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WEIL–PETERSSON GEOMETRY ON MODULI SPACE OF POLARIZED CALABI–YAU MANIFOLDS

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Abstract In this paper, we define and study the Weil–Petersson geometry. Under the framework of the Weil–Petersson geometry, we study the Weil–Petersson metric and the Hodge metric. Among the other results, we represent the Hodge metric in terms of the Weil–Petersson metric and the Ricci curvature of the Weil–Petersson metric for Calabi–Yau fourfold moduli. We also prove that the Hodge volume of the moduli space is finite. Finally, we proved that the curvature of the Hodge metric is bounded if the Hodge metric is complete and the dimension of the moduli space is 1.

Keywords: Schwarz–Yau Lemma; Calabi–Yau manifolds; Weil–Petersson metric

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Contents

1. Introduction

Moduli spaces of general polarized algebraic varieties are studied extensively by algebraic geometers. However, there are two classes of moduli spaces where the methods of differential geometry are equally powerful. These are the moduli spaces of curves and the moduli spaces of polarized Calabi–Yau manifolds. Both spaces are complex orbifolds. The Weil–Petersson metric is the main tool for investigating the geometry of such moduli spaces. Under the Weil–Petersson metrics, these moduli spaces are Kähler orbifolds.

The GIT construction of the (coarse) moduli space (see [**27**]) of Mumford is as follows. Let X be a Calabi–Yau manifold and let L be an ample line bundle over X. The pair (X, L) is called a polarized Calabi–Yau manifold. Choose a large m such that L^{m} is very ample. In this way X is embedded into a complex projective space $\mathbb{C}P^N$. Let $\mathrm{Hilb}(X)$ be the Hilbert scheme of X. It is a compact complex variety. The group $G = PSL(N+1, C)$ acts on $\mathrm{Hilb}(X)$ and the moduli space $\mathcal{M}$ is the quotient of the stable points of $\mathrm{Hilb}(X)$ by the group G. For the purpose of this paper, we assume that $\mathcal{M}$ is connected.

The curvature of these moduli spaces with respect to the Weil–Petersson metric has been studied by many people. For the moduli space of curves, Wolpert [**30**] gave an explicit formula for the curvature and proved that the (Riemannian) sectional curvature of the Weil–Petersson is negative. Siu [**23**] generalized the result to the moduli spaces of Kähler–Einstein manifolds with $c_1 < 0$. Schumacher [**21**], using Siu's methods, computed the curvature tensor of the moduli spaces of Kähler–Einstein manifolds in the case of $c_1 > 0$ and $c_1 < 0$, respectively*. Furthermore, Strominger [**24**] gave the curvature formula for the moduli space of Calabi–Yau threefolds using the Yukawa couplings. Generalizing the formula, Wang [**29**] proved the curvature formula for Calabi–Yau n-folds where there are no Yukawa couplings. His proof is purely Hodge theoretic and is also true on Weil–Petersson varieties.

It is important and interesting to know the geometry of moduli space at infinity. In [**9**], Jost and Yau were able to understand the moduli spaces of curves at infinity using the Schwarz–Yau Lemma [**31**]. For moduli space of polarized Calabi–Yau manifolds, similar results could be found in [**11**]. In order to make use of the Schwarz–Yau Lemma, we need some natural metric on the moduli spaces whose holomorphic sectional curvature is negative away from zero.

Unlike the case of moduli space of curves, the sectional curvature of the Weil–Petersson metric on moduli space of polarized Calabi–Yau manifolds is not negative, even in the case when the moduli space is one dimensional. The curvature of the Weil–Petersson metric can either be positive or negative (cf. [**2**, p. 65]) on the moduli space of Calabi–Yau threefolds which are mirror manifolds of the quintic hypersurfaces in CP^4. This fact prevents us from using the Schwarz–Yau Lemma directly.

In [**12**], the first author introduced the Hodge metric on the moduli space of polarized Calabi–Yau manifolds. The Hodge metric is a Kähler metric on the moduli space. Its holomorphic bisectional curvature is non-positive and both of its Ricci and holomorphic sectional curvature are negative away from zero. The Hodge metric on moduli space of Calabi–Yau manifolds is the counterpart of the Weil–Petersson metric on Teichmüller space. In § 4, we took a further step by defining the 'partial Hodge metric'. We computed the curvature of the 'partial Hodge metric'. The formula is parallel to the curvature formula of Wolpert [**30**] on Teichmüller space. In the case of the moduli space of Calabi–Yau threefolds and fourfolds, we proved that the 'partial Hodge metric' is the same as the Hodge metric, up to a constant.

