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ON THE LOWER BOUND ESTIMATES OF SECTIONS OF THE CANONICAL BUNDLES OVER A RIEMANN SURFACE

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We give a lower bound estimate of the sum of the square norm of the sections of the pluricanonical bundles over a Riemann surface of genus greater than 2 and Gauss curvature -1. Such an estimate must depend on the injective radius of the Riemann surface. However, using this estimate, we give a uniform estimate of the corona problem on Riemann surface. Here "uniform" means that the estimate depends only on the genus of Rieman surface, not on the injective radius.

Keywords: Riemann surface; Teichmüller space; Collar theorem.

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1. Introduction

Suppose M is an n-dimensional Kähler manifold and L is an ample line bundle over M. Let the Kähler form of M be ω_g and a Hermitian metric of L be H. We assume that ω_g is the curvature of H, that is, $\omega_g = \mathrm{Ric}(H)$. The Kähler metric of ω_g is called a polarized Kähler metric on M.

Using H and ω_g, for any positive integer m, $H^0(M, L^m)$ becomes a Hermitian inner product space. We use the following notations: suppose that $S, T \in H^0(M, L^m)$. Let $\langle S, T\rangle_{H^m}$ be the pointwise inner product and

$$(S,T) = \int_M \langle S,T\rangle_{H^m} \frac{\omega_g^n}{n!}$$

be the inner product of $H^0(M, L^m)$. Let

$$\|S\| = \sqrt{\langle S,S\rangle_{H^m}}$$

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be the pointwise norm. In particular, $\|S\|(x)$ denotes the pointwise norm at $x \in M$. Let

$$\|S\|_{L^2} = \sqrt{(S,S)}$$

be the L^2-norm of S.

Let $\{S_1, \ldots, S_{d(m)}\}$ be an orthonormal basis of $H^0(M, L^m)$. The quantity (see [10])

$$\sum_{i=1}^{d(m)} \|S_i\|^2 \tag{1.1}$$

plays an important role in Kähler–Einstein geometry and stability of complex manifolds. A lot of work has been done by several authors [3, 8, 10, 12] and [7] on the estimates of (1.1). However, these works are concentrated on a single manifold. For a single manifold, we know from [12] that there is an asymptotic expansion

$$\sum_{i=1}^{d(m)} \|S_i\|^2 \sim m^n \left(a_0 + \frac{a_1}{m} + \frac{a_2}{m^2} + \cdots\right)$$

where n is the complex dimension of M. In [7], the author proved that the smooth functions $a_i (i = 1, 2, 3, \ldots)$ are polynomials of the curvature and its derivatives. In this point of view, we see that (1.1) splits into two parts. One part is the asymptotic expansion itself. The other part is the difference of the asymptotic expansion from (1.1). The second part may depend on the global geometry of the manifold.

In this paper, we shall study the behavior of (1.1) on Riemann surfaces with constant curvature -1. We established some kind of uniform lower bound estimate of (1.1) depending explicitly on the geometric quantities of the Riemann surfaces (Theorem 1.3). The motivation here is to get something controllable when the Riemann surfaces, considered as points on the Teichmüller space, go to infinity.

One application of Theorem 1.3 is as follows: for a smooth Riemann surface M of genus $g \geq 2$, it is well known that its coordinate ring is finitely generated. That is, there is a positive integer m_0, such that for any $S \in H^0(M, K_M^m)$ with $m > m_0$, we can find $U_i \in H^0(M, K_M^{m_0})$, $i \in I$ and $T_i \in H^0(M, K_M^{m-m_0})$, $i \in I$ such that

$$S = \sum_{i \in I} U_i T_i\,,$$

where I is a finite set. Let $S_i = U_i T_i$. Then $(S_i, i \in I)$ is a decomposition of S. In studying the behavior of Riemann surfaces near the boundary of the Teichmüller space, some uniform estimates of $(S_i, i \in I)$ are needed (because then we can take limit). In this paper, we obtain such uniform estimates by modifying Wolff's [11] proof of Carleson's theorem together with the $\bar{\partial}$-estimates.

The above setting is similar to that in the corona problem in complex analysis. The corona problem on the unit disk was studied by Carleson in [2]. Carleson's result stimulates many ideas which proved to be useful for other problems. An extensive discussion of the Carleson's corona theorem can be found in [5].

Our work is also related to the work of Brüning and Lesch [1], where they gave the asymptotic expansion of the heat kernel of the Laplacian on functions for singular Riemann surfaces and showed that information on the singularities can be found in the expansion. If we consider the Laplacian on the canonical bundle K_M on a Riemann surface M, then the Laplacian can be naturally extended to K_M^m for any positive integer m. Let $k_m(t)$ be the heat kernel of the Laplacian on K_M^m, then (1.1) is the limit when $t \to +\infty$ while the Brüning–Lesch type expansion is valid when t is small. Thus it is interesting to study the asymptotic expansion of (1.1) when M is singular and to hope that the information of the singularities is reflected in the expansion. We will give the result in this direction in the next paper.

The organization of the paper is as follows: in Sec. 2, we give a lower bound of (1.1) in terms of the genus g and the injective radius δ of M. In Sec. 3, we give a counterexample which shows that the lower bound must depend on δ. In Sec. 4, we give the partial uniform estimate. That is, a lower bound of (1.1) at $x \in M$ depending only on the injective radius δ_x of x. In Sec. 5, we solve the uniform corona problem by the partial uniform estimate.

The main results of this paper are the following:

Theorem 1.1. *Let M be a Riemann surface of the genus $g \geq 2$. Let K_M be the canonical line bundle of M endowed with a Hermitian metric H. Let the curvature ω_g of H be positive. ω_g gives a Kähler metric of M. Let the curvature K of ω_g satisfy*

$$-K_1 \leq K \leq K_2$$

for nonnegative constants K_1, $K_2 \geq 0$ and let δ' be the injective radius of M. Let

$$\delta = \min\left(\delta', \frac{1}{\sqrt{K_1 + K_2}}\right).$$

Then there is an absolute constant $C > 0$ such that for $m \geq 2$,

$$\sum_{i=1}^{d(m)} \|S_i\|^2 \geq e^{-\frac{Cg^3}{\delta^6}},$$

where $\{S_1, \ldots, S_{d(m)}\}$ is an orthonormal basis of $H^0(M, K_M^m)$.

It is clear that when the injective radius δ goes to zero, the above estimate becomes useless. However, by the following theorem, we know that we cannot expect a uniform lower bound for all Riemann surfaces.

Theorem 1.2. *For any $\varepsilon > 0$ and $m \geq 2$, there is a Riemann surface M of genus $g \geq 2$ with the constant Gauss curvature -1 such that*

$$\inf_{x \in M} \sum_{i=1}^{d(m)} \|S_i\|^2 \leq \varepsilon.$$

In the following theorem, we prove that (1.1) has a lower bound which depends only on the local information and is independent of the injective radius of M. For this reason, we call the result partial uniform estimate.

Theorem 1.3. *Let M be a Riemann surface of genus $g \geq 2$ and constant curvature -1. Then there are absolute constants $m_0 > 0$ and $D > 0$ such that for any $m > m_0$ and any $x_0 \in M$, there is a section $S \in H^0(M, K_M^m)$ with $\|S\|_{L^2} = 1$ such that*

$$\|S\|(x_0) \geq \frac{\sqrt{m}}{D\left(1 + \frac{1}{\sqrt{m}\delta_{x_0}^2} e^{\frac{\pi}{\delta_{x_0}}}\right)}, \tag{1.2}$$

where δ_{x_0} is the injective radius of x_0.

Using the above result, we solve the uniform corona problem on Riemann surfaces:

Theorem 1.4. *Let M be a Riemann surface as above. Then there is an $m_0 > 0$ such that for any $m > m_0$ and $S \in H^0(M, K_M^m)$, there is a decomposition*

$$S = \sum_{i=1}^{d} S_i$$

of $S_i \in H^0(M, K_M^m)(i = 1, \ldots, d)$ such that

$$\begin{aligned} \|S_i\|_{L^2} &\leq C(m, m_0, g)\|S\|_{L^2}\,, \\ \|S_i\|_{L^\infty} &\leq C(m, m_0, g)\|S\|_{L^\infty}\,, \end{aligned} \tag{1.3}$$

for $i = 1, \ldots, d$, and

$$S_i = T_i U_i$$

for a basis $U_1, \ldots, U_d$ of $H^0(M, K_M^{m_0})$ and $T_1, \ldots, T_d \in H^0(M, K_M^{m-m_0})$.

2. A Lower Bound Estimate

Suppose that M is a Riemann surface of genus $g \geq 2$. Let K_M be the canonical line bundle over M with a Hermitian metric H. We assume that the curvature ω_g of H is positive and defines a Kähler metric of M.

Let K be the Gauss curvature of the metric ω_g. Let K_1 and K_2 be two non-negative constants such that

$$-K_1 \leq K \leq K_2\,. \tag{2.1}$$

Let δ' be the injective radius of M with and

$$\delta = \min\left(\delta', \frac{1}{\sqrt{K_1 + K_2}}\right). \tag{2.2}$$

Let $x_0 \in M$ be a fixed point. Let U be the open set

$$U = \{\mathrm{dist}(x, x_0) < \delta\}\,.$$

It is well known that at each point of U there is an isothermal coordinate. In the first part of this section, we prove that there is a holomorphic function z on U which gives the isothermal coordinate of U with the *required* estimate. Consider the equation

$$\begin{cases} \Delta h = \dfrac{K}{2}\,, \\ h|_{\partial U} = 0\,, \end{cases} \tag{2.3}$$

where Δ is the (complex) Laplacian of M. A solution h exists and is unique. Let ω_g be the Riemann metric of U. Then we have

Lemma 2.1. *The metric $e^h ds^2$ on U is a flat metric.*

Proof. A straightforward computation using (2.3). □

Since U is an open set which is differmorphic to an open set in the Euclidean plane, we can assume that there are global frames on U. Let ω^1 and ω^2 be 1-forms on U such that

$$e^h ds^2 = \omega_1^2 + \omega_2^2\,.$$

Let ω_{12} be the connection 1-form defined by

$$\begin{aligned} d\omega_1 &= \omega_{12} \wedge \omega_2\,, \\ d\omega_2 &= -\omega_{12} \wedge \omega_1\,. \end{aligned} \tag{2.4}$$

Then by Lemma 2.1, $d\omega_{12} = 0$. It follows that there is a real smooth function σ on U such that

$$\omega_{12} = d\sigma\,. \tag{2.5}$$

Let

$$\xi = e^{i\sigma}(\omega_1 + i\omega_2)\,.$$

Then by (2.4) and (2.5), we have

$$d\xi = 0\,.$$

Thus there is a function z on U such that

$$\xi = dz\,,$$

and

$$e^h ds^2 = dz d\bar{z}\,. \tag{2.6}$$

Either z or $\bar{z}$ will be holomorphic because it defines a conformal structure of U. Without losing generality, we assume that z is holomorphic and at x_0, $z = 0$. We have the following lemma:

Lemma 2.2. *Let ρ be the distance to the point x_0. $\rho(x) = \mathrm{dist}(x, x_0)$. Then*

$$\frac{1}{3}\rho \le |z| \le 3\rho \tag{2.7}$$

for $\rho < \delta$.

