\cdots+c_1(\underline{\mathcal{F}}_k^1)+c_1(\underline{\mathcal{F}}_k^0). \tag{5.25}$$

Each $\underline{\mathcal{F}}_k^p$ is a sub-bundle of the flat bundle $\underline{\mathcal{F}}_k^0 = PR^k\pi_*\mathbb{C}$. Let $t_1,\cdots,t_m$ be the local holomorphic coordinate of M and let the bundle map

$$\frac{\partial}{\partial t_i}:\underline{\mathcal{F}}_k^p\to\underline{\mathcal{F}}_k^0/\underline{\mathcal{F}}_k^p,\ 1\leq i\leq m$$

be represented by the matrix

$$\frac{\partial\Omega_\alpha}{\partial t_k} = b_{k\alpha\mu}T_\mu,$$

where Ω_α and T_μ are the basis of $\underline{\mathcal{F}}_k^p$ and $\underline{\mathcal{F}}_k^0/\mathcal{F}_k^p$, respectively. Then the first Chern class can be written as [10]

$$c_1(\underline{\mathcal{F}}_k^p) = \frac{\sqrt{-1}}{2\pi}\sum_{\alpha,\mu} b_{k\alpha\mu}\bar{b}_{l\alpha\mu}dt_k\wedge d\bar{t}_l \tag{5.26}$$

for $0\leq p\leq k$. (5.22) follows from the definition of ω_{PH^k}. (5.23) follows from (5.22) and the Lefschetz decomposition theorem. The proof is completed.

□

Corollary 5.1. *Using the above notations, we have*

$$d\omega_{PH^k} = d\omega_{H^k} = 0.$$

In particular, if the generalized Hodge metric is positive definition, then it defines a Kähler metric.

□

In what follows, for the sake of simplicity, we assume that $\underline{\mathcal{H}}^{k+2,-2} = \underline{\mathcal{H}}^{k+1,-1} = \underline{\mathcal{H}}^{-1,k+1} = \underline{\mathcal{H}}^{-2,k+2} = 0$ and $\underline{\mathcal{F}}_k^{k+1} = 0$, $\underline{\mathcal{F}}_k^{-1} = \underline{\mathcal{F}}^0$.

Fix $k\leq n$, $p\leq k$ and $q = k-p$. Let $\{\Omega_{p,i}\}$, $i = 1,\cdots,h^{p,q}$ be a local holomorphic frame of $\underline{\mathcal{H}}^{p,q}$.

Definition 5.2. *Let $(t_1,\cdots,t_m)$ be a holomorphic local coordinate at a point of M. We define $\nabla_\alpha\Omega_{p,i}\in H^{p-1,q+1}$ to be the projection of $\partial_\alpha\Omega_{p,i} = \frac{\partial}{\partial t_\alpha}\Omega_{p,i}$ to $H^{p-1,q+1}$ with respect to the bilinear form $S(\ ,\ )$.*

For simplicity, we shall use $(\ ,\ )$ in stead of the bilinear form S. With the above notation,

$$(g_p)_{i\bar{j}} = \langle\Omega_{p,i},\overline{\Omega_{p,j}}\rangle = (\sqrt{-1})^{q-p}(\Omega_{p,i},\overline{\Omega_{p,j}}) \tag{5.27}$$

is the Hermitian metric matrix of $\underline{\mathcal{H}}^{p,q}$ for $p = 0,\cdots,k$. It is thus easy to see that (cf. [6])

Proposition 5.2. *For fixed k, the generalized Hodge metric matrix for the local coordinate system $(t_1,\cdots,t_m)$ with respect to $\underline{\mathcal{PH}}^k$, defined in Definition 5.1, is*

$$h_{\alpha\bar{\beta}} = \sum_{p=0}^{k}(\sqrt{-1})^{q-p+2}g_p^{i\bar{j}}(\nabla_\alpha\Omega_{p,i},\overline{\nabla_\beta\Omega_{p,j}}), \tag{5.28}$$

where $(g_p^{i\bar{j}})$ is the inverse of $(g_p)_{i\bar{j}}$.

□

Now we are prepared to state the following result, which can be viewed as a degenerated version of Yau's Schwarz lemma [22]. The result is a slight generalization of [6, Theorem A.1].

Theorem 5.1. *Let $\tau = \frac{\sqrt{-1}}{2\pi}\tau_{\alpha\bar{\beta}}dt_\alpha\wedge d\bar{t}_\beta$ be a Kähler metric on M such that*

[10] This is essentially due to [9].

