

PROOF OF THE NORMAL SCALAR CURVATURE CONJECTURE

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1. INTRODUCTION

Let M^n be an n -dimensional manifold isometrically immersed into the space form $N^{n+m}(c)$ of constant sectional curvature c . Define the normalized scalar curvature ρ (resp. ρ^\perp) for the tangent bundle (resp. the normal bundle) as follows:

$$(1) \quad \begin{aligned} \rho &= \frac{2}{n(n-1)} \sum_{1=i<j}^n R(e_i, e_j, e_j, e_i), \\ (\text{resp.}) \quad \rho^\perp &= \frac{2}{n(n-1)} \left(\sum_{1=i<j}^n \sum_{1=r<s}^m \langle R^\perp(e_i, e_j)\xi_r, \xi_s \rangle^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where $\{e_1, \dots, e_n\}$ (resp. $\{\xi_1, \dots, \xi_m\}$) is an orthonormal basis of the tangent (resp. normal) bundle, and R (resp. R^\perp) is the curvature tensor for the tangent (resp. normal) bundle.

In the study of submanifold theory, De Smet, Dillen, Verstraelen, and Vrancken [5] made the following *normal scalar curvature conjecture*¹:

Conjecture 1. *Let h be the second fundamental form, and let $H = \frac{1}{n}$ trace h be the mean curvature tensor. Then*

$$\rho + \rho^\perp \leq |H|^2 + c.$$

Let $x \in M$ be a fixed point and let (h_{ij}^r) ($i, j = 1, \dots, n$ and $r = 1, \dots, m$) be the entries of (the traceless part of) the second fundamental form under the orthonormal bases of both the tangent bundle and the normal bundle. Then by [10], or [6], Conjecture 1 can be formulated as an inequality with

Date: August 26, 2007.

2000 Mathematics Subject Classification. Primary: 58C40; Secondary: 58E35.

Key words and phrases. DDVV Conjecture, normal scalar curvature conjecture.

The author is partially supported by NSF Career award DMS-0347033 and the Alfred P. Sloan Research Fellowship.

¹Also known as the DDVV conjecture.

respect to the coefficients (h_{ij}^r) as follows:

$$(2) \quad \sum_{r=1}^m \sum_{1=i<j}^n (h_{ii}^r - h_{jj}^r)^2 + 2n \sum_{r=1}^m \sum_{1=i<j}^n (h_{ij}^r)^2 \\ \geq 2n \left(\sum_{1=r<s}^m \sum_{1=i<j}^n \left(\sum_{k=1}^n (h_{ik}^r h_{jk}^s - h_{ik}^s h_{jk}^r) \right)^2 \right)^{\frac{1}{2}}.$$

Suppose that A_1, A_2, \dots, A_m are $n \times n$ symmetric real matrices. Let

$$\|A\|^2 = \sum_{i,j=1}^n a_{ij}^2,$$

where (a_{ij}) are the entries of A , and let

$$[A, B] = AB - BA$$

be the commutator. Then the inequality (2), in terms of matrix notations, can be formulated as

Conjecture 2. *For $n, m \geq 2$, we have*

$$(3) \quad \left(\sum_{r=1}^m \|A_r\|^2 \right)^2 \geq 2 \left(\sum_{r<s} \| [A_r, A_s] \|^2 \right).$$

Fixing n, m , we call the above inequality Conjecture $P(n, m)$. Note that Conjecture 1 is equivalent to Conjecture 2, which is purely linear algebraic.

A weaker version of Conjecture 1, $\rho \leq |H|^2 + c$, was proved in [2]. An alternate proof is in [9].

The following special cases of Conjecture 2 were known. $P(2, m)$ and $P(n, 2)$ were proved in [5]; $P(3, m)$ was proved in [4]; and $P(n, 3)$ was proved in [8], the previous version of this paper. In [6], a weaker version of $P(n, m)$ was proved by using an algebraic inequality in [7] (see also [3]). In the same paper, $P(n, m)$ was proved under the addition assumption that the submanifold is either *Lagrangian H-umbilical*, or *ultra-minimal* in \mathbb{C}^4 .