Proof. By the Gauss Lemma [4, p. 8], the Riemann metric ds^2 can be written as

$$ds^2 = d\rho^2 + f^2(\rho,\theta)d\theta^2$$

for the polar coordinate (ρ, θ) where $f(\rho,\theta)$ is a smooth function satisfying

$$f(0,\theta) = 0\,, \quad \frac{\partial f}{\partial \rho}(0,\theta) = 1\,,$$

and

$$\frac{\partial^2 f}{\partial \rho^2} = -Kf\,.$$

By the Hessian comparison theorem [9, p. 4], we have

$$\Delta\rho \ge \frac{\sqrt{K_2}}{4}\cot\sqrt{K_2}\rho\,.$$

In particular, $\Delta\rho \ge 0$ on U. Noting that Δ is the complex Laplacian, we have

$$\Delta\rho^2 = \frac{1}{2}|\nabla\rho|^2 + 2\rho\Delta\rho \ge \frac{1}{2}\,. \tag{2.8}$$

By (2.8), we have

$$\Delta(h + K_1\rho^2) \ge \frac{K}{2} + \frac{K_1}{2} \ge 0\,,$$

$$\Delta(h - K_2\rho^2) \le \frac{K}{2} - \frac{K_2}{2} \le 0\,.$$

By the maximal principle, we have

$$-1 \le -K_2\delta^2 \le h|_{\partial U} - K_2\rho^2 \le h \le h|_{\partial U} + K_1\rho^2 \le K_1\delta^2 \le 1\,. \tag{2.9}$$

Let $ds_1^2 = e^h ds^2$ denotes the flat metric. Then

$$e^{-1}ds^2 \le ds_1^2 \le e ds^2\,.$$

By (2.6), $|z|$ is the distance to the point x_0 with respect to the metric ds_1^2. Thus by (2.9),

$$\frac{1}{3}\rho \le e^{-1}\rho \le |z| \le e\rho \le 3\rho\,. \qquad \square$$

Proof of Theorem 1.1. Define a smooth function $\eta : \mathbb{R}^+ \to \mathbb{R}$ such that

$$\eta(t) = \begin{cases} 0\,, & t \ge 1\,, \\ 1\,, & 0 \le t \le \frac{1}{2}\,. \end{cases} \tag{2.10}$$

We assume that $|\eta'| \le 4$ and $|\eta''| \le 4$.

In the rest of this paper $C_1, C_2, \ldots,$ are absolute constants, unless otherwise stated.

Let $\delta_1 = \frac{1}{4}\delta$. Define the smooth function r on M by setting

$$r = \begin{cases} \eta\left(\frac{|z|}{\delta_1}\right)\log\left(\frac{|z|}{\delta_1}\right) & x \in U\,, \\ 0 & x \notin U\,. \end{cases} \tag{2.11}$$

r is well defined. For if $x \in \partial U$, then $\rho = \delta$. By the Lemma 2.2, $|z| \geq \frac{4}{3}\delta_1$ and thus $r|_{\partial U} = 0$ using either expression. □

Note that if $z \neq 0$, then $\Delta \log(|z|) = 0$. Define the function ψ by setting $\psi = \Delta r$ for $z \neq 0$ and $\psi = 0$ for $z = 0$. We have

Lemma 2.3. *There is a constant $C_1 > 0$ such that*

$$\begin{cases} |\psi| \leq \dfrac{C_1}{\delta^2}\,, \\ \displaystyle\int_M |\psi| \leq C_1\,. \end{cases} \tag{2.12}$$

Proof. A straightforward computation gives

$$\psi = \Delta r = \frac{1}{4}e^h\left(\frac{1}{\delta_1^2}\eta''\log\left(\frac{|z|}{\delta_1}\right) + \eta'\frac{1}{\delta_1|z|}\log\left(\frac{|z|}{\delta_1}\right) + 2\eta'\frac{1}{\delta_1|z|}\right) \tag{2.13}$$

for $\frac{\delta_1}{2} < z < \delta_1$. Using (2.9) and (2.10), we have the estimate

$$|\psi| \leq \frac{C_2}{\delta^2}$$

for some constant C_2. To get the estimate of the $\int_M |\psi|$, we first see that by the volume comparison theorem [9, p. 11],

$$\mathrm{vol}(U) \leq 2\pi\left(\frac{\cosh\sqrt{K_1}\delta - 1}{K_1}\right).$$

Since $\sqrt{K_1}\delta \leq 1$, there is a constant C_3 such that

$$\mathrm{vol(U)} \leq C_3\delta^2\,. \tag{2.14}$$

The lemma follows with $C_1 = \max(C_2, C_2C_3)$. □

Let $G(x,y)$ be the Green's function of M. That is,

$$\begin{cases} \Delta_x G(x,y) = \dfrac{1}{4}\left(-\delta_x(y) + \dfrac{1}{\mathrm{vol}(M)}\right), \\ \displaystyle\int_M G(x,y)dx = 0\,, \end{cases}$$

where Δ_x is the (complex) Laplacian with respect to $x$and $\delta_x(\cdot)$ is the Dirac function. Let b be a function on M such that

$$\begin{cases} \Delta b = \dfrac{K}{4} + \dfrac{1}{2}\,, \\ \displaystyle\int_M b = 0\,. \end{cases}$$

Since the Kähler metric $\omega_g \in -c_1(M)$, the above equation has a unique solution.

Let the function $a : M \to \mathbb{R}$ be defined by

$$a = G(x, x_0) + \frac{1}{2\pi}\left(r - \frac{1}{\mathrm{vol}(M)}\int_M r\right) + b\,, \tag{2.15}$$

where x_0 is the fixed point of M and r is defined in (2.13). Then a is a smooth function on M. We have

$$\begin{cases} \Delta a = \dfrac{1}{4\,\mathrm{vol}(M)} + \dfrac{1}{2\pi}\psi + \dfrac{K}{4} + \dfrac{1}{2}\,, \\ \displaystyle\int_M a = 0\,. \end{cases} \tag{2.16}$$

Lemma 2.4. *There is a constant C_5 such that*

$$|a| \le \frac{C_5 g^3}{\delta^6}\,,$$

where g is the genus of the Riemann surface M.

Proof. Let λ_1 be the first eigenvalue of M, then by the Poincaré inequality, we have

$$\lambda_1 \int_M a^2 \le \int_M |\nabla a|^2\,. \tag{2.17}$$

Integration by parts using (2.16), we have

$$\int_M |\nabla a|^2 \le \int_M \left| a\left(\frac{1}{2\pi}\psi + \frac{1}{4\,\mathrm{vol}(M)} + \frac{K_1 + K_2}{4} + \frac{1}{2}\right)\right|. \tag{2.18}$$

Let g be the genus of M. By the Gauss–Bonnet theorem, $\mathrm{vol}(M) = 4\pi(g-1)$. On the other hand, $K_1 + K_2 \le \frac{1}{\delta^2}$. Let $a(x') = \sup|a|$ where x' is the point such that a reaches maximum. By (2.17), (2.18) and Lemma 2.3,

$$\int_M a^2 \le \frac{(C_1 + 6\pi)g}{\lambda_1 \delta^2} a(x')\,. \tag{2.19}$$

Consider a neighborhood U' of x' defined by

$$U' = \{x | \mathrm{dist}(x, x') < \delta\}\,.$$

Let z be the holomorphic function in Lemma 2.2 such that $z(x') = 0$. Let

$$U_1 = \left\{|z| < \frac{1}{3}\delta\right\}.$$

Let $\tilde{\Delta} = \frac{\partial^2}{\partial z \partial \bar{z}}$ be the Euclidean Laplacianon U_1. Then by (2.9), (2.12) and (2.16), we have

$$|\tilde{\Delta} a| \leq 3 \left(2 + \frac{C_1 + 1}{\delta^2} \right). \tag{2.20}$$

It follows from an the Poisson formula fact that there is a constant C_4 such that

$$a(x') \leq C_4 \left(\log \frac{1}{\delta} + \frac{1}{\delta} \left(\int_{U_1} a^2(x) \right)^{\frac{1}{2}} \right). \tag{2.21}$$

On the other hand, Cheeger's inequality [9, p. 91] gives

$$\lambda_1 \geq \frac{1}{4g^2} \delta^2 . \tag{2.22}$$

Combining (2.19), (2.21) and (2.22), we have the required estimate. □

Let

$$\varphi = -4\pi (G(x, x_0) + b) . \tag{2.23}$$

Then for $x \neq x_0$, we have

$$\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \varphi \geq \left(-\frac{1}{2 \operatorname{vol}(M)} - \frac{K}{2} - 1 \right) \omega_g . \tag{2.24}$$

Lemma 2.5. *There is a constant C_6 such that*

$$\varphi \leq \frac{C_6 g^3}{\delta^6} ,$$

and for $|z| \leq \delta_1$,

$$\varphi \geq -\frac{C_6 g^3}{\delta^6} + 2 \log |z| .$$

Proof. By (2.15),

$$\varphi = -4\pi \left(a - \frac{1}{2\pi} \left(r - \frac{1}{\operatorname{vol}(M)} \int_M r \right) \right) .$$

The lemma follows from Lemma 2.4 and (2.11). □

We need the following proposition from Demailly (see [10]):

Proposition 2.1. *Suppose that (M, g) is a complete Kähler manifold of complex dimension n, L is a line bundle on M with the Hermitian metric h, and φ is a function on M, which can be approximated by a decreasing sequence of smooth functions $\{\varphi_l\}_{1 \leq l < +\infty}$. If*

$$\left\langle \partial \bar{\partial} \varphi_l + \frac{2\pi}{\sqrt{-1}} (\operatorname{Ric}(h) + \operatorname{Ric}(g)),\ v \wedge \bar{v} \right\rangle_g \geq C \|v\|_g^2$$

for any tangent vector v of type $(1,0)$ at any point of M and for each l, where $C>0$ is a constant independent of l, and $\langle\cdot,\cdot\rangle_g$ is the inner product induced by g, then for any C^∞ L-valued $(0,1)$-form u_1 on M with $\bar\partial u_1=0$ and $\int_M \|u_1\|^2 e^{-\varphi} dV_g$ finite, there exists a C^∞ L-valued function u on M such that $\bar\partial u = u_1$ and

$$\int_M \|u\|^2 e^{-\varphi} dV_g \le \frac{1}{C}\int_M \|u_1\|^2 e^{-\varphi} dV_g\,,$$

where dV_g is the volume form g and the norm $\|\cdot\|$ is induced by h. The function φ is called the weight function.

Let $L=K_M^m$ for $m\ge 2$. H^m gives a positive Hermitian metric on K_M^m. Let ω_g be the Kähler form defined by the curvature of H on K_M. Let $\varphi_l=\max(\varphi,-l)$ for $l\in\mathbb{Z}^+$, where φ is defined in (2.23). Then by (2.24), we have

$$\left\langle \partial\bar\partial\varphi_l + \frac{2\pi}{\sqrt{-1}}(\mathrm{Ric}(H)+\mathrm{Ric}(\omega_g)),\ v\wedge\bar v\right\rangle \ge \left(m-1-\frac{1}{2\ \mathrm{vol}(M)}\right)\|v\|^2\,. \quad (2.25)$$

In order to prove Theorem 1.1, we need to proved that for any $m\ge 2$ and $x_0\in M$, there is a section $S\in H^0(M,K_M^m)$ such that

$$\frac{\|S\|^2(x_0)}{\|S\|^2_{L^2}} \ge e^{-\frac{Cg^3}{\delta^6}}\,.$$

We will use Proposition 2.1 to construct such a section.