(1) τ is a complete metric;
(2) The Ricci curvature of τ has a lower bound.

There there is a constant C, depending only on the dimension of M and the lower bound of the Ricci curvature of τ, such that

$$\omega_{H^k} \leq C\tau.$$

Proof. Let ξ be the smooth function defined by $\xi = \tau^{\alpha\overline{\beta}} h_{\alpha\overline{\beta}}$, where $\tau^{\alpha\overline{\beta}}$ is the inverse matrix of $\{\tau_{\alpha\overline{\beta}}\}$. Then by the Bochner type formula in [6, Appendix A], there is a constant $C_{>}0$, depending only on the dimension of M and the lower bound of the Ricci curvature of τ, such that

$$\Delta\xi \geq \frac{1}{C}\xi^2 - C\xi.$$

Using the generalized maximum principle [3] (**double check the reference!!!**), ξ is a bounded function. This completes the proof.

□

A typical choice of the metric τ is the so-called Poincaré metric. There is a Kähler metric ω_P on M, which is called the Poincaré metric. At any point $p \in \overline{M}\backslash M$, there is a neighborhood U of p such that $U \cap M$ can be identified as $(\Delta^*)^n \times \Delta^m$. The metric ω_P on $(\Delta^*)^n \times \Delta^m$ is asymptotically to the Poincaré metric

$$\omega_P \sim \frac{\sqrt{-1}}{2\pi}\left(\sum_{i=1}^{n} \frac{dt_i \wedge d\bar{t}_i}{|t_i|^2(\log\frac{1}{|t_i|})^2} + \sum_{i=m+1}^{n+m} dt_i \wedge d\bar{t}_i\right).$$

See [17, Section 5] for details.

In our terminology, Theorem A.1 of [6] can be written as

Corollary 5.2. *The Hodge metrics are Poincaré bounded.*

□

We turn to the proof of integrability and Chern number inequalities. We use the following definition from [20]. We changed it slightly for our needs:

Definition 5.3. *A (k,k) current u on M is said to be positive, if for any $(1,0)$ differential forms $w_{k+1}, \cdots, w_s$,*

$$u \wedge \frac{\sqrt{-1}}{2\pi} w_{k+1} \wedge \overline{w}_{k+1} \wedge \cdots \frac{\sqrt{-1}}{2\pi} w_s \wedge \overline{w}_s$$

is a nonnegative volume form.

Let u, v be two currents. Then we use

$$|u| \leq v$$

to denote that both $v - u$ and $v + u$ are positive currents.

Now we assume that M is quasi-projective, and there is a compact manifold $\overline{M}$ such that $M \subset \overline{M}$ and $\overline{M}\backslash M$ is a divisor of normal crossings.

We have the following result:

Theorem 5.2. *Let $c_r(g_p)$ be the r-th Chern-Weil form of the Hodge bundle $\underline{\mathcal{H}}^{p,q}$ with respect to the metric g_p. Then there is a constant $C > 0$ such that*

$$|c_r(g_p)| \leq C\omega_{PH^k}^r.$$

Proof. Let ω_0 be any Kähler metric of M. Then for any $\varepsilon > 0$, $\omega_{PH^k}+\varepsilon\omega_0$ is a Kähler metric. Suppose $(t_1, \cdots, t_s)$ be holomorphic normal coordinate system at $p \in M$, then the curvature of $\underline{H}^{p,q}$ is given by

$$(R_p)_{i\bar{j}\gamma\bar{\gamma}} = (\sqrt{-1})^{q-p}(\nabla_\gamma\Omega_{p,i}, \overline{\nabla_\gamma\Omega_{p,j}}) - (\sqrt{-1})^{q-p}(\bar{\partial}_\gamma\Omega_{p,i}, \overline{\bar{\partial}_\gamma\Omega_{p,j}}).$$

For fixed γ, δ, we have

$$|R(g_p)_{i\bar{j}\gamma\bar{\delta}}| \leq 2\sqrt{h_{\gamma\bar{\gamma}}h_{\delta\bar{\delta}}}.$$

By definition, $h_{\gamma\bar{\gamma}} \leq 1$. Thus we have

$$|R(g_p)_{i\bar{j}\gamma\bar{\delta}}| \leq 2.$$

Thus there is a constant C such that

$$|c_r| \leq C\omega_{PH^k}^r.$$

□

Apparently, the Ricci curvature of ω_P is lowerly bounded and (M, ω_P) is of finite volume.