Perhaps it is useful to make further comments on the motivations of this paper. We go back to the idea of Griffiths. In [**4**, **5**], Griffiths defined the period map. It is a holomor-

* Schumacher's method also yields the curvature formula in the case of $c_1 = 0$.

phic map from a moduli space to the 'classifying space' defined by Griffiths. The image of the period map is an integral subvariety of the horizontal distribution by the Griffiths transversality. The idea of Griffiths is that by studying the integral submanifold of the horizontal distribution, one can partially recover the properties of the moduli space without having the knowledge of the varieties the moduli space parametrized.

In the case of moduli space of polarized Calabi–Yau manifolds, we can do better. By a theorem of Tian [**25**], the Weil–Petersson metric can be defined by the curvature of the first Hodge bundle. This implies that the Weil–Petersson metric can be defined without the detailed knowledge of the Calabi–Yau manifolds. The presence of the Weil–Petersson metric gives severe restrictions on integral submanifold of the horizontal distribution.

In § 8, we define the Weil–Petersson geometry. This is defined to be an integral submanifold of the horizontal distribution with the Weil–Petersson metric on it. We further axiomatize the results of Viehweg [**27**] and Schmid [**20**] in defining the Weil–Petersson geometry. Of course, the axioms will give further restrictions of the integral submanifolds of the horizontal distribution. It has not been comprehensively studied how these results interact with the geometry of the integral submanifolds with the Weil–Petersson metrics.

One of the motivation of this paper is to make a firm foundation to study these interactions.

Before giving the main results of this paper, we give a short definition of Weil–Petersson, Hodge and partial Hodge metrics. For detailed definitions, see [**25**, **26**] for Weil–Petersson metrics, [**12**, **13**] for Hodge metrics and § 4 for partial Hodge metrics.

All of these three metrics are Hodge theoretic in the sense that they depend on the variation of the Hodge structures only. Let F^n be the first Hodge bundle over $\mathcal{M}$. Then the Weil–Petersson metric is defined as

$$\omega_{\mathrm{WP}} = c_1(F^n) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log Q(\Omega,\bar{\Omega}),$$

where Ω is a local holomorphic non-zero section of F^n and Q is the polarization (see (2.2)). The Hodge metric is defined as follows. Given the period map

$$\mathcal{M} \to D,$$

where D is the Griffiths classifying space, let $D = G/V$ for the real semisimple group G defined by the polarization Q. Let K be the connected components of the maximal compact subgroup of G containing V. The space G/K is a (Riemannian) symmetric space which carries the unique invariant metric $\mathrm{d}s^2$ (up to a constant). Let π be the composition map $\mathcal{M} \to D \to G/K$. Then the Hodge metric is defined by $\pi^*(\mathrm{d}s^2)$. It is Kählerian.

The partial Hodge metric is defined by

$$\omega_\mu = \mu\omega_{\mathrm{WP}} + \mathrm{Ric}(\omega_{\mathrm{WP}})$$

for positive number $\mu > m+1$, where m is the dimension of the moduli space. In the case of Calabi–Yau threefolds and fourfolds, with the suitable choice of μ, the partial Hodge metric is the Hodge metric. If the dimension of the Calabi–Yau manifolds is greater than or equal to five, there is no direct link between the Hodge and the partial Hodge metric.

As the first result of this paper, we have the following explicit formula for the curvature of the partial Hodge metric.

Theorem 1.1. *Let $\mathcal{M}$ be a moduli space of polarized Calabi–Yau manifolds. Let the dimension of $\mathcal{M}$ be m. Let ω_{WP} be the Kähler form of the Weil–Petersson metric. Then the metric $\omega_\mu = \mu\omega_{\mathrm{WP}} + \mathrm{Ric}(\omega_{\mathrm{WP}})$ is Kähler for $\mu > m+1$ and the curvature tensor of ω_μ is*