Let e^p be the local representation of the metric H. That is,

$$-\frac{\sqrt{-1}}{2\pi}\partial\bar\partial p = \omega_g\,.$$

Let

$$u_1 = \bar\partial_z \eta\left(\frac{\sqrt{m}|z|}{\delta}\right) e^{-m\frac{\partial p}{\partial \bar z}(x_0)\cdot z}(dz)^m$$

where η is a cut-off function. Then $u_1\in\Gamma(M,K_M^m)$. By (2.6), (2.7) and (2.10), we have

$$\|u_1\|^2 \le \frac{12m}{\delta^2} e^{m(p-2\ \mathrm{Re}\frac{\partial p}{\partial\bar z}(x_0)z)} \quad (2.26)$$

for $\frac{\delta}{2\sqrt{m}}\le|z|\le\frac{\delta}{\sqrt{m}}$. Let $U_1=\{x \| z| < \frac{1}{3}\delta\}$. Let $\tilde\Delta$ be the Euclidean Laplacian. By (2.9), we see that

$$|\tilde\Delta p|\le 3$$

on U_1. Using the Poisson formula we see that there is a constant C_7 such that

$$|\tilde\nabla^2 p(z)| \le \frac{C_7}{\delta^2}$$

for $|z|\le\frac{1}{2}\delta$. Thus

$$|p-2\ \mathrm{Re}\frac{\partial p}{\partial\bar z}(x_0)z - p(x_0)| \le \frac{C_7}{m}$$

for $|z| < \frac{\delta}{m}$. Using this estimate and (2.26), we have

$$\|u_1\|^2 \leq \frac{12m}{\delta^2} e^{C_7} e^{mp(x_0)}$$

for $\frac{\delta}{2\sqrt{m}} \leq |z| \leq \frac{\delta}{\sqrt{m}}$.Thus by Lemma 2.5 and (2.14),

$$\int_M \|u_1\|^2 e^{-\varphi} = \int_{\frac{\delta}{2\sqrt{m}} \leq |z| \leq \frac{\delta}{\sqrt{m}}} \|u_1\|^2 e^{-\varphi} \leq \frac{48C_3 m}{\delta^2} e^{C_7} e^{\frac{C_6 g^3}{\delta^6}} e^{mp(x_0)} .$$

By Proposition 2.1 and (2.25), there is a $u \in \Gamma(M, K_M^m)$ such that $\bar{\partial} u = u_1$ and,

$$\int_M \|u\|^2 e^{-\varphi} \leq \frac{1}{m - 1 - \frac{1}{2\ \mathrm{vol}(M)}} \int_M \|u_1\|^2 e^{-\varphi} .$$

Using Lemma 2.5 again, for $m \geq 2$, there is a C_8 such that

$$\int_M \|u\|^2 \leq \frac{C_8}{\delta^2} e^{\frac{2C_6 g^3}{\delta^6}} e^{mp(x_0)} . \tag{2.27}$$

On the other hand, we have

$$\int_M \left\| \rho\left(\frac{m|z|}{\delta}\right) e^{-m\frac{\partial p}{\partial \bar{z}_0}(x_0) z} (dz)^m \right\|^2$$

$$\leq e^{C_7} \int_{|z| \leq \frac{\delta}{\sqrt{m}}} e^{mp(x_0)} \leq e^{mp(x_0)} C_3 e^{C_7} \frac{\delta^2}{m} . \tag{2.28}$$

Let $S = \eta(\frac{|z|}{\delta_1}) e^{-m\frac{\partial p}{\partial \bar{z}}(x_0) z} (dz)^m - u$. Then $\bar{\partial} S = 0$. Since $\int_M e^{-\varphi} = +\infty$, $u(x_0) = 0$. In particular, $S \neq 0$. Using (2.27) and (2.28), we have

$$\frac{\|S\|^2(x_0)}{\|S\|_{L^2}^2} \geq \frac{1}{\left(2C_3 e^{C_7} \frac{\delta^2}{m} + 2\frac{C_8}{\delta^2} e^{\frac{2C_6 g^3}{\delta^6}}\right)} .$$

Thus for $m \geq 2$, there is a C such that

$$\frac{\|S\|}{\|S\|_{L^2}} \geq e^{-\frac{Cg^3}{\delta^6}} .$$

This completes the proof of Theorem 1.1.

3. A Counterexample

In the last section, we gave a lower bound estimate of (1.1) in terms of the injective radius of M. In this section, we give a counterexample that the uniform estimate is not true. More precisely, we are going to disapprove the following:

Conjecture 3.1. Let K_M be the canonical line bundle of a Riemann surface M of genus $g \geq 2$ and constant Gauss curvature -1. Then for m sufficiently large, there is a number $C(m, g) > 0$, depending only on m and g, such that for any orthonormal basis $S_1, \ldots, S_d$ of $H^0(M, K_M^m)$, we have

$$\inf \sum_{i=1}^{d} \|S_i\|^2 \geq C(m, g) .$$

In order to give the counterexample, we use the following Collar theorem of Keen [6, p. 264]:

Theorem 3.1 (Keen). *Consider the region T of U, the upper half plane, bounded by the curve $r = 1$, $r = e^l$, $\theta = \theta_0$ and $\theta = \pi - \theta_0$. Let γ be a closed geodesic on M with length l. Then there is a conformal isometric mapping $\varphi : T \to M$ such that $\varphi(iy) = \gamma$. The image $\varphi(T)$ of T is called a collar. Then we can choose θ_0 small enough such that the area of the collar is at least $\frac{8}{\sqrt{5}}$.*

The following theorem gives the counterexample and implies Theorem 1.2:

Theorem 3.2. *For any $\varepsilon > 0$ and $m \geq 2$, there is a Riemann surface M of constant curvature -1 and genus $g \geq 2$ such that there is a point $x_0 \in M$ satisfying*

$$\|S\|(x_0) \leq \varepsilon$$

for any $S \in H^0(M, K_M^m)$ with $\|S\|_{L^2} = 1$.

The idea of the proof is that when the length of a closed geodesic line tends to zero, the collar must become longer and longer in order to have its area bounded below. Topologically, a collar is a cylinder. By expanding the functions on the collar using the Fourier series, we can find the suitable x_0 and the estimates. We begin by discussing some elementary properties of a collar.

Let $R > 0$ be a large real number. Let $(\rho, \theta) \in (-R, R) \times \mathbb{R}$. Let the group $\mathbb{Z}$ act on the space $(-R, R) \times \mathbb{R}$ by

$$(n, \rho, \theta) \mapsto (\rho, \theta + n\delta)$$

for $n \in \mathbb{Z}$, where $\delta > 0$ satisfies

$$\delta \sinh R = \varepsilon_1 \left(= \frac{8}{\sqrt{5}} \right)$$

as in Theorem 3.1. Define the metric

$$ds^2 = d\rho^2 + (\cosh \rho)^2 d\theta^2$$

on $(-R, R) \times \mathbb{R}$ which descends to a metric on

$$C = (-R, R) \times \frac{\mathbb{R}}{\mathbb{Z}}.$$

The curvature of the metric is -1. Note that on C, ρ is a global function but θ is only locally defined.

We call C a collar of parameter δ.

Let

$$\begin{cases} x = \theta\,, \\ y = 2\ \arctan e^{\rho} - \dfrac{\pi}{2}\,. \end{cases} \tag{3.1}$$

Define $z = x + iy$. Clearly z is not a global function of C. But it defines a complex structure of C.

Let

$$w = e^{\frac{2\pi i}{\delta}(\theta + 2i(\arctan e^{\rho} - \frac{\pi}{4}))} = e^{\frac{2\pi i}{\delta} z}\,. \tag{3.2}$$

Then w is a global holomorphic function on C. Consequently

$$dz = \frac{\delta}{2\pi i}\frac{dw}{w} \tag{3.3}$$

is a global holomorphic 1-form on C.

Let f be a holomorphic function on a neighborhood of $\bar{C}$. Then f is a periodic function on $[-R, R] \times \mathbb{R}$, satisfying

$$f(\rho, \theta \pm \delta) = f(\rho, \theta)\,.$$

Let the Fourier expansion of $f(-R, \theta)$ and $f(R, \theta)$ be

$$f(-R, \theta) = \sum_{k=-\infty}^{+\infty} A_k e^{\frac{2\pi i}{\delta} k\theta}\,;$$

$$f(R, \theta) = \sum_{k=-\infty}^{+\infty} B_k e^{\frac{2\pi i}{\delta} k\theta}\,.$$

Define

$$\begin{aligned} g_1 &= \sum_{k=1}^{\infty} A_k e^{-\frac{4\pi}{\delta} k(\frac{\pi}{4} - \arctan e^{-R})} w^k\,, \\ g_2 &= B_0\,, \\ g_3 &= \sum_{k=-\infty}^{-1} B_k e^{-\frac{4\pi}{\delta} k(\frac{\pi}{4} - \arctan e^{R})} w^k\,, \end{aligned} \tag{3.4}$$

where w is in (3.2). We have the following lemma:

Lemma 3.1. *With the notations as above,* g_1, g_2, g_3 *are holomorphic functions on* C. *Furthermore*

$$f = g_1 + g_2 + g_3\,.$$

Proof. g_2 is a constant. So it is automatically holomorphic. By Eq. (3.2), we have

$$|w| \le e^{-\frac{4\pi}{\delta}(\arctan e^{\rho} - \frac{\pi}{4})}\,.$$

Thus we have

$$
\begin{aligned}
|g_1| &= \left|\sum_{k=1}^{\infty} A_k e^{-\frac{4\pi}{\delta}k(\frac{\pi}{4}-\arctan e^{-R})} w^k\right| \\
&\leq \sum_{k=1}^{\infty} |A_k| e^{-\frac{4\pi}{\delta}k(\arctan e^{\rho}-\arctan e^{-R})} \\
&\leq \left(\sum_{k=1}^{\infty} |A_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} e^{-\frac{8\pi}{\delta}k(\arctan e^{\rho}-\arctan e^{-R})}\right)^{\frac{1}{2}} .
\end{aligned}
$$

By the Bessel inequality, we have

$$
\sum_{k=1}^{+\infty} |A_k|^2 \leq \frac{1}{\delta}\int_M |f(-R,\theta)|^2 d\theta .
$$

Thus if $\rho > -R$, the series is absolutedly convergent. So g_1 defines a holomorphic function on $\{-R < \rho < R\}$.

By the same argument, g_3 is also holomorphic. □

In order to prove that

$$
f = g_1 + g_2 + g_3 ,
$$

we just need to prove that on the set $\{\rho = 0\}$, $f = g_1 + g_2 + g_3$. Define

$$
p_k(\rho) = \int_0^{\delta} f(\rho,\theta) e^{-\frac{2\pi i}{\delta}\theta k} d\theta \tag{3.5}
$$

for $k \in \mathbb{Z}$. Apparently

$$
p_k(R) = B_k\delta , \quad p_k(-R) = A_k\delta
$$

for $k \in \mathbb{Z}$. By the definition of z in (3.1), we have

$$
\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\sqrt{-1}\cosh\rho\frac{\partial}{\partial\rho} + \frac{\partial}{\partial\theta}\right) .
$$

Using the equation $\frac{\partial f}{\partial \bar{z}} = 0$, from (3.5), we have

$$
p_k' = -\frac{2\pi k}{\delta\cosh\rho} p_k .
$$

Solving the above differential equation gives

$$
\begin{aligned}
& p_k(0) = A_k e^{-\frac{4\pi}{\delta}k(\frac{\pi}{4}-\arctan e^{-R})} , \quad k = 1, 2, \dots , \\
& p_0(0) = B_0 , \\
& p_k(0) = B_k e^{-\frac{4\pi}{\delta}k(\frac{\pi}{4}-\arctan e^{R})} , \quad k = -1, -2, \dots .
\end{aligned} \tag{3.6}
$$

From (3.4) and (3.6), we see that

$$
f|_{\rho=0} = (g_1 + g_2 + g_3)|_{\rho=0} .
$$

Thus

$$f = g_1 + g_2 + g_3$$

on C because both sides are holomorphic functions. □

Let M be a Riemann surface of curvature -1 and genus $g \geq 2$. Assume that there is a closed geodesic γ on M such that length$(\gamma) = \delta > 0$. Assume that δ is small enough. Let θ be the arc length parameter and ρ be the distance function to the geodesic. Then

Lemma 3.2. *(ρ, θ) is the local coordinate system of M as long as*

$$\sinh \rho \cdot \delta \leq \varepsilon_1 = \frac{8}{\sqrt{5}}\,.$$

Proof. Note that the area of $\{-R_0 < \rho < R_0\}$ is $\delta \sinh R_0$ for any $R_0 > 0$. The lemma follows from Theorem 3.1. □

Let $C = \{-R < \rho < R\}$, where R satisfies $\delta \sinh R = \varepsilon_1$. Then $z = x + iy$ defines a complex structure of C where x and y are in (3.1). We have

Lemma 3.3. *Either $z = x + iy$ or $z = x - iy$ is holomorphic on M.*

Proof. A straightforward computation gives

$$ds^2 = (\cosh \rho)^2 dz d\bar{z}\,.$$

Thus z defines a conformal structure which is the same as the one on M. So either z or $\bar{z}$ is holomorphic. □

Without losing generality, we assume that z is holomorphic. Fixing $m \geq 2$. Let $S \in H^0(M, K_M^m)$. We choose an $x_0 \in M$ as follows: let $\rho_0 < 0$ be the number such that

$$\delta(\cosh \rho_0)^2 = \varepsilon_1\,.$$

Then for such a choice of ρ_0 we have $\rho_0 \to -\infty$, $\rho_0 + R \to +\infty$ as $\delta \to 0$. By (3.3), we see that $(dz)^m \in H^0(C, K_M^m)$. Furthermore $(dz)^m \neq 0$ on C. Thus for any $S \in H^0(M, K_M^m)$, there is a holomorphic function f on C such that

$$S|_C = f(dz)^m\,.$$

Let

$$f = g_1 + g_2 + g_3\,,$$

where g_1, g_2, g_3 are defined in Lemma 3.1. Let

$$S_i = g_i(dz)^m\,, \quad i = 1, 2, 3\,. \tag{3.7}$$

Lemma 3.4. *With the notations as above, let* $x_0 = (\rho_0, 0)$. *Then*

$$\lim_{\delta\to 0} \frac{\|S_i\|^2(x_0)}{\|S_i\|^2_{L^2(C)}} = 0$$

for $i = 1, 2, 3$.