Corollary 5.3. *There is a contant $C > 0$ such that*

$$|c_r(g_p)| \leq C\omega_P^r.$$

Remark 5.3. The result was proved in [2, Corollary 5.23] using the SL_2 orbit theorem in several variables. Here we provide a different proof that using the Hodge metrics and the Schwarz lemma.

6. Chern classes on Weil-Petersson geometry

For the rest of this paper, we assume that Z is a Calabi-Yau manifold. Let M be a family of polarized Calabi-Yau manifolds. We assume that $M \to D$ is an immersion. Let $\underline{F}^s$ be the first Hodge bundle on M. Then $\underline{F}^s$ is a line bundle because $\dim H^{s,0}(Z) = 1$ for Calabi-Yau manifolds. By Griffiths, we know that

$$c_1(\underline{F}^s) > 0.$$

We make the following definition.

Definition 6.1. *We call*

$$\omega_{WP} = c_1(\underline{F}^s)$$

the Weil-Petersson metric of M.

Remark 6.1. The original definition of the Weil-Petersson metric is as follows: Let $Z \in M$ be a polarized Calabi-Yau manifolds with Ricc flat Kähler metric μ. Let $X, Y \in H^1(Z, T^{1,0}Z)$. Define the L^2 inner product as

$$(X, Y) = \int_Z \langle X, Y \rangle dV_\mu.$$

Via the Kodaira-Spencer map: $T_Z M \to H^1(Z, T^{1,0}Z)$, the above inner product defines a metric on M. By a theorem of Tian, the metric happens to be the Weil-Petersson metric.

We adopt the above definition because from the definition, the Weil-Petersson metric is automatically Kählerian and depending only on the variation of Hodge structures.

The curvature of the Weil-Petersson metric was given by Strominger for the moduli space of Calabi-Yau threefolds and by Wang for the n-dimensional cases;

Theorem 6.1. *Let Ω be a local holomorphic section of $\underline{F}^s$. Then*

$$R(\omega_{WP})_{\alpha\overline{\beta}\gamma\overline{\delta}} = g_{\alpha\overline{\beta}} g_{\gamma\overline{\delta}} + g_{\alpha\overline{\delta}} g_{\gamma\overline{\beta}} - \frac{(\nabla_\alpha \nabla_\gamma \Omega, \overline{\nabla_l \nabla_\delta \Omega})}{(\Omega, \overline{\Omega})}$$

for any $1 \leq \alpha, \beta, \gamma, \delta \leq s$.

Remark 6.2. In the case of CY threefolds, Strominger was able to represent the curvature tensor in terms of the Yukawa coupling:

$$R(\omega_{WP})_{\alpha\overline{\beta}\gamma\overline{\delta}} = g_{\alpha\overline{\beta}} g_{\gamma\overline{\delta}} + g_{\alpha\overline{\delta}} g_{\gamma\overline{\beta}} - \sum_{\xi,\eta} \frac{1}{(\Omega, \overline{\Omega})^2} g^{\xi\overline{\eta}} F_{\alpha\gamma\xi} \overline{F_{\beta\delta\eta}}.$$

Note that in general, the Weil-Petersson curvature is only L^1. Nevertheless, we are able to prove the following result which will lead to the finiteness of the characteristic classes:

Proposition 6.1. *There is a constant $C > 0$ such that*

$$|c_r(\omega_{WP})| \leq C \omega_H^r.$$

Here the above inequality means that for any $v_1, \cdots, v_r \in TM$, we have

$$|c_r(\omega_{WP})(v_1, \cdots, v_r, \overline{v}_1, \cdots, \overline{v}_r)| \leq C \prod_{i=1}^r ||v_i||^2,$$

where the norm is with respect to the metric ω_H.

Proof. First we choose a normal coordinate at $p \in M$ so that

$$g_{i\overline{j}}(p) = \delta_{ij}, \quad dg_{i\overline{j}}(p) = 0. \tag{6.29}$$

Let

$$R_i^j = \sum_{kl} R_{ik\overline{l}}^j dz^k \wedge d\overline{z}^l, \quad \text{where } R_{ik\overline{l}}^j = g^{j\overline{p}} R_{i\overline{p}k\overline{l}}.$$

Then the r-th Chern class is given by

$$c_r(\omega_{WP}) = \frac{(-1)^r}{r!} \sum_{\tau \in S_r} sgn(\tau) R_{i_1}^{i_{\tau(1)}} \wedge \cdots \wedge R_{i_r}^{i_{\tau(r)}}, \tag{6.30}$$

where S_r is the symmetric group on the set $\{1, 2, \ldots, r\}$.