$$\begin{aligned}\tilde{R}_{i\bar{j}k\bar{l}} = {} & (\mu - m - 1)(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}) - (\mu - m)F_{i\bar{j}k\bar{l}} + F_{i\bar{q}\alpha\bar{l}}F_{p\bar{j}k\bar{\beta}}g^{\alpha\bar{\beta}}g^{p\bar{q}} \\ & + F_{\alpha\bar{q}k\bar{l}}F_{i\bar{j}p\bar{\beta}}g^{\alpha\bar{\beta}}g^{p\bar{q}} + \frac{(D_kD_\alpha D_i\Omega, D_l\overline{D_\beta}D_j\Omega)}{(\Omega,\bar{\Omega})}g^{\alpha\bar{\beta}} \\ & + \frac{(E_{k\alpha i}, \overline{E_{l\beta j}})}{(\Omega,\bar{\Omega})}g^{\alpha\bar{\beta}} - h^{s\bar{t}}\frac{(E_{k\alpha i}, D_{\beta}\bar{D}_t\Omega)}{(\Omega,\bar{\Omega})}\frac{(D_\gamma D_s\Omega, \overline{E_{l\tau j}})}{(\Omega,\bar{\Omega})}g^{\alpha\bar{\beta}}g^{\gamma\bar{\tau}}. \qquad (1.1)\end{aligned}$$

(For notations, see § 4.)

The obvious feature of the above expression is that the high-order terms of $R_{i\bar{i}i\bar{i}}$ dominates the high-order terms of the rest of the curvature tensor. Using this, we can control the Riemannian sectional curvature by the scalar curvature in the case of Calabi–Yau threefolds (cf. [**13**]) and of Calabi–Yau fourfolds (Theorem 5.4).

In the case of moduli space of Calabi–Yau fourfolds, we have the following result in § 4.

Theorem 1.2. *We use the notations as in the above theorem. Let $\mu = m+2$. Then the bisectional curvature of the Kähler metric ω_μ is non-positive. The Ricci and the holomorphic sectional curvature are all negatively bounded by the constant $-1/(m+4)$, where m is the complex dimension of the moduli space. Furthermore, the partial Hodge metric is the Hodge metric in the case of moduli space of Calabi–Yau fourfolds, up to a constant.*

Remark 1.3. The Hodge metric was first defined in [**12**]. Using Theorem 6.2, one can prove that it is Kähler. The fact that the holomorphic sectional curvature is negative away from zero also follows from the classical paper of Griffiths and Schmid [**7**]. The non-positivity of the holomorphic bisectional curvature is from [**12**]. The contribution here is that we find the explicit relation between the Hodge metric and the Weil–Petersson metric in the moduli space of Calabi–Yau fourfolds, and we find out the optimal constant for the upper bound of the holomorphic sectional curvature of the Hodge metric.

We remark that the corresponding result of Theorem 1.2 in the case of Calabi–Yau threefold was proved in [**13**]. In the fourfold case, we do not have the result of Bryant and Griffiths [**1**] about the integral submanifold of the horizontal distribution. However, we are still able to prove that in the case of fourfold, the 'partial Hodge metric' is the Hodge metric.

Using the above theorems and the Schwarz–Yau Lemma, we have the following global result in § 5.

Theorem 1.4. *Let $\mathcal{M}$ be the moduli space of the polarized Calabi–Yau manifolds. Then the Hodge volume on any subvarieties of $\mathcal{M}$ is finite. The Riemannian sectional curvature is L^1 bounded with respect to the Weil–Petersson metric on any subvarieties. In particular, the moduli space of the polarized Calabi–Yau manifolds has finite Weil–Petersson volume.*

Remark 1.5. Since we do not know the boundedness of the curvatures of the Weil–Petersson or Hodge metric at infinity, it seems to be interesting to prove that the integral of the curvature is bounded. In the one-dimensional case, if a complete Riemann surface has bounded total Gauss curvature, then it is S^2 removing finite many points. In high dimensions, we wish to find the geometric implications of the fact that the total curvature is finite.

A more ambitious problem is to prove that the volume and the integration of the curvatures of the Weil–Petersson metric are rational numbers. The same problem on the moduli space of curves was studied by many people (cf. [**10**,**16**,**22**,**32**–**34**]). The difficulty in the case of moduli space of Calabi–Yau manifolds is that the compactification is not known to be 'good' in the sense of Mumford [**18**, § 1]. The results of the Weil–Petersson volume on moduli space of Calabi–Yau manifolds will be in our next paper [**14**].