In this paper, we prove the conjecture for any $n, m \geq 1$.

2. INVARIANCE

Let A_1, \dots, A_m be $n \times n$ symmetric matrices. Let $G = O(n) \times O(m)$. Then G acts on matrices (A_1, \dots, A_m) in the following natural way: let $(p, q) \in G$, where p, q are $n \times n$ and $m \times m$ orthogonal matrices, respectively. Let $q = \{q_{ij}\}$. Then

$$(p, I) \cdot (A_1, \dots, A_m) = (pA_1p^{-1}, \dots, pA_mp^{-1}),$$

and

$$(I, q) \cdot (A_1, \dots, A_m) = \left(\sum_{j=1}^m q_{1j} A_j, \dots, \sum_{j=1}^m q_{mj} A_j \right).$$

It is easy to verify the following

Proposition 1. *Conjecture $P(n, m)$ is G invariant. That is, in order to prove inequality (3) for (A_1, \dots, A_m) , we just need to prove the inequality for any $\gamma \cdot (A_1, \dots, A_m)$ where $\gamma \in G$. Moreover, the expressions of both sides of (3) are G invariant.*

□

Corollary 1. *We can prove Conjecture 2 under the following additional assumptions on the matrices:*

- (1) A_1 is diagonal;
- (2) $\langle A_\alpha, A_\beta \rangle = 0$ if $\alpha \neq \beta$;
- (3) $\|A_1\| \geq \dots \geq \|A_m\|$.

Note that under the above assumptions, $A_k = 0$ if $k > \frac{1}{2}n(n+1)$.

3. PROOF OF $P(n, m)$.

In this section, we prove Conjecture 2. We first establish some lemmas which are themselves interesting.

Lemma 1. *Suppose η_1, \dots, η_n are real numbers and*

$$\eta_1 + \dots + \eta_n = 0, \quad \eta_1^2 + \dots + \eta_n^2 = 1.$$

Let $r_{ij} \geq 0$ be nonnegative numbers for $i < j$. Then we have

$$(4) \quad \sum_{i < j} (\eta_i - \eta_j)^2 r_{ij} \leq \sum_{i < j} r_{ij} + \text{Max}(r_{ij}).$$

Proof. We assume that $\eta_1 \geq \dots \geq \eta_n$. If $\eta_1 - \eta_n \leq 1$ or $n = 2$, then (4) is trivial. So we assume $n > 2$, and

$$\eta_1 - \eta_n > 1.$$

We observe that $\eta_i - \eta_j \leq 1$ for $2 \leq i < j \leq n-1$. Otherwise, we could have

$$1 \geq \eta_1^2 + \eta_n^2 + \eta_i^2 + \eta_j^2 > \frac{1}{2}((\eta_1 - \eta_n)^2 + (\eta_i - \eta_j)^2) > 1,$$

which is a contradiction.

Using the same reason, if $\eta_1 - \eta_{n-1} > 1$, then we have $\eta_2 - \eta_n \leq 1$; and if $\eta_2 - \eta_n > 1$, then we have $\eta_1 - \eta_{n-1} \leq 1$. Replacing η_1, \dots, η_n by $-\eta_n, \dots, -\eta_1$ if necessary, we can always assume that $\eta_2 - \eta_n \leq 1$. Thus $\eta_i - \eta_j \leq 1$ if $2 \leq i < j$, and (4) is implied by the following inequality

$$(5) \quad \sum_{1 < j} (\eta_1 - \eta_j)^2 r_{1j} \leq \sum_{1 < j} r_{1j} + \text{Max}_{1 < j} (r_{1j}).$$