We define

$$h'_{\alpha\overline{\beta}} = \delta_{\alpha\beta} + \sum_\gamma (\nabla_\alpha \nabla_\gamma \Omega, \overline{\nabla_\beta \nabla_\gamma \Omega}).$$

Then $(h'_{\alpha\overline{\beta}})$ defines a Kähler metric ω'.[11]

By Proposition 5.2, we have

$$\omega' \leq \omega_H.$$

Thus in order to prove the proposition, we need only to prove that

$$|c_r(\omega_{WP}| \leq C(\omega')^r.$$

Let $A_{ij} = \sum_k (\nabla_i \nabla_k \Omega, \overline{\nabla_j \nabla_k \Omega})$. Since the matrix (A_{ij}) is hermitian, after suitable unitary change of basis, we can assume

$$A_{ij}(p) = \begin{cases} \lambda_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Since $(A_{ij}(p))$ is positive-semidefinite, $\lambda_i \geq 0$, and we have

$$h'_{i\bar{j}}(p) = \delta_{ij}(1 + \lambda_i). \tag{6.31}$$

Clearly, we have

$$(1 + \lambda_{i_1}) \cdot \ldots \cdot (1 + \lambda_{i_\alpha}) \leq \det h' \tag{6.32}$$

for any $1 \leq i_1 < i_2 < \ldots < i_\alpha \leq n$, where $\det h' = \det(h'_{\alpha\overline{\beta}})$. We assume that $v_i = \frac{\partial}{\partial z_{k_i}}$, then by (6.30), we have

$$|c_r(\omega_{WP})(v_1, \cdots, v_r, \overline{v}_1, \cdots, \overline{v}_r)| \leq C \operatorname{Max} |R^{i_1}_{j_1 k_1 \overline{\sigma(k_1)}} \cdots R^{i_r}_{j_r k_r \overline{\sigma(k_r)}}|,$$

For fixed i, j, k, l, by the Cauchy-Schwartz inequality, we have

$$\begin{aligned} \left|R^j_{ik\bar{l}}\right| &\leq |\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} - (\nabla_i \nabla_k \Omega, \overline{\nabla_j \nabla_l \Omega})| \\ &\leq 2 + \sqrt{(\nabla_i \nabla_k \Omega, \overline{\nabla_i \nabla_k \Omega})(\nabla_j \nabla_l \Omega, \overline{\nabla_j \nabla_l \Omega})} \\ &\leq 2 + \sqrt{h_{i\bar{i}} h_{j\bar{j}}} \leq 2\sqrt{(1 + \lambda_i)(1 + \lambda_j)} \end{aligned}$$

So we get

$$\left| R^{i_1}_{j_1 k_1 \overline{k_{\sigma(1)}}} \ldots R^{i_r}_{j_r k_r \overline{k_{\sigma(r)}}} \right| \leq 2^m \prod_{\alpha=1}^m \left(\sqrt{(1 + \lambda_{k_i})(1 + \lambda_{\sigma(k_i)})} \right)$$

[11] The metric is equivalent to the partial Hodge metric in [16, Section 4]. But we don't need this fact here.

□

Corollary 6.1. *Let $r_, \cdots, r_t \geq 0$ such that $\sum r_i = s$. Then*

$$c_{r_1}(\omega_{WP}) \wedge \cdots c_{r_t}(\omega_{WP}) \wedge \omega_{WP}^{r_0}$$

is integrable.

Proof. Using the same method as above, we have

$$|c_{r_1}(\omega_{WP}) \wedge \cdots c_{r_t}(\omega_{WP}) \wedge \omega_{WP}^{r_0}| \leq C\omega_H^s$$

Since the Hodge volume is finite by [16, Theorem 5.2], the corollary follows.

□

7. Incompleteness of the Weil-Petersson metric

We don't have examples of Calabi-Yau moduli with complete Weil-Petersson metric. Thus the main topic of this section is to study the Weil-Petersson completeness (or incompleteness) at one point. Nevertheless, we have the following interesting result from the gradient estimate.

Theorem 7.1. *If M is the parameter space of a family of polarized Calabi-Yau manifolds, then there is a constant C such that*

$$\omega_{PH^k} \leq C\omega_{WP}$$

for any k.

