PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 128, Number 3, Pages 767–774
S 0002-9939(99)05059-5

A NOTE ON THE ANALYTIC FAMILIES OF COMPACT SUBMANIFOLDS OF COMPLEX MANIFOLDS

ZHIQIN LU

(Communicated by Leslie Saper)

Abstract. In this paper, we prove a result of the deformation of the complex structure of a submanifold. Our result is a modification of the result of Kodaira (Ann. Math 75(1), 146–162, 1962).

1. Introductions

It is a classical theorem of Kodaira [1] that if the first cohomological group of the normal bundle is zero, then the deformation of a compact complex submanifold within the ambient complex manifold is unobstructed. However, it is usually difficult to check if such a cohomological group is indeed zero. Furthermore, as showed in §2, in some cases, it will never be zero.

In this short note, we are going to prove: if the deformation of the complex structure of a compact complex manifold M is unobstructed in the sense of Kodaira and Spencer, and if $M \to V$ is an embedding to the complex manifold V and $H^1(M, T_V|_M) = 0$, then any fiber in a neighborhood of the universal deformation space U at M can be embedded holomorphically to V.

Contrary to the case of normal bundle, it will be relatively easy to check the vanishing of the group $H^1(M, T_V|_M)$. For example, if the curvature of the manifold V has some kinds of positivity along M, then the group vanishes.

The typical examples of M are compact Calabi-Yau manifolds. Those manifolds admit Kähler metrics with zero Ricci curvature. In Tian [3], it is proved that the deformation of the complex structure of a Calabi-Yau manifold is unobstructed. The moduli space of a polarized Calabi-Yau manifold is then a complex orbifold.

We use the similar method as that of Kodaira [1]. That is, we construct a formal power series which gives the map we want. Then we prove the convergence of the power series. Fortunately, in our case, the original method in [1] of proving the convergence can be applied here.

Received by the editors January 23, 1998 and, in revised form, April 22, 1998.
1991 *Mathematics Subject Classification.* Primary 32G05; Secondary 58H15.
Key words and phrases. Deformation of the complex structure, Kodaira-Spencer theory, Calabi-Yau manifold.

2. Analytic families of compact submanifolds

Suppose M is a compact complex manifold. We assume that the deformation of the complex structure of M is unobstructed. That is, the universal deformation space U of M is a complex manifold near M with complex dimension dim $H^1(M, T_M)$, where T_M denotes the holomorphic tangent bundle of M.

If M is a Calabi-Yau manifold, then the deformation of the complex structure on M is unobstructed by the theorem of Tian [3].

We use the notations and definitions in [1].

Definition 2.1. Suppose N is a complex manifold of dimension $r+n$. By an analytic family of compact submanifolds of dimension n of N we shall mean a pair $(\mathcal{M}, U)$ of a complex manifold U and a complex analytic submanifold $\mathcal{M}$ of $N \times U$ of co-dimension r which satisfies the following two conditions:

1). For Each Point $t \in U$, the intersection $\mathcal{M} \cap N \times t$ is a connected, compact submanifold of $N \times t$ of dimension n.

2). For each point $p \in \mathcal{M}$, there exist r holomorphic functions $f_1 = f_1(w,t), \cdots, f_r = f_r(w,t)$ defined on a neighborhood $\mathcal{U}_p$ of p in $N \times U$ such that

$$rank\, \frac{\partial(f_1, \cdots, f_r)}{\partial(w^1, \cdots, w^{r+d})} = r$$

and in $\mathcal{U}_p$, the submanifold $\mathcal{M}$ is defined by the simultaneous equations

$$f_1(w,t) = f_2(w,t) = \cdots = f_r(w,t) = 0.$$

We call U the parameter manifold or the base space of the family (N, U). We denote the family (N, U) simply by N when we need not indicate the base space U. For each point $t \in U$, we set

$$M_t \times t = \mathcal{M} \cap N \times t;$$

the submanifold M_t of N thus defined will be called the fiber of $\mathcal{M}$ over t. We may identify $M_t \times t$ with M_t and consider M_t as a *family consisting of compact submanifolds* M_t, $t \in U$, of N.