In the second part of this paper, we study the asymptotic behaviour of the curvature of the Hodge metric at infinity for moduli space of dimension one. The problem is related to the compactification of the moduli space of Calabi–Yau manifolds. By the theorem of Viehweg [**27**], the moduli space is a quasi-projective variety. Other than this result, we do not know much of the asymptotic behaviour of the moduli space. Yau suggested that one can compactify the moduli space by completing the moduli space using the Weil–Petersson metric first and then compactifying it. Under his suggestion, we study the problem. It seems to us that it is easier to complete the moduli space using the Hodge metric. After the completion of the moduli space using the Hodge metric, one would get a metric space which is not worse than a complex orbifold. We wish to study the curvature of the Hodge metric near the infinity of the moduli space in order to study the Siegel-type theorem [**17**] and wish, by using this, we can give a differential geometric proof of the compactification theorem of Viehweg. The full results will appear at [**14**]. In this paper, we have the following result.

Theorem 1.6. *Assume the moduli space $\mathcal{M}$ of polarized Calabi–Yau threefolds is one dimensional. If Δ^* is a holomorphic chart of $\mathcal{M}$ such that Δ^* is complete at 0 with respect to the Hodge metric, then the Gauss curvature of the Hodge metric is bounded*.*

2. Preliminaries

Let X be a compact Kähler manifold of dimension n. A C^∞ form on X decomposes into (p,q)-components according to the number of $\mathrm{d}z$ and $\mathrm{d}\bar{z}$. Denoting the C^∞ n-forms and the $C^\infty(p,q)$ forms on X by $A^n(X)$ and $A^{p,q}(X)$, respectively, we have the following

* The referee pointed out that the result is also true for partial Hodge metric.

decomposition:

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X).$$

The cohomology group is defined as

$$\begin{aligned} H^{p,q}(X) &= \{\text{closed } (p,q)\text{-forms}\}/\{\text{exact } (p,q)\text{-forms}\} \\ &= \{\varphi \in A^{p,q}(X) \mid \mathrm{d}\varphi = 0\}/\mathrm{d}A^{n-1}(X) \cap A^{p,q}(X). \end{aligned}$$

The relations between the groups $\{H^{p,q}(X)\}$ and the de Rham cohomology is the following Hodge decomposition.

Theorem 2.1 (Hodge Decomposition Theorem). *Let X be a compact Kähler manifold of dimension n. Then the nth complex de Rham cohomology group of X can be written as the direct sum*

$$H^n(X,\mathbb{Z}) \otimes \mathbb{C} = H^n_{\mathrm{DR}}(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X). \tag{2.1}$$

A $(1,1)$ form ω is called a polarization of X if $[\omega]$ is the first Chern class of an ample line bundle over X. The pair (X,ω) is called a polarized algebraic variety.

Using ω, one can define

$$L : H^k(X,\mathbb{C}) \to H^{k+2}(X,\mathbb{C}), \qquad [\alpha] \mapsto [\alpha \wedge \omega]$$

to be the multiplication by ω for $k = 0,\ldots,2n-2$.

The following two famous Lefschetz Theorems give a filtration of the Hodge groups and thus are extremely important in defining the classifying space and the period map.

Theorem 2.2 (hard Lefschetz Theorem). *On a polarized algebraic variety (X,ω) of dimension n,*

$$L^k : H^{n-k}(X,\mathbb{C}) \to H^{n+k}(X,\mathbb{C})$$

is an isomorphism for every positive integer $k \leqslant n$.

The primitive cohomology $P^k(X,\mathbb{C})$ is then defined to be the kernel of L^{n-k+1} on $H^k(X,\mathbb{C})$.

Theorem 2.3 (Lefschetz Decomposition Theorem). *On a polarized algebraic variety (X,ω) of dimension n, we have the following decomposition:*

$$H^n(X,\mathbb{C}) = \bigoplus_{k=0}^{[n/2]} L^k P^{n-2k}(X,\mathbb{C}).$$

Let $H_Z = P^n(X,\mathbb{C}) \cap H^n(X,\mathbb{Z})$ and $H^{p,q} = P^n(X,\mathbb{C}) \cap H^{p,q}(X)$ for $0 \leqslant p,q \leqslant n$. Then we have

$$H_Z \otimes \mathbb{C} = \sum H^{p,q}, \quad H^{p,q} = \overline{H^{q,p}},$$

for $p+q=n$. Set $H = H_Z \otimes \mathbb{C}$. We call $\{H^{p,q}\}$ the Hodge decomposition of H.

Remark 2.4. We define a filtration of $H_Z \otimes \mathbb{C} = H$ by

$$0 \subset F^n \subset F^{n-1} \subset \cdots \subset F^1 = H$$

such that

$$H^{p,q} = F^p \cap \overline{F^q}, \qquad F^p \oplus \overline{F^{n-p+1}} = H.$$

The sets $\{H^{p,q}\}$ and $\{F^p\}$ are equivalent in describing the Hodge decomposition of H (cf. (2.1)). We will use both notations interchangeably for the rest of this paper.