Let $s_j = r_{1j}$ for $j = 2, \dots, n$. The the above inequality becomes

$$(6) \quad \sum_{1 < j} (\eta_1 - \eta_j)^2 s_j \leq \sum_{1 < j} s_j + \text{Max}_{1 < j} (s_j).$$

In order to prove the above inequality, we define the matrix P as follows

$$P = \begin{pmatrix} \sum_{1 < j} s_j & -s_2 & \cdots & -s_n \\ -s_2 & s_2 & & \\ \vdots & & \ddots & \\ -s_n & & & s_n \end{pmatrix}.$$

We claim that the maximum eigenvalue of A is no more than $r = \sum_j s_j + \text{Max}(s_j)$. To see this, we compute the determinant of the matrix

$$\begin{pmatrix} y - \sum_{1 < j} s_j & s_2 & \cdots & s_n \\ s_2 & y - s_2 & & \\ \vdots & & \ddots & \\ s_n & & & y - s_n \end{pmatrix}.$$

Using the Cramer's rule, the answer is

$$(y - s_2) \cdots (y - s_n) \left(y - \sum_{1 < j} s_j - \sum_{1 < j} \frac{s_j^2}{y - s_j} \right).$$

For any $y > r$, we have $y - s_j > \sum_{s=2}^n s_j$. Thus the above expression is greater than

$$(y - s_2) \cdots (y - s_n) \left(y - \sum_{1 < j} s_j - \left(\sum_{1 < j} s_j \right)^{-1} \sum_{1 < j} s_j^2 \right) > 0.$$

Let $\eta = (\eta_1, \dots, \eta_n)^T$, we then have

$$\sum_{1 < j} (\eta_1 - \eta_j)^2 s_j = \eta^T P \eta \leq r = \sum_{1 < j} s_j + \text{Max}(s_j).$$

□

Lemma 2. *Let A be an $n \times n$ diagonal matrix of norm 1. Let A_2, \dots, A_m be symmetric matrices such that*

- (1) $\langle A_\alpha, A_\beta \rangle = 0$ if $\alpha \neq \beta$;
- (2) $\|A_2\| \geq \cdots \geq \|A_m\|$.

Then we have

$$(7) \quad \sum_{\alpha=2}^m \|[A, A_\alpha]\|^2 \leq \sum_{\alpha=2}^m \|A_\alpha\|^2 + \|A_2\|^2.$$

Proof. Replacing each A_α with its off-diagonal component won't change the left hand side of the above inequality, but will decrease the right hand side of the above. Thus without loss of generality, we assume that each A_α has zero diagonal component. Furthermore, we assume that each A_α is not zero.

Let $A_\alpha = ((a_\alpha)_{ij})$, where $(a_\alpha)_{ij}$ are the entries for $\alpha = 2, \dots, m$. Let

$$\delta = \text{Max}_{i \neq j} \sum_{\alpha=2}^m (a_\alpha)_{ij}^2.$$

Let

$$A = \begin{pmatrix} \eta_1 & & \\ & \ddots & \\ & & \eta_m \end{pmatrix}.$$

Then by the previous lemma, we have

$$(8) \quad \sum_{\alpha=2}^m \|[A, A_\alpha]\|^2 \leq \sum_{\alpha=2}^m \|A_\alpha\|^2 + 2\delta.$$

Thus it remains to prove that

$$(9) \quad 2\delta \leq \|A_2\|^2.$$

To see this, we identify each A_α with the (column) vector \vec{A}_α in $\mathbb{R}^{\frac{1}{2}n(n+1)}$. Let μ_α be the norm of the vector. Then we have

$$(10) \quad \mu_\alpha^2 = \frac{1}{2} \|A_\alpha\|^2$$

for $\alpha = 2, \dots, m$. Extending the set of vectors $\{\vec{A}_\alpha/\mu_\alpha\}_{2 \leq \alpha \leq m}$ into an orthonormal basis of $\mathbb{R}^{\frac{1}{2}n(n+1)}$