Definition 2.2. We say $(\mathfrak{X}, U)$ is the local total family of M, if U is a neighborhood of C^d with $d = \dim H^1(M, T_M)$ where T_M is the holomorphic tangent bundle of M, and there exists a projection

$$\pi : \mathfrak{X} \to U$$

such that it is holomorphic, surjective, of rank d and such that for all $t \in U$, $\pi^{-1}(t)$ is a deformation of complex structure of the center fiber $\pi^{-1}(0) = M$.

We state the main result of this paper.

Theorem 2.1. *Suppose M is a compact complex manifold whose deformation of its complex structure is unobstructed. Let N be another complex manifold. Suppose that*

$$i : M \to N$$

is a holomorphic embedding. If $H^1(M, T_N|_M) = 0$ where $T_N|_M$ is the restriction of the holomorphic tangent bundle of N to M, then there is a holomorphic map

$$f : \mathfrak{X} \to N$$

such that $f|_M = i$. Here $(\mathfrak{X}, U)$ is the local total family of M. Furthermore, $f|_{\pi^{-1}(t)}$ is an embedding of $\pi^{-1}(t)$ to N for $t \in U$.

Proof. Suppose $\bigcup_{j\in I} U_j \supset \mathfrak{X}$ is an open covering. On each U_j, $j \in I$, suppose $(z_j^1, \cdots, z_j^n, t)$ is a local coordinate such that

$$\pi(z_j^1, \cdots, z_j^n, t) = t, \qquad t \in U \subset C^d.$$

It is obvious that, for fixed t, $(z_j^1, \cdots z_j^n)$ will be local coordinate for $\pi^{-1}(t) = M_t$. We further assume that U_j is defined by

$$U_j = \{|z_j| = Max_\alpha |z_j^\alpha| < 1\}.$$

We have, however, holomorphic functions g_{jk} such that

$$z_j^\alpha = g_{jk}^\alpha(z_k, t), \qquad j, k \in I,$$

for $\alpha = 1, \cdots, n$ and for $U_j \cap U_k \neq \emptyset$.

Now we suppose $\bigcup V_A \supset N$ is an open covering of N. Suppose

$$i(U_j) \subset V_{A(j)}, \qquad j \in I,$$

for some $A(j)$ of j. Suppose $(w_A^1, \cdots, w_A^r)$ is the local holomorphic coordinate chart of N on V_A. And we have transition functions h_{AB} on $V_A \cap V_B \neq \emptyset$,

$$w_A^s = h_{AB}^s(w_B)$$

for $s = 1, \cdots r + n = \dim N$. And again, we assume

$$V_j = \{|w_j| = \ Max|w_j^s| < 1\}.$$

For the sake of simplicity, we denote j for $A(j)$. In order to construct f, we need only have to construct holomorphic mappings (f_j) such that

$$f_j : U_j \to V_j, \qquad j \in I,$$

satisfying

$$h_{jk}^s(f_k(z_k, t)) = f_j^s(g_{jk}(z_k, t), t) \qquad \text{on} \quad U_j \cap U_k \neq \emptyset, \quad j, k \in I, \tag{2.1}$$

for $s = 1, \cdots, r + n$.

We set up some notations. Let

$$f_k(z, t) = f_{k|0} + f_{k|1} + \cdots + f_{k|m} + \cdots$$

be the decomposition of $f_k(z_k, t)$ into homogeneous polynomials of t of degree m. Of course, each f_k's and $f_{k|m}$'s are vector valued functions $f_k = (f_k^s), f_{k|m} = (f_{k|m}^s), s = 1, \cdots, r + n$. Suppose

$$f_k^m = f_{k|0} + \cdots + f_{k|m}, \qquad k \in I,$$

and let $a \equiv_m b$ mean $a - b$ is a polynomial of t of degree bigger than or equal to $m + 1$.

We construct $f_{k|m}$ inductively. First, set

$$f_{j|0} = i_j(z_j), \qquad j \in I.$$

It is easy to check that

$$h_{jk}(f_k^0(z_k, t)) \equiv_0 f_j^0(g_{jk}(z_k, t), t), \qquad U_j \cap U_k \neq \emptyset,$$

where $h_{jk} = (h_{jk}^s)_{s=1,\cdots,r+n}$.