Proof. We let

$$\omega' = \omega_{PH^k} + \varepsilon\omega_{WP},$$

where $\varepsilon > 0$. Then by, we have

$$R(\omega')_{\alpha\overline{\alpha}\beta\overline{\beta}} \geq \varepsilon R(\omega_{WP})_{\alpha\overline{\alpha}\beta\overline{\beta}} + \delta^2_{\alpha]bar\alpha}.$$

We let

$$f_\varepsilon = g^{i\overline{j}}(h'_{i\overline{j}}).$$

Then since the Ricci curvature of the Weil-Petersson metric is lowerly bounded, by the Bochner formula, we have

$$\Delta f_\varepsilon \geq -Cf_\varepsilon + \delta f_\varepsilon^2 + \varepsilon R(\omega)_{\alpha\overline{\alpha}\beta\overline{\beta}}$$

Let $\varepsilon \to 0$ we get

$$\Delta f_0 \geq -Cf_0 + \delta f_0^2.$$

The theorem follows from the following generalized maximal principal.

□

Theorem 7.2. *Let M be a Kähler manifold with the Ricci curvature bounded below. Let ϕ be a nonnegative smooth function on M satisfying*

$$\Delta\phi \geq C_1\phi^\alpha - C_2\phi - C_3 \quad \text{on } M,$$

where $\alpha > 1$, $C_1 > 1, C_2, C_3 \geq 0$ are constants. Then

$$\sup\phi \leq \mathrm{Max}\left\{1, \left(\frac{C_2 + C_3}{C_1}\right)^\alpha\right\}.$$

Next, we study the incompleteness of the Weil-Petersson metric. We will use the same notations as in the previous sections. In particular, $B = (\Delta^*)^n \times \Delta^m$ and $(z_1, \cdots, z_{n+1}, \cdots, z_{n+m}) \in B$ are the complex coordinates of D. We let ω_{WP}, ω_P, and ω_0 be the Weil-Petersson metric, the Poincaré metric and the Euclidean metric on B, respectively.

We make the following

Definition 7.1. *We say the Weil-Petersson metric is complete on B at $o = (0, \cdots, 0)$, if there is a sequence $\{x_r\} \subset B$ such that*

(1) $\{x_r\}$ is a Cahchy sequence of D with respect to ω_{WP};

(2) $x_r \to o$, as $r \to \infty$ under the Euclid distance.

If the metric is not incomplete at o, we say that it is complete at 0.

In the case of $n = 1$ and $m = 0$, the following result of Wang gave a satisfactory result of the incompleness of the Weil-Petersson metric.

Theorem 7.3 (Wang). *Let $n = 1, m = 0$. Write the local section Ω of $F^s \to D$ as*

$$\Omega = \exp(\frac{\sqrt{-1}}{2\pi} \log \frac{1}{z} N) A(z)$$

for an analytic vector-valued function $A(z)$. Then the Weil-Petersson metric is incomplete if and only if $NA(0) = 0$.

Sketch of the Proof. By definition, we have

$$\omega_{WP} = -\frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log(\Omega, \overline{\Omega}).$$

If $NA(0) \neq 0$, then

$$(\Omega, \overline{\Omega}) = f(\log \frac{1}{r}) + B(z),$$

where $f(x)$ is a nonconstant polynomial and

$$|B(z)| \leq Cr^b$$

for constant $C, b > 0$, where $r = |z|$. Thus there is a constant $C_1 > 0$ such that

$$\omega_{WP} \geq C_1 \frac{\sqrt{-1} dz \wedge d\overline{z}}{r^2 (\log \frac{1}{r})^2}$$

and thus the metric is complete.

If $NA(0) = 0$, then by the Nilpotent orbit theorem, we know that $\exp(\frac{\sqrt{-1}}{2\pi} \log \frac{1}{z} N) A(0) = A(0)$ is a Nilpotent orbit. In particular for $A(0) \in B$. Thus we have

$$(A(0), \overline{A(0)}) \neq 0.$$

A straightforward computation gives

$$\omega_{WP} \leq C_2 \frac{\sqrt{-1} dz \wedge d\overline{z}}{r (\log \frac{1}{r})^{-t}}$$

for some $t \geq -2$. Thus the Weil-Petersson metric is incomplete.