Proof. By (3.2), we have

$$w(x_0) = e^{-\frac{4\pi}{\delta}(\arctan e^{\rho_0} - \frac{\pi}{4})}.$$

Thus

$$\|S_1\|^2(x_0) = \frac{1}{(\cosh\rho_0)^{2m}} \left| \sum_{k=1}^{\infty} A_k e^{-\frac{4\pi}{\delta}k(\arctan e^{\rho_0} - \arctan e^{-R}) + \frac{2\pi i}{\delta}k\theta} \right|^2 \leq \frac{1}{(\cosh\rho_0)^{2m}}$$
$$\cdot \sum_{k=1}^{\infty} |A_k|^2 e^{-\frac{8\pi}{\delta}k(\arctan e^{\rho_0-1} - \arctan e^{-R})} \cdot \sum_{k=1}^{\infty} e^{-\frac{8\pi}{\delta}k(\arctan e^{\rho_0} - \arctan e^{\rho_0-1})}$$
$$\leq \sum_{k=1}^{\infty} |A_k|^2 e^{-\frac{8\pi}{\delta}k(\arctan e^{\rho_0-1} - \arctan e^{-R})} \cdot \sum_{k=1}^{\infty} e^{-\frac{4\pi k}{\delta\cosh(\rho_0-1)}}. \tag{3.8}$$

We assume that δ is so small that

$$\frac{4\pi}{\delta\cosh(\rho_0-1)} \geq \frac{\mu}{\sqrt{\delta}} > 0,$$

where μ is an absolute constant. In addition, assume that $e^{-\frac{\mu}{\sqrt{\delta}}} < \frac{1}{2}$. Then

$$\sum_{k=1}^{\infty} e^{-\frac{4\pi k}{\delta\cosh(\rho_0-1)}} \leq \sum_{k=1}^{\infty} e^{-\frac{\mu k}{\sqrt{\delta}}} \leq 2e^{-\frac{\mu}{\sqrt{\delta}}}. \tag{3.9}$$

On the other hand

$$\|S_1\|^2_{L^2(C)} = \delta \int_{-R}^{R} \frac{1}{(\cosh\rho)^{2m-1}} \sum_{k=1}^{\infty} |A_k|^2 e^{-\frac{8\pi k}{\delta}(\arctan e^{\rho} - \arctan e^{-R})} d\rho.$$

Assuming $\rho_0 - 1 + R > 1$ and $\cosh R < 2\sinh R$, we have

$$\|S_1\|^2_{L^2(C)} \geq \delta \int_{-R}^{\rho_0-1} \frac{1}{(\cosh R)^{2m-1}} \sum_{k=1}^{\infty} |A_k|^2 e^{-\frac{8\pi k}{\delta}(\arctan e^{\rho_0-1} - \arctan e^{-R})} d\rho$$
$$\geq \frac{\delta}{(\frac{\varepsilon_1}{2\delta})^{2m-1}} \sum_{k=1}^{\infty} |A_k|^2 e^{-\frac{8\pi k}{\delta}(\arctan e^{\rho_0-1} - \arctan e^{-R})}. \tag{3.10}$$

By (3.8)–(3.10), we have

$$\frac{\|S_1\|^2(x_0)}{\|S_1\|^2_{L^2(C)}} \leq \frac{2^{2m-1}\varepsilon_1^{2m-1}}{\delta^{2m} e^{\frac{\mu}{\sqrt{\delta}}}} \to 0 \tag{3.11}$$

as $\delta \to 0$.

The idea for estimating S_2 and S_3 is almost the same. By (3.7), we have

$$\begin{aligned} \|S_2\|^2(x_0) &= \frac{1}{(\cosh\rho_0)^{2m}}|B_0|^2\,, \\ \|S_2\|^2_{L^2(C)} &= \delta\int_{-R}^{R}\frac{1}{(\cosh\rho)^{2m-1}}|B_0|^2 d\rho\,. \end{aligned} \tag{3.12}$$

If $\delta\to 0$, then $R\to+\infty$. Thus if R is large enough, we have

$$\int_{-R}^{R}\frac{1}{(\cosh\rho)^{2m-1}}d\rho\geq\mu_1>0\,,$$

where μ_1 is a constant depending only on m. Thus

$$\frac{\|S_2\|^2(x_0)}{\|S_2\|^2_{L^2(C)}}=\frac{1}{\mu_1}\frac{1}{\delta(\cosh\rho_0)^{2m}}=\frac{1}{\mu_1\varepsilon_1^m}\delta^{m-1}\to 0 \tag{3.13}$$

for $\delta\to 0$.

For S_3, we have

$$\begin{aligned} \|S_3\|^2(x_0) &= \frac{1}{(\cosh\rho_0)^{2m}}\left|\sum_{k=-\infty}^{-1}B_k e^{-\frac{4\pi}{\delta}k(\arctan e^{\rho_0}-\arctan e^{R})}\right|^2 \\ &\leq \sum_{k=-\infty}^{-1}|B_k|^2 e^{-\frac{8\pi}{\delta}k(\frac{\pi}{4}-\arctan e^{R})}\sum_{k=-\infty}^{-1}e^{-\frac{8\pi}{\delta}k(\arctan e^{\rho_0}-\frac{\pi}{4})}\,. \end{aligned} \tag{3.14}$$

Since $\rho_0\to-\infty$, we can assume

$$\arctan e^{\rho_0}<\frac{\pi}{8}\,.$$

Thus

$$\sum_{k=-\infty}^{-1}e^{-\frac{8\pi}{\delta}k(\arctan e^{\rho_0}-\frac{\pi}{4})}\leq\sum_{k=-\infty}^{-1}e^{\frac{\pi^2}{\delta}k}\leq 2e^{-\frac{\pi^2}{\delta}} \tag{3.15}$$

for $e^{-\frac{\pi}{\delta^2}}<\frac{1}{2}$. On the other hand,

$$\begin{aligned} \|S_3\|^2_{L^2(C)} &= \delta\int_{-R}^{R}\frac{1}{(\cosh\rho)^{2m-1}}\sum_{k=-\infty}^{-1}|B_k|^2 e^{-\frac{8\pi k}{\delta}(\arctan e^{\rho}-\arctan e^{R})}d\rho \\ &\geq \delta\int_{0}^{R}\frac{1}{(\cosh\rho)^{2m-1}}\sum_{k=-\infty}^{-1}|B_k|^2 e^{-\frac{8\pi k}{\delta}(\frac{\pi}{4}-\arctan e^{R})}d\rho \\ &\geq \frac{\delta R}{(\cosh R)^{2m-1}}\sum_{k=-\infty}^{-1}|B_k|^2 e^{-\frac{8\pi k}{\delta}(\frac{\pi}{4}-\arctan e^{R})}\,. \end{aligned} \tag{3.16}$$

By (3.14)–(3.16), we have

$$\frac{\|S_3\|^2(x_0)}{\|S_3\|^2_{L^2(C)}} \leq \frac{2e^{-\frac{\pi^2}{\delta}}}{\delta R}(\cosh R)^{2m-1} \leq \frac{2^{2m}}{\delta^{2m}} 2e^{-\frac{\pi^2}{\delta}} \to 0\,. \tag{3.17}$$

Thus for any $\varepsilon > 0$ and $m \geq 2$, from (3.11), (3.13) and (3.17), we can find M such that there is a closed geodesic with the length sufficiently small and an $x_0 \in M$ such that

$$\frac{\|S_i\|^2(x_0)}{\|S_i\|^2_{L^2(C)}} \leq \varepsilon$$

for $i = 1, 2, 3$.

One can check that

$$(S_i, S_j)_{L^2(C)} = 0\,.$$

Thus for any $S \in H^0(M, K_M^m)$ with $\|S\|_{L^2(M)}$ and $x_0 \in M$,

$$\|S\|^2(x_0) \leq \frac{\|S\|^2(x_0)}{\|S\|^2_{L^2(C)}} \leq 3\frac{\sum_{i=1}^3 \|S_i\|^2(x_0)}{\sum_{i=1}^3 \|S_i\|^2_{L^2(C)}} \leq 3\varepsilon.$$

Theorem 3.2 is proved. □

4. Partial Uniform Estimates

Let M be a Riemann surface of genus g and constant curvature -1. In this section, we prove that there is a (positive) lower bound of (1.1) depending only on the injective radius of the point. More precisely, we prove the following

Theorem 4.1. *Let M be a Riemann surface of genus $g \geq 2$ and constant curvature -1. Then there are absolute constants $m_0 > 0$ and $D > 0$ such that for any $m > m_0$ and any $x_0 \in M$, there is a section $S \in H^0(M, K_M^m)$ with $\|S\|_{L^2} = 1$ such that*

$$\|S\|(x_0) \geq \frac{\sqrt{m}}{D\left(1 + \frac{1}{\sqrt{m}\delta_{x_0}^2} e^{\frac{\pi}{\delta_{x_0}}}\right)}\,, \tag{4.1}$$

where δ_{x_0} is the injective radius of x_0.

Note that in the result the lower bound does not depend on the injective radius of M, which will go to zero as M approaches the boundary of the Teichmüller space.

We use all the notations of Sec. 3 about the collars and the functions on them. The following proposition is a corollary of the collar theorem:

Proposition 4.1. *Let M be a Riemann surface of genus $g \geq 2$ and constant curvature -1. Let $\gamma_1, \ldots, \gamma_s$ be the closed geodesics on M such that*

$$\text{length}(\gamma_i) \leq \frac{1}{1000}\,, \quad 1 \leq i \leq s\,.$$

Let $C_{\gamma_i}(1 \leq i \leq s)$ be the corresponding collars embedded in M (Theorem 3.1). Then for any $x \in M \backslash U_{i=1}^s C_{\gamma_i}$, there is an absolute constant $\varepsilon_2 > 0$ such that

$$\delta_x \geq \varepsilon_2 \,.$$

Proof. Let $\delta = \frac{1}{4000}$. Let $x \in M \backslash \bigcup_{i=1}^s C_{\gamma_i}$ and $\operatorname{inj}(x) \geq \delta_x > 0$. Then there is a point $y \in M$ such that there are two geodesics l_1 and l_2 connecting x and y but l_1 and l_2 are not homotopic to each other. If $\delta_x > 1$, the theorem has been proved. Otherwise, let γ' be the shortest closed curve homotopic to the closed curve $l_1^{-1} l_2$ in

$$D' = \frac{M}{\bigcup_{\text{length}(\gamma_i) < 2\delta}} C_{\gamma_i}(R_i - 2) \,,$$

where

$$C_{\gamma_i}(R_i - 2) = \{x | \operatorname{dist}(x, \gamma_i) < R_i - 2\} \,.$$

Since D' is a compact set. If γ' does not touch any of the boundary $\partial C_{\gamma_i}(R_i - 2)$ for any i, then γ' must be a closed geodesic and by the definition, we have $\text{length}(\gamma) \geq \frac{1}{1000}$ and thus $\delta_x \geq \frac{1}{2000}$. Otherwise either $\delta_x > 1$ or $\gamma' \subset C_{\gamma_i}(R_i - 1) \backslash C_{\gamma_i}(R_i - 2)$ for some i. In the latter case, since γ' is not homotopic to zero, we see that

$$\text{length}(\gamma') \geq \text{length}(\gamma_i) \cosh(R_i - 2) \geq \frac{1}{18} \varepsilon_1$$

(remember $\text{length}(\gamma_i) \sinh R_i = \varepsilon_1$). Thus

$$\delta_x \geq \frac{1}{2} \text{length}(\gamma') \geq \frac{1}{36} \varepsilon_1 \geq \varepsilon_2$$

for $\varepsilon_2 = \frac{1}{36} \varepsilon_1$. □

Using the above lemma, we know that outside the collars whose shortest closed geodesics are small, the injective radius has a lower bound and the weight function in Proposition 2.1 can be constructed in the ordinary way. If $x \in C_{\gamma_i}$ for some i, we are going to construct the weight functions having the compact support within C_{γ_i}. For this reason, we first assume that C_δ is a collar with $\delta < \frac{1}{1000}$ and do some analysis on it.