Now suppose for integer m, f_k^m is constructed, and

$$h_{jk}(f_k^m(z_k, t)) \equiv_m f_j^m(g_{jk}(z_k, t), t), \qquad U_j \cap U_k \neq \emptyset. \tag{2.2}$$

Define

$$\Psi_{jk}(z_k,t)\equiv_{m+1} h_{jk}(f_k^m(z_k,t))-f_j^m(g_{jk}(z_k,t),t),\quad U_j\cap U_k\neq\emptyset. \tag{2.3}$$

Then

Claim.

$$\Psi_{ik}=\Psi_{ij}+\frac{\partial w_i}{\partial w_j^s}|_{t=0}\Psi_{jk}^s\qquad \textit{on}\quad U_i\cap U_j\cap U_k\neq\emptyset.$$

Proof of the Claim. By Equation (2.3), we have

$$f_i^m(g_{ik}(z_k,t),t)\equiv_{m+1} h_{ik}(f_k^m(z_k,t))-\Psi_{ik}(z_k,t),\qquad U_i\cap U_k\neq\emptyset.$$

Thus on $U_i\cap U_j\cap U_k\neq\emptyset$, we have

$$\begin{aligned}&h_{ij}(f_j^m(g_{jk}(z_k,t),t))\equiv_{m+1} h_{ij}(h_{jk}(f_k^m(z_k,t))-\Psi_{jk}(z_k,t))\\&\equiv_{m+1} h_{ij}(h_{jk}(f_k^m(z_k,t)))-\frac{\partial w_i}{\partial w_j^s}\Psi_{jk}^s(z_k,t)\\&\equiv_{m+1} h_{ik}(f_k^m(z_k,t))-\frac{\partial w_i}{\partial w_j^s}\Psi_{jk}^s(z_k,t).\end{aligned}$$

Note that $\Psi_{jk}^s(z_k,t)\equiv_m 0$. So

$$\frac{\partial w_i}{\partial w_j^s}\Psi_{jk}^s(z_k,t)\equiv_{m+1}\frac{\partial w_i}{\partial w_j^s}|_{t=0}\Psi_{jk}^s(z_k,t).$$

We have

$$\Psi_{ij}(g_{jk}(z_k,t),t)\equiv_{m+1}\Psi_{ik}(z_k,t)-\frac{\partial w_i}{\partial w_j^s}|_{t=0}\Psi_{jk}^s(z_k,t),\qquad U_j\cap U_k\neq\emptyset.$$

On the other hand

$$g_{jk}(z_k,t)\equiv_0 z_j,\qquad U_j\cap U_k\neq\emptyset.$$

So

$$\Psi_{ij}(z_j,t)\equiv_{m+1}\Psi_{ij}(g_{jk}(z_k,t),t).$$

Thus we have

$$\Psi_{ij}(z_j,t)=\Psi_{ik}(z_k,t)-\frac{\partial w_i}{\partial w_j^s}|_{t=0}\Psi_{jk}^s(z_k,t),\qquad U_j\cap U_k\neq\emptyset. \tag{2.4}$$

The claim is proved. □

Suppose $\mathcal{U}=(U_j)$ is the covering of M; we see from Equation (2.4) $\{\Psi_{ij}\}_{i,j\in I}$ defined a cocycle of $Z^1(\mathcal{U},T_N|_M)$. Thus $\{\Psi_{ij}\}_{i,j\in I}\in H^1(\mathcal{U},T_N|_M)$. With a good covering $\mathcal{U}$, we have $H^1(\mathcal{U},T_N|_M)=H^1(M,T_N|_M)$ and by the assumption, the latter is zero. So we can find $\{\Psi_j\}$ such that

$$\Psi_{jk}=\Psi_j-\frac{\partial w_j}{\partial w_k^s}|_{t=0}\Psi_k^s,\qquad U_j\cap U_k\neq\emptyset.$$