□

Corollary 7.1. *For $n=1, m=0$, the following statements are equivalent:*

1. *(1) The Weil-Petersson metric is incomplete at o;*
2. *(2) the period map extends across to o (Wang);*
3. *(3) The Hodge length of Ω is bounded near o;*
4. *(4) There doesn't exist a constant $C>0$ such that $\omega_{WP} \geq C\omega_P$.*

□

In order to state the theorem, we first observe the following fact: let

$$\Omega = \exp(\sum \frac{\sqrt{-1}}{2\pi} \log \frac{1}{z_i} N_i) A(z_1, \cdots, z_{n+m})$$

be the holomorphic section of F^s near $o \in B$. Let

$$\Omega_0 = \exp(\sum \frac{\sqrt{-1}}{2\pi} \log \frac{1}{z_i} N_i) A(0, \cdots, 0)$$

be the corresponding Nilpotent orbit. Then we have the following:

Lemma 7.1. *Using the above notations, we have*

$$-\frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log(\Omega_0, \overline{\Omega_0}) \geq 0$$

for $\operatorname{Im} z_i, (1 \leq i \leq n)$ *sufficiently large.*

Proof. A straighforward computation gives

$$-\partial_i \overline{\partial}_j \log(\Omega_0, \overline{\Omega_0}) = -(\Omega_0, \overline{\Omega_0})^{-1} (\nabla_i \Omega_0, \overline{\nabla_j \Omega_0}),$$

where

$$\nabla_i \Omega_0 = \partial_i \Omega_0 - \frac{(\partial_i \Omega_0, \overline{\Omega_0})}{(\Omega_0, \overline{\Omega_0})} \Omega_0.$$

The map being horizontal, we conclude that

$$\nabla_i \Omega_0 \in H^{s-1,1}.$$

By the second Hodge-Riemann relations, we conclude that $(\Omega_0, \overline{\Omega_0}) > 0$ and $-(\nabla_i \Omega_0, \overline{\nabla_j \Omega_0})$ is a semi-positive matrix. The lemma thus follows.

□

We define

$$\omega' = -\frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log(\Omega_0, \overline{\Omega_0}).$$

One of the main result of this section is

Theorem 7.4. *There is a constant C, depending on B such that*

$$\omega' - C\omega_0 \leq \omega_{WP} \leq \omega' + C\omega_0.$$

Corollary 7.2. *The Weil-Petersson metric is incomplete at o if and only if the metric $\omega' + \omega_0$ is incomplete at o.*

Note that for the case $n=1, m=0$, the above corollary follows from the theorem of Wang. On the other hand, we need the detailed information of the Nilpotent orbit. Since the proof is complicated, we first state and prove the following special case.

Theorem 7.5. *Let $n = 1$. Then the necessay and sufficient condition for ω_{WP} to be incomplete is that $NA(0, \cdots, 0) = 0$, where $N = N_1$ in eqref.*

Remark 7.1. The above result is equivalen to theorem. By the Nilpotent orbit theorem, we know that

Proof. Let
$$\omega_{WP} = \frac{\sqrt{-1}}{2\pi} g_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$
Let $\mathcal{W}$ be the set of smooth curves $\sigma : [0, 1) \to B$ such that
$$\lim_{t \to 1} \sigma(t) = o$$
under the Euclidean distance. We claim that ω_{WP} is incomplete if and only if there is a curve $\sigma \in \mathcal{W}$ with finite length with respect to the metric $\omega_{WP} + \omega_0$.

Let $\sigma = (\sigma^1, \cdots, \sigma^{n+m})$. Let the length of σ under the Weil-Petersson metric be $L(\sigma)$. Then we have
$$L(\sigma) = 2 \int_0^1 \sqrt{g_{i\bar{j}} \frac{\partial \sigma^i}{\partial t} \overline{\frac{\partial \sigma^j}{\partial t}}} dt.$$
We observe that
$$g_{i\bar{j}} \frac{\partial \sigma^i}{\partial t} \overline{\frac{\partial \sigma^j}{\partial t}} \geq \left| g_{1\bar{1}} \overline{\frac{\partial \sigma^1}{\partial t}} + \frac{1}{\sqrt{g_{1\bar{1}}}} \sum_{j=2}^{n+m} g_{1\bar{j}} \overline{\frac{\partial \sigma^j}{\partial t}} \right|^2$$
This follows from the fact that the matrix
$$g_{i\bar{j}} - \frac{g_{i\bar{1}} g_{1\bar{j}}}{g_{1\bar{1}}}$$
is always semi-positive definite. Using the fact, we get
$$\frac{1}{2} L(\sigma) \geq \int_0^1 \sqrt{g_{1\bar{1}}} \left| \frac{\partial \sigma^1}{\partial t} \right| dt - \sum_{j=2}^{n+m} \int_0^1 \frac{|g_{1\bar{j}}|}{\sqrt{g_{1\bar{1}}}} \left| \frac{\partial \sigma^j}{\partial t} \right| dt$$

8. Parallel results in special Kähler manifolds

Let (M, g) be a Kähler manifold of dimension s with a Kähler metric g.