Now suppose that Q is the quadratic form on H_Z induced by the cup product of the cohomology group $H^n_{\mathrm{DR}}(X, \mathbb{C})$. Q can be represented by

$$Q(\varphi, \psi) = (-1)^{n(n-1)/2} \int_X \varphi \wedge \psi \tag{2.2}$$

for $\varphi, \psi \in H$. Q is a non-degenerate quadratic form, and is skew-symmetric if n is odd and is symmetric if n is even. On H, the form Q satisfies the two Hodge–Riemann relations on the space $H^{p,q}$ of primitive harmonic (p,q) forms:

(1) $Q(H^{p,q}, H^{p',q'}) = 0$ unless $p' = n - p$, $q' = n - q$; and

(2) $(\sqrt{-1})^{p-q} Q(\varphi, \bar{\varphi}) > 0$ for any non-zero element $\varphi \in H^{p,q}$.

Definition 2.5. A polarized Hodge structure of weight n, denoted by $\{H_Z, F^p, Q\}$, is given by a lattice H_Z, a filtration of $H = H_Z \otimes \mathbb{C}$,

$$0 \subset F^n \subset F^{n-1} \subset \cdots \subset F^0 \subset H,$$

such that

$$H = F^p \oplus \overline{F^{n-p+1}},$$

together with a bilinear form

$$Q : H_Z \otimes H_Z \to \mathbb{Z},$$

which is skew-symmetric if n is odd and symmetric if n is even such that it satisfies the two Hodge–Riemann relations:

(3) $Q(F^p, F^{n-p+1}) = 0$ for $p = 1, \ldots n$; and

(4) $(\sqrt{-1})^{p-q} Q(\varphi, \bar{\varphi}) > 0$ if $\varphi \in H^{p,q}$ and $\varphi \neq 0$, where $H^{p,q}$ is defined by

$$H^{p,q} = F^p \cap \overline{F^q}$$

for $p + q = n$.

Definition 2.6. The classifying space D for the polarized Hodge structure is the set of all filtrations

$$0 \subset F^n \subset \cdots \subset F^1 \subset H, \qquad F^p \oplus \overline{F^{n-p+1}} = H,$$

or the set of all the decompositions

$$\sum H^{p,q} = H, \quad H^{p,q} = \overline{H^{q,p}},$$

on which Q satisfies the two Hodge–Riemann relations (1), (2) or (3), (4) above.

Let

$$G_{\mathbb{R}} = \{\xi \in \mathrm{Hom}(H_{\mathbb{R}}, H_{\mathbb{R}}) \mid Q(\xi\varphi, \xi\psi) = Q(\varphi, \psi)\}. \tag{2.3}$$

Then D can also be written as the homogeneous space

$$D = G/V, \tag{2.4}$$

where V is the compact subgroup of G which leaves a fixed Hodge decomposition $\{H^{p,q}\}$ invariant. Apparently, G is a semisimple real Lie group.

Over the classifying space D we have the holomorphic vector bundles $\underline{F}^n, \ldots, \underline{F}^1, \underline{H}$ whose fibres at each point are the vector spaces $F^n, \ldots, F^1, H$, respectively. These bundles are called Hodge bundles.

In § 6, we identify the holomorphic tangent bundle $T^{1,0}(D)$ as a subbundle of $\mathrm{Hom}(\underline{H}, \underline{H})$,

$$T^{1,0}(D) \subset \oplus \,\mathrm{Hom}(\underline{F}^p, \underline{H}/\underline{F}^p) = \bigoplus_{r>0} \mathrm{Hom}(\underline{H}^{p,q}, \underline{H}^{p-r,q+r}),$$

such that the following compatible condition holds:

$$\begin{array}{ccc} F^p & \longrightarrow & F^{p-1} \\ \big\downarrow & & \big\downarrow \\ H/F^p & \longleftarrow & H/F^{p-1} \end{array}$$

Definition 2.7. A subbundle $T_{\mathrm{h}}(D)$ is called the horizontal distribution of D if

$$T_{\mathrm{h}}(D) = \{\xi \in T^{1,0}(D) \mid \xi F^p \subset F^{p-1},\ p = 1, \ldots, n\}.$$

For any point $x \in D$ such that x is defined as subspaces $\{H^{p,q}\}$ of H, define the two vector spaces

$$\begin{aligned} H^+ &= H^{n,0} + H^{n-2,2} + \cdots, \\ H^- &= H^{n-1,1} + H^{n-3,3} + \cdots. \end{aligned}$$

We fix a point $x_0 \in D$. Suppose that the corresponding vector spaces are $\{H_0^{p,q}\}$ and $\{H_0^+, H_0^-\}$. Define K to be the connected compact subgroup of G leaving H_0^+ invariant. We give the basic properties of the classifying spaces in the following three lemmas. The proofs are easy and are omitted.