We then define f_k^{m+1} inductively as

$$f_k^{m+1}(z_k,t)=f_k^m(z_k,t)+\Psi_k(z_k,t),\qquad k\in I. \tag{2.5}$$

With this definition, we have

$$\begin{aligned}
&h_{jk}(f_k^{m+1}(z_k,t)) \equiv_{m+1} h_{jk}(f_k^m(z_k,t)+\Psi_k(z_k,t))\\
&\equiv_{m+1} h_{jk}(f_k^m(z_k,t)) + \frac{\partial w_j}{\partial w_k^s}\Psi_k^s(z_k,t)\\
&\equiv_{m+1} \Psi_{jk}(z_k,t) + \frac{\partial w_j}{\partial w_k^s}\Psi_k^s(z_k,t) + f_j^m(g_{jk}(z_k,t),t)\\
&\equiv_{m+1} \Psi_j(z_j,t) + f_j^m(g_{jk}(z_k,t),t)\\
&\equiv_{m+1} f_j^{m+1}(g_{jk}(z_k,t),t).
\end{aligned}$$

Now we have got a formal series

$$f_{k|0} + f_{k|1} + \cdots + f_{k|m} + \cdots, \qquad k \in I,$$

which satisfies Equation (2.2) for any m. If it converges, then f_k's and f will be holomorphic and f will satisfy Equation (2.1). We put off the proof of the convergence to the next section. At this moment, we assume the convergence is true.

By continuity, for fixed t, t being sufficiently small, f will be an immersion on $\pi^{-1}(t)$. We claim that for sufficiently small t, f is an embedding. Suppose not, then we can find $t_n \to 0$ and $x_n, y_n \in \pi^{-1}(t_n)$ such that $x_n \neq y_n$ but $f(x_n,t) = f(y_n,t)$. Suppose $x_n \to x$ and $y_n \to y$. We see $x = y$; otherwise it will contradict the fact that i is an embedding. But if $x = y$, it will contradict the fact that f is an immersion on each fiber.

Now we give an important example of manifolds such that $H^1(M, T_N|_M) = 0$.

Proposition 2.1. *Let M be a simply connected Calabi-Yau threefold. $M \to CP^D$ is an embedding. Then we have*

$$H^1(M, T_{CP^D}|_M) = 0.$$

Proof. From the Euler exact sequence

$$0 \to C \to \oplus_{D+1}\mathcal{O}(1) \to T_{CP^D} \to 0. \tag{2.6}$$

We have the long exact sequence

$$\cdots \to H^1(M, \oplus_{d+1}\mathcal{O}(1)) \to H^1(M, T_{CP^D}|_M) \to H^2(M, \mathcal{O}) \to \cdots.$$

By the Kodaira Vanishing theorem

$$H^1(M, \oplus_{D+1}\mathcal{O}(1)) = 0.$$

By the Dolbeault theorem and Serre Duality $H^2(M, \mathcal{O}) = 0$, we have

$$H^1(M, T_{CP^D}|_M) = 0.$$

□

Corollary 2.1. *Suppose M is a simply connected Calabi-Yau threefold. If M is embedded to some CP^D, then $M_t = \pi^{-1}(t)$ can also be embedded to the same CP^D for small t.*

The following example showed that, in general, $H^1(M, T_{CP^D}|_M/T_M) \neq 0$.

Example. Suppose $\mathcal{N} = T_{CP^D}|_M/T_M$ is the normal bundle of M in CP^D in the previous proposition. Then, in general, $H^1(M, \mathcal{N}) \neq 0$.

Proof. We have the exact sequence:

$$0 \to T_M \to T_{CP^D}|_M \to \mathcal{N}|_M \to 0.$$

So the long exact sequence gives

$$\begin{aligned}&\cdots \to H^1(M, T_{CP^D}|_M) \to H^1(M,\mathcal{N}) \to H^2(M,T_M)\\ &\to H^2(M,T_{CP^D}|_M) \to \cdots .\end{aligned}$$

However, by Serre Duality, $H^2(M,T_M) = H^{1,1}(M)$. And it is easy to see from the Euler Sequence (2.6) that $\dim H^2(M,T_{CP^D}|_M) = 1$. Thus in general $\dim H^1(M,\mathcal{N}) \geq \dim H^{1,1}(M) - 1$ and is not zero.