Definition 8.1. *A special Kähler manifold (M, g, ∇, J) is a Kähler manifold with a real flat torsion-free simplectic connection ∇ such that*
$$d_\nabla J = 0,$$
where J is the complex structure on M.

Let F be a holomorphic symmetric cubic tensor defined by
$$F = -\omega(\pi^{(1,0)}, \nabla \pi^{(1,0)}) \in H^0(M, \mathrm{Sym}^3 T^* M),$$

where $\pi^{(1,0)} \in \Omega^{(1,0)}(T_{\mathbb{C}}M)$ is a projection of the complexified tangent space $T_{\mathbb{C}}M$ into the holomorphic tangent space $T^{(1,0)}M$. Locally, it can be written as F_{ijk} such that it is symmetric in i, j, k. By [7], we have

$$R(\omega)_{\alpha\overline{\beta}\gamma\overline{\delta}} = -g^{\xi\overline{\eta}} F_{\alpha\gamma\xi}\overline{F_{\beta\delta\eta}}, \tag{8.33}$$

where ω is the Kähler form of g. It is clear that the curvature operator of ω is nonnegative.

Define

$$h_{\alpha\overline{\beta}} = F_{\alpha\gamma\xi}\overline{F_{\beta\delta\eta}} g^{\gamma\overline{\delta}} g^{\xi\overline{\eta}}.$$

and let

$$\omega_h = \frac{\sqrt{-1}}{2\pi} h_{\alpha\overline{\beta}} dz_\alpha \wedge d\overline{z}_\beta. \tag{8.34}$$

Similar to the cases of Calabi-Yau moduli, we make the following definition:

Definition 8.2. *Suppose the cubic form F is not identically zero. Then ω_h defines a Kähler metric on the dense open set*

$$U = \{p \mid F(p) \neq 0\} \subset M.$$

We call it the Hodge metric on M.

Similar to the cases of Hodge metric on Calabi-Yau moduli, we have the following

Theorem 8.1. *The curvatures of ω_h satisfy the following properties:*

(1) The holomorphic bisectional curvature of ω_h is nonpositive;
(2) The holomorphic sectional curvature of ω_h is negative away from zero;
(3) The Ricci curvature of ω_h is negative away from zero.

Proof. We choose a local coordinate around $p \in M$ such that $g_{i\overline{j}}(p) = \delta_{ij},\ dg_{i\overline{j}}(p) = 0$. By equation (8.33) we obtain

$$\begin{aligned} R(\omega_h)_{\alpha\overline{\beta}\gamma\overline{\delta}} &= \sum_{\xi,\mu} F_{\alpha\xi\mu,\gamma}\overline{F_{\beta\xi\mu,\delta}} + 2\sum_{\xi,\eta,\mu,\nu} F_{\alpha\xi\mu}\overline{F_{\beta\xi\nu}} F_{\nu\gamma\eta}\overline{F_{\mu\delta\eta}} \\ &\quad - \sum_{\tau,\varepsilon} h^{\tau\overline{\varepsilon}} \left[\sum_{\xi,\eta} F_{\xi\eta\alpha,\gamma}\overline{F_{\xi\eta\varepsilon}}\right]\left[\overline{\sum_{\xi,\eta} F_{\xi\eta\beta,\delta}\overline{F_{\xi\eta\tau}}}\right], \end{aligned}$$

where $F_{\alpha\beta\gamma,\delta} = \frac{\partial F_{\alpha\beta\gamma}}{\partial z^\delta}$.