Lemma 2.8. *K is the maximal compact subgroup of G containing V. In particular, V itself is a compact subgroup.*

Define the Weil operator

$$C : H^{p,q} \to H^{p,q}, \qquad C|_{H^{p,q}} = (\sqrt{-1})^{p-q}.$$

Then we have

$$C|_{H^+} = (\sqrt{-1})^n, \qquad C|_{H^-} = -(\sqrt{-1})^n.$$

Let

$$Q_1(x,y) = Q(Cx, \bar{y}).$$

Then we have the following result.

Lemma 2.9. *Q_1 is an Hermitian inner product.*

Lemma 2.10. *Let*

$$D_1 = \{H^{n,0} + H^{n-2,2} + \cdots \mid \{H^{p,q}\} \in D\}.$$

Then the group G acts on D_1 transitively with the stable subgroup K at H_0^+, and D_1 is a (Riemannian) symmetric space.

Definition 2.11. We call map p,

$$p : G/V \to G/K, \qquad \{H^{p,q}\} \mapsto H^{n,0} + H^{n-2,2} + \cdots,$$

the natural projection of the classifying space. Using the notation of coset, $p(aV) = aK$ for any $a \in G$.

With the above discussions, we can prove the following result.

Proposition 2.12. *Suppose $T_v(D)$ is the distribution of the tangent vectors of the fibres of the canonical map*

$$p : D \to G/K.$$

Then

$$T_v(D) \cap T_{\mathrm{h}}(D) = \{0\}.$$

Proof. Let $\mathfrak{g}$ be the Lie algebra of the Lie group G. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition such that $\mathfrak{k}$ is the Lie algebra of K. Then

$$T_v(D) = G \times_V \mathfrak{v}_1,$$

where $\mathfrak{k} = \mathfrak{v} + \mathfrak{v}_1$ and $\mathfrak{v}_1$ is the orthonormal complement of the Lie algebra $\mathfrak{v}$ of V. On the other hand, $T_{\mathrm{h}}(D) \subset G \times_V \mathfrak{p}$. So we have $T_v(D) \cap T_{\mathrm{h}}(D) = \{0\}$. □

Definition 2.13. A horizontal slice $\mathcal{M}$ of D is a complex integral submanifold of the distribution $T_{\mathrm{h}}(D)$.

Definition 2.14. Let U be an open neighbourhood of the universal deformation space of X. Assume that U is smooth. Then for each X' near X, we have an isomorphism $H^n(X', \mathbb{C}) = H^n(X, \mathbb{C})$. Under this isomorphism, $\{H^{p,q}(X') \cap P^n(X', \mathbb{C})\}_{p+q=n}$ can be considered as a point of D. The map

$$U \to D, \qquad X' \mapsto \{H^{p,q}(X') \cap P^n(X', \mathbb{C})\}_{p+q=n}$$

is called the period map. If $\Gamma_1 \to \Gamma$ is a homomorphism between two discrete groups and the period map is equivariant with respect to the two groups, then we also call the induced map

$$\Gamma_1 \setminus U \to \Gamma \setminus D$$

a period map.

The most important property of the period map is the following [**6**].

Theorem 2.15 (Griffiths). *The period map $p : U \to D$ is holomorphic. Furthermore, it is an immersion and $p(U)$ is a horizontal slice of the classifying space.*

From the above theorem and Proposition 2.12 in this section, we can prove the following.

Corollary 2.16. *With the notations as above, the map*

$$p : U \subset D \to D_1 = G/K$$

is a (*real*) *immersion.*

Definition 2.17. Using the above notations, let h be the invariant Kähler metric on D_1. The Hodge metric is defined as the (Riemannian) metric p^*h on the horizontal slice U.

Remark 2.18. In [**12**], the first author proved that the Hodge metric of U is Kähler.