We fix some notations: there are absolute constants $\varepsilon_3, \varepsilon_4 > 0$ such that

$$\varepsilon_3 < \delta \cosh R \,, \quad \delta \cosh(R \pm 4) \,, \quad \delta \sinh(R \pm 4) \,, \quad \delta e^{R \pm 4} < \varepsilon_4 \,. \tag{4.2}$$

Let (ρ, θ) be the local coordinate of the collar $C = C_\delta$ as in Sec. 3. Then

$$w = e^{\frac{2\pi i}{\delta}\theta - \frac{4\pi}{\delta}(\arctan e^\rho - \frac{\pi}{4})} \tag{4.3}$$

is the holomorphic function on C_δ. Let x_0 and p_0 be the points on C_δ such that the local coordinate of x_0 and p_0 can be represented as: $x_0 = (\rho_0, 0)$ for $R - 4 > \rho_0 \geq 0$ and $p_0 = (R - 1, 0)$. The function w at x_0 and p_0 has the values

$$\begin{aligned} w_0 &= e^{-\frac{4\pi}{\delta}(\arctan e^{\rho_0} - \frac{\pi}{4})} \,, \\ w_{p_0} &= e^{-\frac{4\pi}{\delta}(\arctan e^{R-1} - \frac{\pi}{4})} \,, \end{aligned} \tag{4.4}$$

at x_0 and p_0 respectively. Let

$$\alpha = \frac{2 \arctan e^{\rho_0}}{\pi}, \tag{4.5}$$

and define the functions φ_1, φ_2 and φ_3 on C_δ to be

$$\begin{aligned} \varphi_1 &= \log \left| \frac{w}{w_0} - 1 \right|, \\ \varphi_2 &= \log \left| \frac{w}{w_{p_0}} - 1 \right|, \\ \varphi_3 &= \varphi_1 - \alpha \varphi_2 . \end{aligned} \tag{4.6}$$

The Riemann metric on C_δ can be represented as

$$ds^2 = d\rho^2 + (\cosh \rho)^2 d\theta^2 . \tag{4.7}$$

Let the injective radius at x_0, p_0 and x be δ_{x_0}, δ_{p_0} and δ_x. Then we have an absolute constant $\varepsilon_5 > 0$ such that

$$\begin{cases} \frac{1}{2}\delta \cosh \rho_0 > \delta_{x_0} > \varepsilon_5 \delta \cosh \rho_0 , \\ \frac{1}{2}\varepsilon_1 > \delta_{p_0} > \varepsilon_5 , \\ \frac{1}{2}\delta \cosh \rho > \delta_x > \varepsilon_5 \delta \cosh \rho . \end{cases} \tag{4.8}$$

We establish some elementary properties of the function φ_3. Let $d = d(x)$ be the distance function to the point x_0. Then we have

Lemma 4.1. *With the notations as above, there are constants* C_9, $C_{10} > 0$ *such that*

$$\varphi_3 \leq C_9 , \tag{4.9}$$

for $-R + 1 \leq \rho \leq R - 2$ *and*

$$\varphi_3 \geq \log d(x) - \frac{4\pi}{\delta e^\rho} - C_{10} \tag{4.10}$$

for $d(x) \leq \delta_{x_0}$.

Proof. By (4.2) and (4.3), we have

$$\begin{aligned} \frac{w}{w_0} &= e^{\frac{2\pi i}{\delta}\theta - \frac{4\pi}{\delta}(\arctan e^\rho - \arctan e^{\rho_0})} , \\ \frac{w}{w_{p_0}} &= e^{\frac{2\pi i}{\delta}\theta - \frac{4\pi}{\delta}(\arctan e^\rho - \arctan e^{R-1})} . \end{aligned} \tag{4.11}$$

From (4.10), we have

$$\begin{cases} \log\left|\dfrac{w}{w_0}-1\right| \le \log 2\,, & \rho_0 \le \rho \le R-2\,, \\ \log\left|1-\dfrac{w_0}{w}\right| \le \log 2\,, & -R+1 \le \rho \le \rho_0\,, \\ \log\left|\dfrac{w}{w_{p_0}}\right| \ge 0\,, & -R+1 \le \rho \le R-1\,, \\ \log\left|1-\dfrac{w_{p_0}}{w}\right| \ge \log\left(1-e^{-\frac{2\pi}{\varepsilon_4}}\right), & -R+1 \le \rho \le R-2\,. \end{cases} \tag{4.12}$$

If $\rho_0 \le \rho \le R-2$, then we can write

$$\varphi_3 = \log\left|\frac{w}{w_0}-1\right| - \alpha\log\left|\frac{w}{w_{p_0}}\right| - \alpha\log\left|1-\frac{w_{p_0}}{w}\right|.$$

By (4.11), we have

$$\varphi_3 \le \log 2 - \log\left(1-e^{-\frac{2\pi}{\varepsilon_4}}\right). \tag{4.13}$$

If $-R+1 \le \rho \le \rho_0$. We can write

$$\varphi_3 = \log\left|\frac{w}{w_0}\right| - \alpha\log\left|\frac{w}{w_{p_0}}\right| + \log\left|1-\frac{w_0}{w}\right| - \alpha\log\left|1-\frac{w_{p_0}}{w}\right|. \tag{4.14}$$

Using (4.10), we have

$$\begin{aligned} \log\left|\frac{w}{w_0}\right| - \alpha\log\left|\frac{w}{w_{p_0}}\right| &= -\frac{8}{\delta}\left(\frac{\pi}{2}-\arctan e^{\rho_0}\right)\arctan e^{\rho} \\ &\quad + \frac{8}{\delta}\arctan e^{\rho_0}\left(\frac{\pi}{2}-\arctan e^{R-1}\right). \end{aligned} \tag{4.15}$$

Thus by (4.1) and (4.7)

$$-\frac{4\varepsilon_5}{\delta_{x_0}} \le \log\left|\frac{w}{w_0}\right| - \alpha\log\left|\frac{w}{w_{p_0}}\right| \le \frac{4\pi}{\varepsilon_3}. \tag{4.16}$$

By (4.11), (4.13) and (4.15), we have

$$\varphi_3 \le \frac{4\pi}{\varepsilon_3} + \log 2 - \log\left(1-e^{-\frac{2\pi}{\varepsilon_4}}\right). \tag{4.17}$$

Thus by (4.12) and (4.16), we have $\varphi_3 \le C_9$ for

$$C_9 = \frac{4\pi}{\varepsilon_3} + \log 2 - \log\left(1-e^{-\frac{2\pi}{\varepsilon_4}}\right).$$

Now assume that $d(x) < \delta_{x_0}$. Then by the triangle inequality we have

$$\begin{cases} |\rho-\rho_0| + |\theta|\cosh\rho_0 \ge d(x)\,, & 0 \le \theta\cosh\rho_0 < d(x)\,, \\ |\rho-\rho_0| + |\theta-\delta\cosh\rho_0| \ge d(x)\,, & \delta\cosh\rho_0 - d(x) < \theta\cosh\rho_0 < \delta\cosh\rho_0\,. \end{cases} \tag{4.18}$$

Without losing generality we assume that $0 \le \theta \le d(x)$ and

$$|\rho - \rho_0| + |\theta| \cosh \rho_0 \ge d(x).$$

If $\theta \cosh \rho_0 \ge \frac{1}{4} d(x)$, then

$$\left| \frac{w}{w_0} - 1 \right| \ge e^{-\frac{4\pi}{\delta}(\arctan e^{\rho} - \arctan e^{\rho_0})} \sin \frac{2\pi}{\delta}\theta \ge \frac{\pi}{4} e^{-\pi} d(x). \tag{4.19}$$

On the other hand, if $|\rho - \rho_0| \ge \frac{1}{2} d(x)$, then

$$\left| \frac{w}{w_0} - 1 \right| \ge e^{-\pi} \frac{2\pi}{\varepsilon_1} d. \tag{4.20}$$

By (4.18) and (4.19), there is a constant $C_{11} > 0$ such that

$$\log \left| \frac{w}{w_0} - 1 \right| \ge \log d - C_{11}. \tag{4.21}$$

We also have

$$\log \left| \frac{w}{w_{p_0}} - 1 \right| \le \log 2 + \frac{4\pi}{\delta e^{\rho}} \tag{4.22}$$

for $d \le \delta_{x_0}$. By (4.20) and (4.21), from (4.5)

$$\varphi_3 \ge \log d - C_{11} - \left(\log 2 + \frac{4\pi}{\delta e^{\rho}} \right).$$

This completes the proof of Lemma 4.1. □

Lemma 4.2. *There is a constant $C_{12} > 0$ such that*

$$|\varphi_3| \le C_{12}, \quad |\nabla \varphi_3| \le C_{12}$$

for $R - 3 < |\rho| < R - 2$.

Proof. By the above lemma, we see that

$$\varphi_3 \le C_9$$

for the constant C_9. Thus we just need to prove the lower bound of φ_3 and the bound for the derivative of φ_3.

If $R - 3 < \rho < R - 2$, by (4.10), we have

$$\left| \frac{w}{w_0} \right| \le e^{-\frac{2\pi}{\varepsilon_1}}. \tag{4.23}$$

Thus

$$\varphi_1 \ge \log \left(1 - e^{-\frac{2\pi}{\varepsilon_1}} \right). \tag{4.24}$$

Also we have

$$\left| \frac{w}{w_{p_0}} \right| \le e^{\frac{4\pi}{\varepsilon_3}} \tag{4.25}$$

for $R-3<\rho<R-2$. Thus

$$\varphi_3 \geq \log\left(1-e^{-\frac{2\pi}{\varepsilon_1}}\right) - \log\left(1+e^{\frac{4\pi}{\varepsilon_3}}\right). \tag{4.26}$$

If $R-3<-\rho<R-2$, then by (4.14), we have

$$\log\left|\frac{w}{w_0}\right| - \alpha\log\left|\frac{w}{w_{p_0}}\right| \geq -\frac{8}{\delta}\left(\frac{\pi}{2}-\arctan e^{\rho_0}\right)\arctan e^{-(R-3)} \geq -\frac{2}{\varepsilon_2}.$$

By (4.13), we have

$$\varphi_3 \geq -\frac{8}{\varepsilon_3} + \log\left(1-e^{-\frac{\pi^2}{2}}\right) - \log 2. \tag{4.27}$$

Combining (4.25) and (4.26), we get the lower bound of φ_3. Next we consider $\nabla\varphi_3$. Obviously

$$|\nabla\varphi_3| \leq |\nabla\varphi_1| + |\nabla\varphi_2|.$$

Thus we just need to estimate $|\nabla\varphi_1|$ and $|\nabla\varphi_2|$. By (4.6), the Riemann metric under the coordinate w can be written as

$$ds^2 = \frac{\delta^2(\cosh\rho)^2}{4\pi^2|w|^2}dwd\bar{w}.$$

Thus

$$|\nabla\varphi_1|^2 = \frac{4\pi^2|w|^2}{\delta^2(\cosh\rho)^2}\cdot\frac{1}{|w-w_0|^2} = \frac{4\pi^2}{\delta^2(\cosh\rho)^2}\cdot\frac{1}{|1-\frac{w_0}{w}|^2},$$

$$|\nabla\varphi_2|^2 = \frac{4\pi^2|w|^2}{\delta^2(\cosh\rho)^2}\cdot\frac{1}{|w-w_{p_0}|^2} = \frac{4\pi^2}{\delta^2(\cosh\rho)^2}\cdot\frac{1}{|1-\frac{w_{p_0}}{w}|^2}.$$

Using the same elementary estimates as above, we get,

$$\begin{cases} \left|\dfrac{w_0}{w}\right| \geq e^{\frac{2\pi}{\varepsilon_1}}, & R-3<\rho<R-2, \\ \left|\dfrac{w_0}{w}\right| \leq e^{-\frac{\pi^2}{2\delta}}, & R-3<-\rho<R-2, \\ \left|\dfrac{w_{p_0}}{w}\right| \leq e^{-\frac{2\pi}{\varepsilon_1}}, & R-3<\rho<R-2, \\ \left|\dfrac{w_{p_0}}{w}\right| \leq e^{-\frac{\pi^2}{2\delta}}, & R-3<-\rho<R-2. \end{cases} \tag{4.28}$$

Using these results, we get the bound for the gradient of φ_1 and φ_2. This completes the proof of the lemma. □

The following proposition summarizes the technical results of this section.