3. The convergence

Now we shall show that the power series $\sum f_k(z_k,t)$ in Equation (2.5) for all $k \in I$, converge for $|t| < \varepsilon_0$, ε_0 being a sufficiently small positive number, provided that we choose for each m the homogeneous polynomials $f_{k|m+1}(z_k,t), k \in I$, satisfying (2.3) in a proper manner.

For any vector $\xi = (\xi^1, \xi^2, \cdots, \xi^\lambda, \cdots)$, we define

$$|\xi| = Max_\lambda |\xi^\lambda|.$$

Consider a power series

$$\xi(z,u) = \sum \xi_{lm,\cdots,n} u_1^l u_2^m \cdots u_q^n$$

in $u_1, \cdots, u_q$ whose coefficients $\xi_{lm,\cdots,n}$ are vector valued functions of z and a power series

$$a(u) = a_{lm,\cdots,n} u_1^l u_2^m \cdots u_q^n, \qquad a_{lm,\cdots,n} \geq 0.$$

We indicate by writing $\xi(z,u) \ll a(u)$ that

$$|\xi_{lm,\cdots,n}(z)| \leq a_{lm,\cdots,n}.$$

Let

$$A(t) = \frac{a}{16b} \sum_{n=1}^{\infty} \frac{1}{n^2} b^n (t_1 + \cdots + t_l)^n,$$

where a and b are positive constants. We have

$$A(t)^\gamma \ll (\frac{a}{b})^{\gamma-1} A(t), \qquad \gamma = 2, 3, \cdots .$$

For our purpose it suffices to prove the inequalities

$$f_k(z_k,t) \ll A(t), \qquad i \in I.$$

In what follows we denote by $c_0, c_1, c_2, \cdots$ positive constants *which are greater than 1.* We may assume that

$$|\frac{\partial h_{ij}^\lambda}{\partial w_j^\mu}| < c_0, \qquad c_0 > 1.$$

For the sake of simplicity, we denote $f_k^m(z_k,t) - i(z_k)$ by $f_k^m(z_k,t)$. Then for sufficiently large a, we have

$$f_k^1(z_k,t) \ll \frac{a}{16}(t_1 + \cdots + t_d) \ll A(t), \qquad k \in I.$$

Now assuming the inequalities

$$f_k^m(z_k, t) \ll A(t), \qquad k \in I,$$

for an integer $m \geq 1$, we shall estimate the coefficients of the homogeneous polynomials $\Psi_{ik}(z, t)$. We expand h_{ik} and g_{ik} into power series, whose coefficients are vector valued holomorphic functions:

$$h_{ik}(w_k) \ll \sum_{\alpha=0}^{\infty} c_1^{\alpha}(w_k^1 + \cdots w_k^{r+n})^{\alpha},$$

$$g_{ik}(z_k) \ll \sum_{\alpha=0}^{\infty} c_1^{\alpha}(z_k^1 + \cdots + z_k^n)^{\alpha}$$

for $i, k \in I$.

Recall that in Equation (2.3)

$$\Psi_{jk} = [h_{jk}(f_k^m(z_k, t))]_{m+1} - [f_j^m(g_{jk}(z_k, t), t)]_{m+1},$$

where $[a]_{m+1}$ is a polynomial of t of degree bigger than m. First we estimate $[h_{jk}(f_k^m(z_k, t))]_{m+1}$. The terms which are linear in h_{jk} contribute nothing to $[h_{jk}(f_k^m(z_k, t))]_{m+1}$. So we have

$$[h_{jk}(f_k^m(z_k, t))]_{m+1} \ll \sum_{\alpha=2}^{\infty} c_1^{\alpha}(r+n)^{\alpha} A(t)^{\alpha}$$

$$\ll c_1(r+n)A(t) \sum_{\alpha=1}^{\infty} (\frac{c_1(r+n)a}{b})^{\alpha}.$$

Assuming that

$$b > 2c_1(r+n)a,$$

we obtain therefore

$$[h_{jk}(f_k^m(z_k, t))]_{m+1} \ll 2c_1^2 r^2 a b^{-1} A(t). \tag{3.1}$$

On the other hand

$$[f_j^m(g_{jk}(z_k, t), t)]_{m+1} = [f_j^m(g_{jk}(z_k, t), t) - f_j^m(z_j, t)]_{m+1}.$$