$$\vec{A}_2/\mu_2, \dots, \vec{A}_m/\mu_m, \vec{A}_{m+1}, \dots, \vec{A}_{\frac{1}{2}n(n+1)+1},$$

we get an orthogonal matrix. Apparently, each row vector of the matrix is a unit vector. Thus we have

$$\sum_{\alpha=2}^m (\mu_\alpha)^{-2} (a_\alpha)_{ij}^2 \leq 1.$$

Since $\mu_2 \geq \dots \geq \mu_m$, we get

$$\sum_{\alpha=2}^m (a_\alpha)_{ij}^2 \leq \mu_2^2 \leq \frac{1}{2} \|A_2\|^2.$$

This proves (9). □

Remark 1. Let A be a diagonal matrix of unit norm and let B be a symmetric matrix. Let $\|B\|_\infty = \text{Max}(|b_{ij}|)$, where (b_{ij}) are the entries of B . By (8), we get

$$\|[A, B]\|^2 \leq \|B\|^2 + 2\|B\|_\infty^2.$$

Although not used directly in this paper, this is the crucial step of estimate that makes the whole proof work. Note that in [5] (or by $P(n, 2)$), we have a much weaker version of the above inequality

$$\|[A, B]\|^2 \leq 2\|B\|^2.$$

Proof of Conjecture 2. Let $a > 0$ be the largest positive real number such that

$$\left(\sum_{\alpha=1}^m \|A_\alpha\|^2\right)^2 \geq 2a \left(\sum_{\alpha < \beta} \|[A_\alpha, A_\beta]\|^2\right).$$

Since a is maximum, by the invariance, we can find matrices A_1, \dots, A_m such that

$$(11) \quad \left(\sum_{\alpha=1}^m \|A_\alpha\|^2\right)^2 = 2a \left(\sum_{\alpha < \beta} \|[A_\alpha, A_\beta]\|^2\right)$$

with the following additional properties:

- (1) A_1 is diagonal;
- (2) $\langle A_\alpha, A_\beta \rangle = 0$ if $\alpha \neq \beta$;
- (3) $\|A_1\| \geq \|A_2\| \geq \dots \geq \|A_m\|$.

We let $t^2 = \|A_1\|^2$ and let $A' = A_1/|t|$. Then (11) becomes a quadratic expression in terms of t^2 :

$$\begin{aligned} t^4 - 2t^2 \left(a \sum_{1 < \alpha} \|[A', A_\alpha]\|^2 - \sum_{1 < \alpha} \|A_\alpha\|^2 \right) + \left(\sum_{\alpha=2}^m \|A_\alpha\|^2 \right)^2 \\ - 2a \left(\sum_{1 < \alpha < \beta} \|[A_\alpha, A_\beta]\|^2 \right) = 0. \end{aligned}$$

Since the left hand side of the above is non-negative for all t^2 , we have

$$a \sum_{1 < \alpha} \|[A', A_\alpha]\|^2 - \sum_{1 < \alpha} \|A_\alpha\|^2 > 0,$$

and

$$\|A_1\|^2 = a \sum_{1 < \alpha} \|[A', A_\alpha]\|^2 - \sum_{1 < \alpha} \|A_\alpha\|^2.$$

By Lemma 2, we have

$$\sum_{1 < \alpha} \|[A', A_\alpha]\|^2 \leq \sum_{\alpha=2}^m \|A_\alpha\|^2 + \|A_2\|^2 \leq \sum_{\alpha=1}^m \|A_\alpha\|^2,$$

which proves that $a \geq 1$. □

Remark 2. In the study of random matrices, Böttcher and Wenzel [1] posed the following conjecture: let X, Y be $n \times n$ matrices. Then

$$\|[X, Y]\|^2 \leq 2\|X\|^2 \cdot \|Y\|^2.$$

Note that if one of the matrix is symmetric, then the result follows from Conjecture 2². The results of the above conjecture will appear in our next paper.

²which is certainly well known.

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