Proposition 4.2. *Suppose M is a compact Riemann surface of genus $g\geq 2$ and constant curvature -1. Then for any $x_0\in M$, there is a function $\varphi=\varphi_{x_0}$ such that φ is smooth on $M\backslash\{x\}$ and*

(1) *In a neighborhood U_x of x, φ can be written as*

$$\varphi = 2\log d(x) + \psi\,,$$

where ψ is a smooth function on U_x. Consequently,

$$\int_{U_x} e^{-\varphi} = +\infty\,. \tag{4.29}$$

(2) *There is a constant C_{13} such that*

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi \geq -C_{13}\omega_g \tag{4.30}$$

on $M\backslash\{x\}$, where ω_g is the Kähler form of M;

(3) *φ satisfies*

$$\varphi \leq C_{13} \tag{4.31}$$

on M and

$$\varphi \geq 2\log d(x) - \frac{2\pi}{\delta_{x_0}} - C_{13} \tag{4.32}$$

for $d(x) \leq \delta_{x_0}$.

Proof. Let $\gamma_1, \ldots, \gamma_s$ be the closed geodesics such that $\text{length}(\gamma_i) < \frac{1}{1000}$. Let C_{γ_i} be the corresponding collars. Let

$$C_{\gamma_i}(R_i - 4) = \{x | \text{dist}(x, \gamma_i) \leq R_i - 4\} \quad i = 1, \ldots s\,.$$

For any $x_0 \in M$, if $x_0 \in C_{\gamma_i}(R_i - 4)$ for some i, then let $C_\delta = C_{\gamma_i}$ and define $\varphi = \varphi_{x_0} : M \to \mathbb{R}$ as follows

$$\varphi = \begin{cases} 2\eta(\rho - (R-3))\varphi_3 & \rho \geq 0 \quad \text{and} \quad x \in C_\delta\,, \\ 2\eta(-\rho - (R-3))\varphi_3 & \rho < 0 \quad \text{and} \quad x \in C_\delta\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{4.33}$$

where the function η is defined in (2.10). By Lemma 4.1, Lemma 4.2 and the fact that φ_3 is harmonic on $C_\delta\backslash\{x_0\}\backslash\{p_0\}$, it is easy to check that the function φ satisfies all the assertions in the proposition. On the other hand, if $x_0 \notin C_{\gamma_i}(R_i - 4)$ for any $1 \leq i \leq s$, then by Proposition 4.1, $\delta_{x_0} \geq \varepsilon_2$. The argument becomes quite standard: define

$$\varphi = 2\eta\left(\frac{d(x)}{\varepsilon_2}\right)\log\left(\frac{d(x)}{\varepsilon_2}\right). \tag{4.34}$$

Then we can prove that φ satisfies all the requirements by using the same method as in Lemma 2.3. ◻

Theorem 4.2. *Let M be a Riemann surface of genus $g \geq 2$ and constant curvature -1. Then there are absolute constants $m_0 > 0$ and $D > 0$ such that for any $m > m_0$ and any $x_0 \in M$, there is a section $S \in H^0(M, K_M^m)$ with $\|S\|_{L^2} = 1$ such that*

$$\|S\|(x_0) \geq \frac{\sqrt{m}}{D\Big(1 + \frac{1}{\sqrt{m}\delta_{x_0}^2} e^{\frac{\pi}{\delta_{x_0}}}\Big)} . \tag{4.35}$$

Proof. Let $x_0 \in M$ and $U_{x_0} = \{x | \mathrm{dist}(x, x_0) < \delta_{x_0}\}$. Let z_1 be the holomorphic function on U_{x_0} such that the hermitian metric can be represented as

$$ds^2 = \frac{1}{\big(1 - \frac{1}{4}|z_1|^2\big)^2} dz_1 d\bar{z}_1 .$$

For $m > 0$, large enough, let

$$u_1 = \bar{\partial}\left(\eta\left(\frac{2|z_1|}{\delta_{x_0}}\right)\right)(dz_1)^m .$$

Then $\bar{\partial} u_1 = 0$ and

$$\|\bar{\partial} u_1\|^2 \leq \frac{16}{\delta_{x_0}^2}\left(1 - \frac{1}{4}|z_1|^2\right)^{m+1}$$

for $\frac{1}{4}\delta_{x_0} \leq |z_1| \leq \frac{1}{2}\delta_{x_0}$.Thus there is a $C_{14} > 0$ such that

$$\int_M \|\bar{\partial} u_1\|^2 e^{-\varphi_{x_0}} \leq \frac{1}{m} \frac{C_{14}}{\delta_{x_0}^4} e^{\frac{2\pi}{\delta_{x_0}}} .$$

Let $m_0 = C_{13} + 2$. By Proposition 2.1, for $m > m_0$, we can find a $u \in \Gamma(M, K_M^m)$ such that $\bar{\partial} u = u_1$ with

$$\int_M \|u\|^2 e^{-\varphi_{x_0}} \leq \frac{1}{m(m - C_{13} - 1)} \frac{C_{14}}{\delta_{x_0}^4} e^{\frac{2\pi}{\delta_{x_0}}} . \tag{4.36}$$

Let $S = \eta(\frac{2|z_1|}{\delta_{x_0}})(dz_1)^m - u$. Then $\bar{\partial} S = 0$. Thus S is an element of $H^0(M, K_M^m)$. Furthermore, since $\int_M e^{-\varphi_{x_0}} = +\infty$, we must have $u(x_0) = 0$. So

$$\|S\|(x_0) = 1 . \tag{4.37}$$

On the other hand,

$$\|S\|_{L^2}^2 \leq 2\left(\int_M \|u\|^2 + \int_M \|\eta(dz_1)^m\|^2\right) .$$

By (4.35) and (4.30), we have

$$\int_M \|u\|^2 \leq \frac{1}{m(m - C_{13} - 1)} \cdot \frac{C_{14} e^{C_{13}}}{\delta_{x_0}^4} e^{\frac{2\pi}{\delta_{x_0}}} . \tag{4.38}$$

We also have

$$\int_M \|\eta(dz_1)^m\|^2 \leq \frac{\pi}{m} . \tag{4.39}$$

The theorem follows from (4.36)–(4.38). ∎

5. The Uniform Corona Problem

Let M be a Riemann surface of genus $g \geq 2$. It is well known that the coordinate ring $\bigoplus_{m=0}^{\infty} H^0(M, K_M^m)$ is finitely generated. That is, there is an $m_0 > 0$ such that for any $m > 0$ and $S \in H^0(M, K_M^m)$, S can be represented by

$$S = \sum_{i=1}^{d} U_i T_i \,, \tag{5.1}$$

where $U_i \in H^0(M, K_M^{m_0})$ and

$$T_i \in H^0(M, K_M^{m-m_0}) \quad \text{for } i = 1, \ldots, d = \dim H^0(M, K_M^{m_0})\,.$$

Finding a suitable set of $\{T_i\}_{i=1,\ldots,d}$ is called the corona problem (cf. [5]).

We need to consider the case where M approaches the boundary of the moduli space in the Teichmüller theory. So in addition to the existence of U_i and T_i, we need some uniform estimates. In this section, we give the uniform estimate for the corona problem on Riemann surfaces.

Theorem 5.1. *Let M be a Riemann surface of genus g and constant curvature -1. Then there is an $m_0 > 0$ such that for any $m > m_0$ and $S \in H^0(M, K_M^m)$, there is a decomposition*

$$S = \sum_{i=1}^{d} S_i$$

of $S_i \in H^0(M, K_M^m)(i = 1, \ldots, d)$ such that

$$\begin{aligned} \|S_i\|_{L^2} &\leq C(m, m_0, g)\|S\|_{L^2}\,, \\ \|S_i\|_{L^\infty} &\leq C(m, m_0, g)\|S\|_{L^\infty}\,, \end{aligned} \tag{5.2}$$

for $i = 1, \ldots, d$, and

$$S_i = T_i U_i$$

for a basis $U_1, \ldots, U_d$ of $H^0(M, K_M^{m_0})$ and $T_1, \ldots, T_d \in H^0(M, K_M^{m-m_0})$.

Remark 5.1. An estimate on $T_i (i = 1, \ldots, d)$ alone is not expected because of the counterexample in Sec. 3, where $\sum \|U_i\|^2$ can be arbitrarily small.

Throughout this section, we will use the notation $D_1, D_2, \ldots$ to denote the constants depending only on m_0 and the genus g. We also use $A \leq \sim B$ to mean that there is a positive constant $C = C(m_0, g)$, depending only on m_0 and g such that $A \leq CB$. Likewise, we use $A \geq \sim B$ to denote the fact $A \geq CB$ for some constant $C = C(m_0, g)$.

The idea of the proof is that, if the injective radius of M is greater than an absolute constant, then $\sum \|U_i\|^2$ has a lower bound by an absolute positive constant. In this case, we can solve the corona problem exactly using the method in [5]. So we just need to prove the theorem in the case where $\mathrm{inj}(M)$ is arbitrarily small.

By the collar theorem, we know that in this case, there are finite many collars $C_{\delta_1}, \dots, C_{\delta_s}$ (with max δ_i small) embedded in M and they do not intersect each other. By Proposition 4.1, outside the collars, the injective radius has an absolute lower bound. Special care must be taken for the sections over these collars. In order to take care of the collars to get the estimates, we first fix a collar C_δ embedded in M with the parameter δ small. We will use all the notations about collars in Sec. 3. For any $\tilde{R} > 0$, let

$$C_\delta(\tilde{R}) = \{x \| \rho| \le \tilde{R}\}\,.$$

In particular, $C_\delta = C_\delta(R)$ with $\delta \sinh R = \varepsilon_1$.

We choose and fix a number $m_0 > 0$ such that $K_M^{m_0}$ is very ample. Let $\tilde{\eta}$ be the cut-off function of M defined as

$$\tilde{\eta} = \begin{cases} \eta(\rho - (R-1))\,, & \rho \ge 0 \quad \text{and} \quad x \in C_\delta\,, \\ \eta(-\rho - (R-1))\,, & \rho < 0 \quad \text{and} \quad x \in C_\delta\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{5.3}$$

where the function η is defined in (2.10). Let

$$u_1 = \frac{1}{\sqrt{\delta}} \tilde{\eta} (dz)^{m_0} \tag{5.4}$$

be a section of $K_M^{m_0}$ over M using this cut-off function, where dz is defined in (3.3). We can check that

$$\begin{cases} \|\bar{\partial} u_1\|^2 \le \dfrac{4}{\delta (\cosh \rho)^{2m_0}} & R - 1 \le |\rho| \le R\,, \\ \bar{\partial} u_1 = 0 & \text{otherwise}\,. \end{cases}$$

Thus

$$\int_M \|\bar{\partial} u_1\|^2 \le 2 \int_{R-1}^{R} \frac{4}{\delta (\cosh \rho)^{2m_0}} \cosh \rho \delta d\rho \le \frac{8}{(\cosh (R-1))^{2m_0 - 1}}\,.$$

By Proposition 2.1, there is a section u of $K_M^{m_0}$ such that

$$\bar{\partial} u = \bar{\partial} u_1$$

with

$$\int_M \|u\|^2 \le \frac{1}{m_0 - 1} \int_M \|\bar{\partial} u_1\|^2 \le \frac{8}{(m_0 - 1)(\cosh(R-1))^{2m_0 - 1}}\,.$$

By using (4.1), we see that

$$\int_M \|u\|^2 \le \sim \delta^{2m_0 - 1}\,. \tag{5.5}$$

Let

$$U' = u_1 - u\,. \tag{5.6}$$

Then $\bar{\partial} U' = 0$.