Now we introduce the Nilpotent Orbit Theorem of Schmid [**20**]. Let $f : \mathcal{X} \to S$ be a family of compact Kähler manifolds. In order to study the degeneration of the variation of the Hodge structure, we let $S = \Delta^{*l} \times \Delta^{m-l}$, where $l \geqslant 1$, $m \geqslant l$, and Δ and Δ^* are the unit disk and the punctured unit disk in the complex plane, respectively. Consider the period map

$$\Phi : \Delta^{*l} \times \Delta^{m-l} \to \Gamma \setminus D.$$

By going to the universal covering $U^l \times \Delta^{m-l}$, one can lift Φ to a mapping

$$\tilde{\Phi} : U^l \times \Delta^{m-l} \to D,$$

where U is the upper half plane. Corresponding to each of the first l variables, we choose a monodromy transformation $T_i \in \Gamma$, where Γ is the monodromy group, so that

$$\tilde{\Phi}(z_1, \dots, z_i + 1, \dots z_l, w_{l+1}, \dots, w_m) = T_i \circ \tilde{\Phi}(z_1, \dots, z_l, w_{l+1}, \dots, w_m),$$

holds identically in all variables. The T_i commute with each other. We know that all the eigenvalues of the T_i are roots of unity. Let $T_i = T_{i,\mathrm{s}} T_{i,\mathrm{u}}$ be the Jordan decomposition where $T_{i,\mathrm{s}}$ is semisimple and $T_{i,\mathrm{u}}$ is unipotent. We also assume that $T_{i,\mathrm{s}}^{s_i} = I$ for some positive integer s_i, so that we can define

$$N_i = \frac{1}{s_i} \log T_i^{s_i} = \sum_{k \geqslant 1} (-1)^{k+1} \frac{1}{k} (T_i^{s_i} - I)^k.$$

All the N_i are commutative.

Let $z = (z_1, \dots, z_l)$, $sz = (s_1 z_1, \dots, s_l z_l)$ and $w = (w_{l+1}, \dots, w_m)$. The map

$$\tilde{\Psi}(z, w) = \exp\left(- \sum_{i=1}^{l} s_i z_i N_i \right) \circ \tilde{\Phi}(sz, w)$$

remains invariant under the translation $z_i \mapsto z_i + 1$, $1 \leqslant i \leqslant l$. It follows that $\tilde{\Psi}$ drops to a mapping

$$\Psi : \Delta^{*l} \times \Delta^{m-l} \to \check{D}.$$

Theorem 2.19 (Nilpotent Orbit Theorem [20]). *The map* Ψ *extends holomorphically to* Δ^{m}*. For* $w \in \Delta^{m-l}$*, the point*

$$a(w) = \Psi(0, w) \in \check{D}$$

is left fixed by $T_{i,\mathrm{s}}$, $1 \leqslant i \leqslant l$*. For any given number* η *with* $0 < \eta < 1$*, there exist constants* $\alpha, \beta \geqslant 0$*, such that under the restrictions*

$$\operatorname{Im} z_i \geqslant \alpha, \quad 1 \leqslant i \leqslant l, \qquad \text{and} \qquad |w_j| \leqslant \eta, \quad l+1 \leqslant j \leqslant m,$$

the point $\exp(\sum_{i=1}^{l} z_i N_i) \circ a(w)$ *lies in* D *and satisfies the inequality*

$$d\left(\exp\left(\sum_{i=1}^{l} z_i N_i\right) \circ a(w), \tilde{\Phi}(z, w)\right) \leqslant \left(\prod_{i=1}^{l} \operatorname{Im} z_i\right)^{\beta} \sum_{i=1}^{l} \exp(-2\pi s_i^{-1} \operatorname{Im} z_i);$$

here, d *is the* G_R *invariant Riemannian distance function on* D*. Finally, the mapping*

$$(z, w) \mapsto \exp\left(\sum_{i=1}^{l} z_i N_i\right) \circ a(w)$$

is horizontal.

Now we assume that the generic fibre X of the map $f : \mathcal{X} \to S$ is a polarized Calabi–Yau manifold. For the sake of simplicity, we assume that X is compact, simply connected and algebraic with $c_1(X) = 0$. By a theorem of Tian [**25**]* (see also [**26**]), the universal deformation space of X is smooth. Since there are no non-zero holomorphic vector fields on a Calabi–Yau manifold, the moduli space of polarized Calabi–Yau manifolds is an orbifold. The following important theorem of Viehweg gives the compactification of the moduli space.