Denote by U_i^{δ} the subdomain of U_i consisting of all points $z_j = (z_j, 0)$, $|z_j| < 1 - \delta$. We fix a positive number δ such that $\{U_i^{\delta} | i \in I\}$ forms a covering of M. Take a point $z \in U_k \cap U_i^{\delta}$ and let z_k and z_j be the local coordinates of z on U_k and U_j respectively. Obviously, we have

$$z_j = g_{jk}(z_k, 0), \qquad |z_k| < 1, |z_j| < 1 - \delta.$$

Letting $y = (y_1, \cdots, y_n)$, we expand the coefficients of polynomial $f_i^m(z_j + y, t)$ into power series. Suppose $|y| < \delta$; we have

$$[f_j^m(z_j + y, t) - f_j^m(z_j, t)]_{m+1} \ll A(t)(\prod_{\alpha=1}^{n}(1 - \frac{|y_{\alpha}|}{\delta})^{-1} - 1).$$

Now if t is small, and $\mu = Max_{j,k}|g_{jk}(z_k, t) - z_j| < \delta$, then

$$[f_j^m(g_{jk}(z_k, t), t)]_{m+1} \ll ((1 - \frac{\mu}{\delta})^{-n} - 1)A(t). \tag{3.2}$$

So from Equation (3.1) and (3.2), we have

$$\Psi_{jk} \ll (2c_1^2(r+n)^2ab^{-1} + ((1-\frac{\mu}{\delta})^{-n} - 1))A(t).$$

We take an arbitrary point $z \in U_k \cap U_i$ and choose a domain U_j^δ, which contains z; then

$$\Psi_{ik} = \frac{\partial w_i}{\partial w_j}\Psi_{jk} - \frac{\partial w_i}{\partial w_j}\Psi_{ji}.$$

Thus

$$\Psi_{ik} \ll 2c_0(2c_1^2(r+n)^2ab^{-1} + ((1-\frac{\mu}{\delta})^{-n} - 1))A(t).$$

Let $c_3 = 2c_0(2c_1^2(r+n)^2ab^{-1} + ((1-\frac{\mu}{\delta})^{-n} - 1))$. Then

$$\Psi_{jk} \ll c_3A(t), \qquad z \in U_k \cup U_i. \tag{3.3}$$

The following lemma can be proved by an elementary consideration [2].

Lemma 3.1. *We can choose the homogeneous polynomials* $\Psi_i, i \in I$, *satisfying*

$$\Psi_{ik} = \Psi_i - \frac{\partial w_i}{\partial w_k^s}\Psi_k^s$$

in such a way that

$$\Psi_i \ll c_3c_4A(t)$$

where $c_4 > 1$ *is a constant independent of* m. ☐

Since μ is independent of m, and $\mu \to 0$ as $t \to 0$, we can choose t small enough such that $2c_0((1-\mu/\delta)^{-n} - 1) < \frac{1}{2}$. Choosing b large enough so that $4c_0c_1^2(r+n)^2ab^{-1} < \frac{1}{2}$, we have

$$f_j^{m+1} \ll A(t), \qquad j \in I.$$

The convergence of the power series is proved.

Acknowledgment

The author thanks his advisor, Professor G. Tian for the help during the preparation of this paper. He also thanks the referee for pointing out an error at the early version of this paper.

References

[1] Kunihiko Kodaira *A Theorem of Completeness of Characteristic Systems for Analytic Families of Compact Submanifolds of Complex Manifolds* Annals of Mathematics,75(1), 146-162, 1962 MR **24:**A3665b

[2] Kunihiko Kodaira and D. C. Spencer. *A Theorem of Completeness of Characteristic Systems of Complete Continuous Systems*, Amer. J. Math, 81, 477–500, 1959 MR **22:**3011

[3] Gang Tian, *Smoothness of the Universal Deformation Space of Compact Calabi-Yau Manifolds and its Peterson-Weil Metric*, Mathematical aspects of string theory, (1), 629–646, Shing-Tung Yau ed, 1987, World Scientific CMP 20:04

Department of Mathematics, Columbia University, New York, New York 10027
E-mail address: `lu@math.columbia.edu`