We consider the case when $\beta = \alpha, \delta = \gamma$. By the Cauchy inequality, we have

$$\sum_{\xi,\mu} F_{\alpha\xi\mu,\gamma}\overline{F_{\alpha\xi\mu,\gamma}} - \sum_{\tau,\varepsilon} h^{\tau\overline{\varepsilon}} \left[\sum_{\xi,\eta} F_{\xi\eta\alpha,\gamma}\overline{F_{\xi\eta\varepsilon}}\right]\left[\overline{\sum_{\xi,\eta} F_{\xi\eta\alpha,\gamma}\overline{F_{\xi\eta\tau}}}\right] \geq 0.$$

$\square$

From the above inequality, we obtain

$$R(\omega_h)_{\alpha\overline{\alpha}\gamma\overline{\gamma}} \geq 2\sum_{\xi,\eta}\left|\sum_{\mu} F_{\alpha\xi\mu}\overline{F_{\gamma\mu\eta}}\right|^2. \tag{8.35}$$

In particular, $R(\omega_h)_{\alpha\overline{\alpha}\gamma\overline{\gamma}} \geq 0$. Thus the metric has nonpositive sectional curvartures. Furthermore, we let $\alpha = \gamma$. Then we have

$$\begin{aligned} R(\omega_h)_{\alpha\overline{\alpha}\alpha\overline{\alpha}} &\geq 2\sum_{\xi,\eta}\left|\sum_{\mu} F_{\alpha\xi\mu}\overline{F_{\alpha\mu\eta}}\right|^2 \\ &\geq 2\sum_{\xi}\left|\sum_{\mu}|F_{\alpha\xi\mu}|^2\right|^2 \geq \frac{2}{s}\left|\sum_{\xi,\mu}|F_{\alpha\xi\mu}|^2\right|^2 \geq \frac{2}{s}h^2_{\alpha\overline{\alpha}}, \end{aligned}$$

from which the second assertion follows. Finally, the Ricci curvature estimate follows from the bounds of the bisectional curvatures and the holomorphic sectional curvatures.

□

Next we prove a convexity property of the curvature of ω.

let $r_1, \cdots, r_t \geq 0$ be integers such that $\sum r_i = s$. Then we have the following

Proposition 8.1. *There is a constant C such that*

$$0 \leq (-1)^s c_{r_1}(\omega) \wedge \cdots c_{r_t}(\omega) \leq C\omega_h^s.$$

Proof. We use the similar method as in section xxx. Fix a point $p \in M$ and assume that at p, $g_{\alpha\overline{\beta}}(p) = \delta_{\alpha\beta}$ and $dg_{\alpha\overline{\beta}}(p) = 0$. Then by [8], we have

$$0 \leq -R(\omega)_{\alpha\overline{\beta}\gamma\overline{\delta}} \leq \sum_{\xi} F_{\alpha\gamma\xi}\overline{F_{\beta\delta\eta}} \leq \sqrt{h_{\gamma\overline{\gamma}}h_{\delta\overline{\delta}}}.$$

for any fixed α, β, γ and δ.

Since

$$c_r(\omega_{WP}) = \left(\frac{\sqrt{-1}}{2\pi}\right)^r \left[\sum_{\sigma,\tau\in S_r} sgn(\tau) R^{i_{\tau(1)}}_{i_1 k_1 \overline{k_{\sigma(1)}}} \cdots R^{i_{\tau(r)}}_{i_r k_r \overline{k_{\sigma(r)}}}\right] dz_K \wedge d\overline{z}_{\sigma(K)},$$

where

$$dz_K \wedge d\overline{z}_{\sigma(K)} = dz_{k_1} \wedge d\overline{z}_{\sigma(1)} \wedge \cdots dz_{k_r} \wedge d\overline{z}_{k_{\sigma(r)}}.$$

Thus there is a constant $C > 0$ such that

$$|(-1)^s c_{r_1}(\omega) \wedge \cdots \wedge c_{r_t}(\omega)| \leq C\left|\prod_{i=1}^{s} h_{i\overline{i}} dz_1 \wedge \cdots d\overline{z}_s\right| = C\omega_h^s.$$

□

Corollary 8.1. *Using the above notations, suppose that M is quasi-projective, then*

$$\left|\int_M (-1)^s c_{r_1}(\omega)\wedge\cdots\wedge c_{r_t}(\omega)\right| < +\infty$$

Proof. We let U be in definition 6.2. Since M is quasi projective, so is U. Suppose that $\tilde{U}$ is a smooth compactification of U such that $\tilde{U}\backslash U$ is a divisor of normal crossings. Let ω_P be the global Poincaré metric. Then by the Schwarz lemma, we know that there is a constant C such that

$$\omega_h \leq C\omega_P.$$

The corollary follows from the above.

□

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E-mail address, Michael Douglas, NHETC and Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855-0849, USA and IHES, Le Bois-Marie, Bures-sur-Yvette, 91440, France: `mrd@physics.rutgers.edu`

E-mail address, Zhiqin Lu, Department of Mathematics, UC Irvine, Irvine, CA 92697: `zlu@math.uci.edu`