Lemma 5.1. *Let U'' be a holomorphic section of $K_M^{m_0}$ on $C_\delta(R)$ such that $(U'', u_1)_{C_\delta(R-2)} = 0$. That is,*

$$\int_{C_\delta(R-2)} \langle U'', u_1 \rangle = 0\,, \tag{5.7}$$

where u_1 is the section defined in (5.4). *Then there is an absolute constant $\varepsilon_6 > 0$ such that*

$$\|U''\| \leq \sim e^{-\varepsilon_6 e^{R-|\rho|} + m_0(R-|\rho|)} \|U''\|_{L^2(C_\delta)}\,, \tag{5.8}$$

and

$$\|\nabla U''\| \leq \sim e^{-\varepsilon_6 e^{R-|\rho|} + (m_0+1)(R-|\rho|)} \|U''\|_{L^2(C_\delta)}\,. \tag{5.9}$$

We use the notation in Sec. 3. Let

$$w_1 = e^{\frac{2\pi\sqrt{-1}}{\delta}\theta - \frac{4\pi}{\delta}(\arctan e^\rho - \arctan e^{-(R-2)})}\,,$$

$$w_2 = e^{-\frac{2\pi\sqrt{-1}}{\delta}\theta + \frac{4\pi}{\delta}(\arctan e^\rho - \arctan e^{R-2})}\,.$$

Let

$$U'' = (g_1(w_1) + a + g_3(w_2))(dz)^{m_0}$$

be the decomposition similar to that in (3.4) where $g_1(w_1)$ and $g_2(w_2)$ are holomorphic functions of w_1 and w_2 respectively, $g_1(0) = g_2(0) = 0$, a is a constant, and dz is defined in (3.3).

Using (5.7), we see that $a = 0$. By the Schwartz lemma, we have

$$|g_1(w_1)| \leq |w_1| \max_{|w_1|=1} |g_1(w_1)|\,. \tag{5.10}$$

At each point of $\{|w_1| = 1\}$ or $\{\rho = R-2\}$, by the collar theorem, there is an absolute lower bound for the injective radius. Thus by the Cauchy integral formula, we have

$$\max_{|w_1|=1} \|g_1(dz)^{m_0}\| \leq \sim \|g_1(dz)^{m_0}\|_{L^2(C_\delta)}\,. \tag{5.11}$$

Using (5.10) and (5.11), we have

$$\|g_1(dz)^{m_0}\| \leq e^{-\frac{4\pi}{\delta}(\arctan e^\rho - \arctan e^{-(R-2)})} \frac{(\cosh R)^{m_0}}{(\cosh\rho)^{m_0}} \|g_1(dz)^{m_0}\|_{L^2(C_\delta)}\,. \tag{5.12}$$

It is elementary to check that there is an absolute constant $\varepsilon_6 > 0$ such that

$$e^{-\frac{4\pi}{\delta}(\arctan e^\rho - \arctan e^{-(R-2)})} \leq e^{-\varepsilon_6 e^{R-|\rho|}}\,. \tag{5.13}$$

Combining (5.12) and (5.13),

$$\|g_1(dz)^{m_0}\| \leq \sim e^{-\varepsilon_6 e^{R-|\rho|} + m_0(R-|\rho|)} \|g_1(dz)^{m_0}\|_{L^2(C_\delta)}\,. \tag{5.14}$$

Similarly, we have

$$\|g_2(dz)^{m_0}\| \leq \sim e^{-\varepsilon_6 e^{R-|\rho|} + m_0(R-|\rho|)} \|g_2(dz)^{m_0}\|_{L^2(C_\delta)}\,. \tag{5.15}$$

(5.14) and (5.15) give the inequality (5.8).

On the other hand, a straightforward computations gives

$$\nabla U'' = \left(\frac{2\pi\sqrt{-1}}{\delta}(w_1 g_1' + w_2 g_2') - \frac{1}{2} m_0 \sqrt{-1} \sinh\rho (g_1 + g_2)\right)(dz)^{m_0} \otimes dz\,. \quad (5.16)$$

Using the same argument as above, we get (5.9).

Lemma 5.2. *With the notations as above, there is a constant $r > 2$, depending only on m_0 and the genus g, such that $U' \neq 0$ on $C_\delta(R-r)$, where U' is defined in* (5.6).

Proof. By (5.6), we know that u is holomorphic on $C_\delta(R-2)$. Let

$$u = u' + \alpha u_1 \quad (5.17)$$

be the decomposition of u such that $(u', u_1)_{C_\delta(R-2)} = 0$ and α is a constant. Then

$$\int_M \|u\|^2 \geq \int_{C_\delta(R-2)} \|u'\|^2 + \int_{C_\delta(R-2)} |\alpha|^2 \|u_1\|^2\,. \quad (5.18)$$

In particular

$$\int_{C_\delta(R-2)} \|u\|^2 \geq |\alpha|^2 \int_{-(R-2)}^{R-2} \frac{1}{(\cosh\rho)^{2m_0-1}} d\rho\,. \quad (5.19)$$

By (5.5) and (5.19), we have

$$|\alpha| \leq \sim \delta^{m_0 - \frac{1}{2}}\,. \quad (5.20)$$

On the other hand, by Lemma 5.1,

$$\|u'\| \leq \sim e^{-\varepsilon_6 e^{R-|\rho|} + m_0(R-|\rho|)} \delta^{m_0 - \frac{1}{2}}\,. \quad (5.21)$$

Thus by (5.4), (5.20) and (5.21), there are constants D_1 and D_2 such that

$$\|U'(x)\| \geq \frac{1}{\sqrt{\delta}(\cosh\rho)^{m_0}}(1 - D_1 \delta^{m_0 - \frac{1}{2}}) - D_2 e^{-\varepsilon_6 e^{R-|\rho|} + m_0(R-|\rho|)} \delta^{m_0 - \frac{1}{2}} \quad (5.22)$$

for $|\rho| < R - 3$. If r is large enough, then $\|U'(x)\| > 0$ for $|\rho| < R - r$. In particular $U' \neq 0$ on $C_\delta(R-r)$. This completes the proof of the lemma. □

Now we assume the general case. Let $C_{\delta_1}, \ldots, C_{\delta_s}$ be the collars of parameters $\delta_1, \ldots, \delta_s$ respectively embedded into M. We assume that $C_{\delta_i}, (i = 1, \ldots, s)$ do not intersect each other. By Lemma 5.2, there are $R_i, (i = 1, \ldots, s)$ such that we can find $U_i' \in H^0(M, K_m^{m_0}), (i = 1, \ldots, s)$ with $U_i'|_{C_{\delta_i}(R_i)} \neq 0$. Let $\rho_i, dz_i, (i = 1, \ldots, s)$ be defined in Sec. 3 corresponding to $C_{\delta_i} (i = 1, \ldots, s)$, respectively. Let

$$\eta_i = \begin{cases} \eta(\rho_i - (R_i - 1))\,, & \rho_i \geq 0 \quad \text{and} \quad x \in C_{\delta_i}\,, \\ \eta(-\rho_i - (R_i - 1))\,, & \rho_i \leq 0 \quad \text{and} \quad x \in C_{\delta_i}\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

be the cut-off functions for $i = 1, \ldots, s$. Assume that max δ_i is small enough. Then by Lemma 5.2, $U'_1, \ldots, U'_s \in H^0(M, K_M^{m_0})$ have the following properties

(1) $\|U'_i\| \geq \sim \frac{1}{\sqrt{\delta_i}} e^{-m_0|\rho_i|}$ on $C_{\delta_i}(R_i)$ for $i = 1, \ldots, s$ [by (5.22)].

(2) There are decompositions $U'_i|_{C_{\delta_i}(R_i)} = \alpha_i v_{1i} + v_{2i}$ with

$$v_{1i} = \frac{1}{\sqrt{\delta_i}} \eta_i (dz_i)^{m_0}, \quad 1 \leq i \leq s \tag{5.23}$$

and

$$(v_{1i}, v_{2i})_{C_{\delta_i}(R_i)} = 0, \quad 1 \leq i \leq s,$$

where $\alpha_i (i = 1, \cdots, s)$ are a constant such that for $1 \leq i \leq s$,

$$1 \leq \sim \alpha_i \leq \sim 1. \tag{5.24}$$

(3) By (5.4) and (5.5),

$$\int_{M \setminus C_{\delta_i}(R_i)} \|U'_i\|^2 \leq \sim \delta_i^{2m_0 - 1} \tag{5.25}$$

for $i = 1, \ldots, s$.

We have the following lemma:

Lemma 5.3. *With the notations as above, there are holomorphic sections* $U_1, \ldots, U_s \in H^0(M, K_M^{m_0})$ *such that*

(1) $\|U_i\| \geq \sim \frac{1}{\sqrt{\delta_i}} e^{-m_0|\rho_i|}$ *on* $C_{\delta_i}(R_i)$;

(2) $(U_i|_{C_{\delta_j}(R_j)}, v_{1j})_{C_{\delta_j}(R_j)} = 0$ *for* $i \neq j$, $1 \leq i, j \leq s$;

(3) $\int_{M \setminus C_{\delta_i}(R_i)} \|U_i\|^2 \leq \sim \delta_i^{2m_0 - 1}$.

Proof. Let

$$\beta_{ij} = (U'_i, v_{1j})_{C_{\delta_j}(R_j)}, \quad 1 \leq i, \ j \leq s.$$

Then if $i \neq j$, we have

$$|\beta_{ij}| \leq \sim \delta_i^{m_0 - \frac{1}{2}} \tag{5.26}$$

by (5.25) and the definition of $v_{1j}, (j = 1, \ldots, s)$ in (5.23). We also have

$$1 \geq \sim \beta_{ii} \geq \sim 1 \tag{5.27}$$

by (5.24).

Let $B = (\beta_{ij})_{s \times s}$ be the matrix of (β_{ij}) for $1 \leq i, j \leq s$. Then by (5.26) and (5.27), B is an invertible matrix, when $\max \delta_i$ is small enough. Let $A = B^{-1}$ be the inverse matrix and let $A = (\alpha_{ij})_{s \times s}$. Define

$$U_i = \sum_{j=1}^{s} \alpha_{ij} U'_j, \quad 1 \leq i \leq s.$$

Then $U_i (i = 1, \ldots, s)$ satisfies all the requirements in the lemma by the fact that

$$\begin{aligned} |\alpha_{ij}| &\leq \sim \delta_i^{m_0 - \frac{1}{2}} && i \neq j\,, \\ 1 \geq \sim \alpha_{ij} &\geq \sim 1 && i = j\,. \end{aligned}$$

□

Let $U'_{s+1}, \ldots, U'_d$ be an orthonormal basis of the space $\text{span}\{U_1, \ldots, U_s\}^{\perp}$. Assume that

$$(U'_i, U_j) = 0$$

for $s < i \leq d$, $1 \leq j \leq s$. Let

$$U_i = U'_i - \sum_{j=1}^{s} \gamma_{ij} U_j\,, \quad s < i \leq d\,, \tag{5.28}$$

where

$$\gamma_{ik} = \frac{1}{(U_k, v_{1k})_{C_{\delta_k}(R_k)}} (U'_i, v_{1k})_{C_{\delta_k}(R_k)}\,, \quad 1 \leq k \leq s,\; s < i \leq d\,. \tag{5.29}$$

Then we have

$$(U_i|_{C_{\delta_j}(R_j)}, v_{1j})_{C_{\delta_j}(R_j)} = 0$$

for $s < i \leq d$ and $1 \leq j \leq s$.

We have the following lemma:

Lemma 5.4. *If* $x \notin \bigcup_{j=1}^{s} C_{\delta_j}(R_j - 2)$, *then*

$$\sum_{i=1}^{d} \|U_i\|^2 \geq \sim 1\,.$$

Proof. Let

$$l_{ij} = (U_i, U_j)$$

for $1 \leq i, j \leq d$. If $1 \leq i, j \leq s$, we have

$$\begin{aligned} (U_i, U_j) &\leq \sim \max \delta_i && i \neq j\,, \\ 1 \geq \sim (U_i, U_j) &\geq \sim 1 && i = j \end{aligned} \tag{5.30}$$

by (5.26), (5.27) and Lemma 5.3.

