HOW BIG IS THE COMMUTATOR OF TWO MATRICES?

ZHIQIN LU

In the study of random matrices, Böttcher and Wenzel [1] posed the following conjecture:

Conjecture 1. *Let X, Y be two $n \times n$ matrices. Then*

$$||[X,Y]||^2 \leq 2||X||^2||Y||^2,$$

where the norm is defined as

$$||X||^2 = \sum_{i,j=1}^{n} x_{ij}^2.$$

Böttcher and Wenzel proved the following special cases of the conjecture: if $n = 2$, or X is of rank 1, or X is normal, then the conjecture is true. Furthermore, they proved the following weaker version of the conjecture:

$$||[X,Y]||^2 \leq 3||X||^2||Y||^2.$$

In this paper, we prove Conjecture 1.

We fix X and assume that $||X|| = 1$. Let $V = gl(n, \mathbb{R})$. Define a linear map

$$T : V \to V, \quad Y \mapsto [X^T, [X,Y]].$$

Then we have

Lemma 1. *T is a semi-positive definite symmetric linear transformation of V.*

Proof. This is a straightforward computation

$$\langle Y_1, [X^T, [X, Y_2]]\rangle = \langle [X, Y_1], [X, Y_2]\rangle = \langle [X^T, [X, Y_1]], Y_2\rangle.$$

Obviously T is semi-positive.

□

The conjecture is equivalent to the statement that the maximum eigenvalue of T is not more than 2.

We let α be the maximum eigenvalue of T. Then $\alpha > 0$. Let Y be an eigenvector of T with respect to α. Then we have

$$T(Y) = \alpha Y.$$

Date: September 14, 2007.

2000 *Mathematics Subject Classification.* Primary: 58C40; Secondary: 58E35.

Key words and phrases. BW Conjecture, commutator of matrices.

The author is partially supported by NSF Career award DMS-0347033 and the Alfred P. Sloan Research Fellowship.

A straightforward computation gives

$$T([X^T, Y^T]) = \alpha[X^T, Y^T],$$

where X^T is the transpose of X.

We claim that Y and $Y_1 = [X^T, Y^T]$ are linearly independent: first, $Y_1 \neq 0$, and second $\langle Y, Y_1\rangle = 0$. Thus, we have the following conclusion

Proposition 1. *The multiplicity of the eigenvalue α is at least 2.*

□

Let

$$X = Q_1 \Lambda Q_2$$

be the singular decomposition of X, where Q_1, Q_2 are orthogonal matrices and Λ is a diagonal matrix. Let

$$B = Q_2 Y Q_2^{-1}, \quad C = Q_1^{-1} Y Q_1.$$

Then we have

$$||[X, Y]||^2 = ||\Lambda B - C\Lambda||^2.$$

Let

$$\Lambda = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{pmatrix}.$$

Without loss of generality, we assume that $s_1 \geq \cdots \geq s_n$. Since $||X|| = 1$, we have

$$s_1^2 + \cdots + s_n^2 = 1.$$

Assume that $s_1^2 \leq 1/2$. Then we have

$$||\Lambda B - C\Lambda||^2 = \sum_{i,j=1}^n (s_i b_{ij} - s_j c_{ij})^2 \leq \sum_{i,j=1}^n 2(b_{ij}^2 + c_{ij}^2) s_1^2 \leq 2. \tag{1}$$

Thus in this case, the conjecture is trivially true. Now assume that $s_1^2 > 1/2$. By Proposition 1, we can find an eigenvector Y of T such that 1). $||Y|| = 1$, and 2). $b_{11} = 0$.

The conjecture can be proved if we can prove that

$$||[X, Y]||^2 \leq 2.$$

We first have the following equality (because $b_{11} = 0$)

$$||\Lambda B - C\Lambda||^2 = c_{11}^2 s_1^2 + \sum_{i=2}^n (s_i b_{i1} - s_1 c_{i1})^2 + \sum_{j=2}^n (s_1 b_{1j} - s_j c_{1j})^2 + \Delta_1,$$

where we define

$$\Delta = \sum_{i=2}^n b_{1i}^2 + \sum_{i=1}^n c_{i1}^2,$$

and

$$\Delta_1 = \sum_{i,j=2}^{n} (s_i b_{ij} - s_j c_{ij})^2.$$

Apparently we have

$$\Delta_1 \leq \sum_{i,j=2}^{n} (b_{ij}^2 + c_{ij}^2),$$

because $s_2^2 \leq 1/2$. Thus we just need to prove that

$$c_{11}^2 s_1^2 + \sum_{i=2}^{n} (s_i b_{i1} - s_1 c_{i1})^2 + \sum_{i=2}^{n} (s_1 b_{1i} - s_j c_{1i})^2 \leq \Delta + \sum_{i=2}^{n} b_{i1}^2 + \sum_{i=2}^{n} c_{1i}^2.$$

We consider the matrix

$$P = \begin{pmatrix} \Delta & -b_{12}c_{12} - b_{21}c_{21} & \cdots & -b_{1n}c_{1n} - b_{n1}c_{n1} \\ -b_{12}c_{12} - b_{21}c_{21} & b_{21}^2 + c_{12}^2 & & \\ \vdots & & \ddots & \\ -b_{1n}c_{1n} - b_{n1}c_{n1} & & & b_{n1}^2 + c_{1n}^2 \end{pmatrix}.$$

The above inequality is equivalent to that the maximum eigenvalue of the above matrix is no more than $\Delta + \sum_{i=2}^{n} b_{i1}^2 + \sum_{i=2}^{n} c_{1i}^2$. To see this, we let

$$y = \Delta + \sum_{i=2}^{n} b_{i1}^2 + \sum_{i=2}^{n} c_{1i}^2 + \varepsilon$$

for $\varepsilon > 0$. We have

$$\det(yI - P) = \prod_{i=2}^{n} (y - b_{i1}^2 - c_{1i}^2) \left(y - \Delta - \sum_{i=2}^{n} \frac{(b_{1i}c_{1i} + b_{i1}c_{i1})^2}{y - b_{i1}^2 - c_{1i}^2} \right).$$

Let

$$\beta = \max(b_{i1}^2 + c_{1i}^2).$$

Then we have

$$y - \Delta - \sum_{i=2}^{n} \frac{(b_{1i}c_{1i} + b_{i1}c_{i1})^2}{y - b_{i1}^2 - c_{1i}^2} \geq \beta + \varepsilon - \beta \sum_{i=2}^{n} \frac{b_{1i}^2 + c_{i1}^2}{\sum_{i=2}^{n} (b_{1i}^2 + c_{i1}^2)} > 0.$$

The conjecture is proved.

□

References

[1] A. Böttcher and D. Wenzel. How big can the commutator of two matrices be and how big is it typically? *Linear Algebra Appl.*, 403:216–228, 2005.

Department of Mathematics, University of California, Irvine, Irvine, CA 92697

E-mail address, Zhiqin Lu: zlu@math.uci.edu