If $1 \leq i \leq s$, $s < j \leq d$, we have

$$(U_i, U_j) = -\sum_{k=1}^{s} \gamma_{jk} (U_i, U_k)$$

using (5.28). By the definition of γ_{jk} in (5.29), we have

$$|\gamma_{jk}| \leq \sim \max \delta_j \,. \tag{5.31}$$

Thus

$$|(U_i, U_j)| \leq \sim \max \delta_j \tag{5.32}$$

for $1 \leq i \leq s$ and $s < j \leq d$. Finally, if $s < i,\ j \leq d$, then

$$\begin{aligned} |(U_i, U_j)| &\leq \sim \max \delta_i & i \neq j\,, \\ 1 \geq \sim (U_i, U_j) &\geq \sim 1 & i = j\,, \end{aligned} \tag{5.33}$$

by (5.28) and (5.31). Using (5.30), (5.32) and (5.33), we have

$$\begin{aligned} |l_{ij}| &\leq \sim \max \delta_i & i \neq j\,, \\ 1 \geq \sim |l_{ij}| &\geq \sim 1 & i = j\,. \end{aligned}$$

Let $(m_{ij})_{d\times d}$ be the matrix such that

$$\sum_{i=1}^{d} \sum_{t=1}^{d} m_{ji} \overline{m_{tk}} l_{it} = \delta_{jk}$$

for $1 \leq j,\ k \leq d$. We can choose m_{ij} such that

$$\begin{aligned} |m_{ij}| &\leq \sim \max \delta_i & i \neq j\,, \\ 1 \geq \sim m_{ij} &\geq \sim 1 & i = j\,. \end{aligned} \tag{5.34}$$

It is easy to check that $\sum_{j=1}^{d} m_{ij} U_j$ for $1 \leq i \leq d$ forms an orthonormal basis of $H^0(M, K_M^{m_0})$. Thus we have

$$\sum_{i=1}^{d} \left\| \sum_{j=1}^{d} m_{ij} U_j \right\|^2 \geq \sim 1$$

by Theorem 1.3. The lemma thus follows from (5.34) and the fact that $\max \delta_i$ is small. $\square$

We summarize the results up to now in the following

Proposition 5.1. *Let $U_1, \ldots, U_d$ be sections of $H^0(M, K_M^{m_0})$ as above. Then*

$$\|U_i\| + \|\nabla U_i\| \leq \sim 1 \qquad x \notin \bigcup_{j=1}^{s} C_{\delta_j}(R_j - 2)\,, \tag{5.35}$$

$$\|U_i\| + \|\nabla U_i\| \leq \sim e^{-(m_0+1)(R_j - |\rho_j|)} \qquad x \in C_{\delta_j}(R_j - 2),\ 1 \leq j \leq s\,, \tag{5.36}$$

for $i > s$ *and*

$$\|U_i\| + \|\nabla U_i\| \leq \sim \delta_i{}^{m_0 - \frac{1}{2}} \qquad x \notin \bigcup_{j=1}^{s} C_{\delta_j}(R_j - 2)\,, \tag{5.37}$$

$$\|U_i\| + \|\nabla U_i\| \leq \sim \frac{1}{\sqrt{\delta}_i e^{m_0|\rho_i|}} \qquad x \in C_{\delta_i}(R_i - 2)\,, \tag{5.38}$$

$$\|U_i\| + \|\nabla U_i\| \leq \sim \delta_i^{m_0 - \frac{1}{2}} e^{-(m_0+1)(R_j - |\rho_j|)} \quad x \in C_{\delta_j}(R_j - 2),\ j \neq i\,, \tag{5.39}$$

for $1 \leq i \leq s$. *Furthermore,*

$$\sum_{k=1}^{d} \|U_i\|^2 \geq \sim 1 \tag{5.40}$$

for $x \notin \bigcup_{i=1}^{s} C_{\delta_i}(R_i - 2)$.

Proof. If $x \notin \bigcup_{j=1}^{s} C_{\delta_j}(R_j)$, then δ_x has a uniform lower bound. (5.35) follows from the Cauchy integral formula. (5.36) is a corollary of Lemma 5.1. (5.37) follows from (5.5) and the Cauchy integral formula. (5.38) follows from a straightforward computation. (5.39) follows from Lemma 5.1 and (5.25). Finally, (5.40) is just a statement of the conclusion of Lemma 5.4. □

Define an inner product $\langle\, ,\, \rangle$ in the coordinate ring $\bigoplus_{m=0}^{\infty} H^0(M, K_M^m)$. Let $S_1 \in H^0(M, K_M^m)$ and $S_2 \in H^0(M, K_M^{m_1})$. Suppose that $m \geq m_1$. We define a section of $K_M^{m-m_1}$ as follows: let $x \in M$ and U_x is a local trivialization of K_M. Let $S_1|_{U_x} = S' \cdot S''$ for $S' \in \Gamma(U_x, K_M^{m-m_1})$ and $S'' \in \Gamma(U_x, K_M^{m_1})$. Then

$$S_3|_{U_x} = S' \langle S'', S_2 \rangle_{H^{m_1}}\,,$$

where $\langle\, ,\, \rangle_{H^{m_1}}$ is the pointwise inner product.

Proof of Theorem 5.1. We modify the method of Wolff [11] of solving the corona problem on the unit disk. First we construct a C^∞ solution. Let $S \in H^0(M, K_M^m)$ for a fixed $m > m_0$. Let

$$\begin{cases} b_k = \eta_k \dfrac{S}{U_k} + \left(1 - \displaystyle\sum_{j=1}^{s} \eta_j\right) \dfrac{\langle S, U_k \rangle}{\sum_{j=1}^{d} \|U_j\|^2} & 1 \leq k \leq s\,, \\[2ex] b_k = \left(1 - \displaystyle\sum_{j=1}^{s} \eta_j\right) \dfrac{\langle S, U_k \rangle}{\sum_{j=1}^{d} \|U_j\|^2} & k > s\,. \end{cases} \tag{5.41}$$

Here $b_k (1 \leq k \leq s)$ is well defined because of Lemma 5.2. We can check that

$$S = \sum_{k=1}^{d} U_k b_k\,.$$

If $x \notin \bigcup_{j=1}^{s} C_{\delta_j}(R_j - 2)$, then by Lemma 5.4,

$$\begin{cases} \|b_k\| \leq \sim (\sqrt{\delta}_k(\cosh\rho_k)^{m_0} + 1)\|S\| & 1 \leq k \leq s\,, \\ \|b_k\| \leq \sim \|S\| & k > s\,. \end{cases} \tag{5.42}$$

By (5.41), we have

$$\begin{aligned} \bar\partial b_k &= \bar\partial \eta_k \frac{S}{U_k} - \bar\partial \sum_{j=1}^{s} \eta_j \frac{\langle S, U_k\rangle}{\sum_{j=1}^{d} \|U_j\|^2} + \left(1 - \sum_{j=1}^{s} \eta_j\right) \frac{\langle S, \nabla U_k\rangle}{\sum_{j=1}^{d} \|U_j\|^2} \\ &\quad - \left(1 - \sum_{j=1}^{s} \eta_j\right) \frac{\langle S, U_k\rangle \sum_{j=1}^{d} \langle U_j, \nabla U_j\rangle}{\left(\sum_{j=1}^{d} \|U_j\|^2\right)^2} \quad 1 \leq k \leq s\,, \\ \bar\partial b_k &= -\bar\partial \sum_{j=1}^{s} \eta_j \frac{\langle S, U_k\rangle}{\sum_{j=1}^{d} \|U_j\|^2} + \left(1 - \sum_{j=1}^{s} \eta_j\right) \frac{\langle S, \nabla U_k\rangle}{\sum_{j=1}^{d} \|U_j\|^2} \\ &\quad - \left(1 - \sum_{j=1}^{s} \eta_j\right) \frac{\langle S, U_k\rangle \sum_{j=1}^{d} \langle U_j, \nabla U_j\rangle}{\left(\sum_{j=1}^{d} \|U_j\|^2\right)^2} \quad k > s\,. \end{aligned} \tag{5.43}$$

Thus

$$\begin{cases} \|\bar\partial b_k\| \leq \sim \dfrac{1}{\delta_k^{m_0 - \frac{1}{2}}} \|S\| & 1 \leq k \leq s\,, \\ \|\bar\partial b_k\| \leq \sim \|S\| & k > s\,. \end{cases} \tag{5.44}$$

Let

$$c_{ik} = \frac{\langle \bar\partial b_k, U_i\rangle}{\sum_{j=1}^{d} \|U_j\|^2} \tag{5.45}$$

for $i \neq k$ and $1 \leq i,\, k \leq d$. Then by (5.42) and (5.44), we have

$$\begin{cases} \|c_{ik}\| \leq \sim \dfrac{1}{\delta_k^{m_0 - \frac{1}{2}}} \|S\| & 1 \leq k \leq s\,, \\ \|c_{ik}\| \leq \sim \|S\| & k > s\,. \end{cases} \tag{5.46}$$

Consider the equations

$$\bar\partial b_{ik} = c_{ik}$$

for $i \neq k$ and $1 \leq i,\, k \leq d$. By Proposition 2.1, the solutions exist and we may assume that

$$\|b_{ik}\|_{L^2} \leq \|c_{ik}\|_{L^2}$$

for $i \neq k$, $1 \leq i$, $k \leq d$. Thus by (5.46), we have

$$\begin{cases} \|b_{ik}\|_{L^2} \leq \sim \dfrac{1}{\delta_k^{m_0-\frac{1}{2}}}\|S\|_{L^2} & 1 \leq k \leq s\,, \\ \|b_{ik}\|_{L^2} \leq \sim \|S\|_{L^2} & s < k \leq d\,. \end{cases} \tag{5.47}$$

Let

$$T_i = b_i + \sum_{k=1}^{d}(b_{ik} - b_{ki})U_k\,.$$

One can check that $\bar{\partial} T_i = 0$ and

$$S = \sum_{i=1}^{d} T_i U_i\,.$$

In order to prove the theorem, we need to estimate $\|T_iU_i\|$ for $1 \leq i \leq d$. By (5.47) and Proposition 5.1, we have

$$\|T_iU_i\|_{L^2(M\setminus\bigcup_{j=1}^{s} C_\delta(R_j))} \leq \sim \|S\|_{L^2}\,. \tag{5.48}$$

By the Cauchy integral formula and Proposition 4.1

$$\|T_iU_i\|(x) \leq \sim \|S\|_{L^2} \tag{5.49}$$

for $1 \leq i \leq d$ and $x \notin \bigcup_{j=1}^{s} C_{\delta_j}(R_j)$.

Let

$$E_i(\rho) = \{x \in C_{\delta_i}(R_i) \| \rho_i(x)| \geq \rho\}\,, \quad 1 \leq i \leq s$$

for positive number $\rho > 0$. By (5.47) and Proposition 5.1 again, we have

$$\begin{cases} \|T_iU_i - \eta_i S\|_{L^2(E_j(\rho)-E_j(\rho+2))} \leq \sim e^{-(R_i-\rho)}\|S\|_{L^2} & 1 \leq i \leq s\,, \\ \|T_iU_i\|_{L^2(E_j(\rho)-E_j(\rho+2))} \leq \sim e^{-(R_i-\rho)}\|S\|_{L^2} & i > s\,. \end{cases} \tag{5.50}$$

By the Cauchy formula,

$$\begin{cases} \|T_iU_i - \eta_i S\|(x) \leq \sim \dfrac{1}{\delta_x} e^{-(R_i-\rho)}\|S\|_{L^2} & 1 \leq i \leq s\,, \\ \|T_iU_i\|(x) \leq \sim \dfrac{1}{\delta_x} e^{-(R_i-\rho)}\|S\|_{L^2} & i > s\,, \end{cases} \tag{5.51}$$

for any $x \in E_j(\rho + \frac{1}{2}) - E_j(\rho + 1)$. Since

$$\delta_x \geq \varepsilon_5 \delta_j (\cosh \rho_j)$$

by (4.7), we have

$$\begin{cases} \|T_iU_i - \eta_i S\|(x) \leq \sim \|S\|_{L^2} & 1 \leq i \leq s\,, \\ \|T_iU_i\|(x) \leq \sim \|S\|_{L^2} & i > s\,, \end{cases} \tag{5.52}$$

for any $x \in E_j(\rho + \frac{1}{2}) - E_j(\rho + 1)$ and any $|\rho_j| < R_j - 3$, $j = 1, \ldots s$. Thus if $\|S\|_{L^2} = 1$, then by (5.49) and (5.52)

$$\|T_i U_i\|_{L^2} \leq \sim 1$$

for $1 \leq i \leq d$. If $\|S\|_{L^\infty} = 1$, then

$$\|T_i U_i\|_{L^\infty} \leq \sim 1$$

for $1 \leq i \leq d$. These results give the inequality (5.2). □

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