Theorem 2.20 (cf. Theorem 1.13 on p. 21 of [27]). *Let* $\mathcal{M}$ *be the moduli space of polarized Calabi–Yau manifolds and the line bundle* $\underline{F}^n$ *is the Hodge bundle defined right after Definition 2.6. Then* $\mathcal{M}$ *is quasi-projective and the line bundle* $\underline{F}^n$ *extends to an ample line bundle over* $\bar{\mathcal{M}}$*, the compactification of* $\mathcal{M}$*.*

With the classical Hironaka Theorem [**8**], we have the following result.

Corollary 2.21. *Let* $\bar{\mathcal{M}}$ *be the compactification of the moduli space* $\mathcal{M}$ *in the above sense. Then after a smooth resolution, one can assume that* $\bar{\mathcal{M}} \setminus \mathcal{M}$ *is a divisor of normal crossing. In other words, let* $x_0 \in \bar{\mathcal{M}} \setminus \mathcal{M}$*. Then in a neighbourhood of* x_0*, we can write* $\mathcal{M}$ *as*

$$\Delta^{*l} \times (\Delta)^{m-l},$$

where m *is the complex dimension of* $\bar{\mathcal{M}}$*.*

* Tian's proof is more general since one merely assumes the $\partial\bar{\partial}$-lemma hold for X. That is equivalent to assume that the Hodge–de Rham spectral sequence for X degenerates at the E_1 term. See the survey paper of Friedman [**3**] for details.

3. Curvature of Weil–Petersson metrics

For the rest of this paper, we assume that $\mathcal{M}$ is the moduli space of polarized Calabi–Yau manifolds of dimension $n > 2^*$.

Remark 3.1. The following notations and conventions will be used through out the rest of this paper.

Form a Kähler manifold M with metric $g_{i\bar{j}}$, the curvature tensor is given by

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \overline{z_l}} - g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \overline{z_l}}.$$

Using this convention, the Ricci curvature is

$$R_{i\bar{j}} = -g^{k\bar{l}} R_{i\bar{j}k\bar{l}}.$$

Furthermore, the Christoffel symbol of this metric is given by

$$\Gamma_{ij}^k = g^{k\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_j}.$$

We also have

$$\frac{\partial \Gamma_{ij}^k}{\partial \overline{z_j}} = g^{k\bar{q}} R_{i\bar{l}j\bar{q}}.$$

So the holomorphic bisectional curvature of this metric is non-positive, which means $R_{i\bar{i}j\bar{j}} \geqslant 0$ for all i, j.

Now let Ω be a non-zero local holomorphic section of the Hodge bundle $\underline{F}^n$. In this section, we assume $1 \leqslant i, j \leqslant m$ unless otherwise stated, where m is the dimension of the moduli space $\mathcal{M}$. We set†

$$(\Omega, \bar{\Omega}) = (\sqrt{-1})^n Q(\Omega, \bar{\Omega}). \tag{3.1}$$

By the Hodge–Riemann relations we know that $(\Omega, \bar{\Omega}) > 0$. In local coordinates, the Weil–Petersson metric is given by

$$g_{i\bar{j}} = -\partial_i \overline{\partial_j} \log(\Omega, \bar{\Omega}) = -\frac{(\partial_i \Omega, \overline{\partial_j \Omega})}{(\Omega, \bar{\Omega})} + \frac{(\partial_i \Omega, \bar{\Omega})(\Omega, \overline{\partial_j \Omega})}{(\Omega, \bar{\Omega})^2}, \tag{3.2}$$

where ∂_i, $\overline{\partial_j}$ are the operators $\partial/\partial z_i$, $\partial/\partial \bar{z}_j$, respectively. From [**25**], we know that the definition is the same as the Weil–Petersson metric defined in the classical way. In this section, we compute the curvature of the Weil–Petersson metric. We begin with defining

$$K_i = -\partial_i \log(\Omega, \bar{\Omega}) = -\frac{(\partial_i \Omega, \bar{\Omega})}{(\Omega, \bar{\Omega})} \tag{3.3}$$

* For $K3$ surfaces, the Weil–Petersson metric is half of the Hodge metric. Thus we omit this case.
† In fact, we use the notation $(\xi, \eta) = (\sqrt{-1})^n Q(\xi, \eta)$ in the rest of this paper where ξ, η are n-forms. The bilinear form $(\cdot, \cdot)$ is not necessary positive-definite.

and

$$D_i\Omega = \partial_i\Omega + K_i\Omega \tag{3.4}$$

for $1 \leqslant i \leqslant m$. Then $g_{i\bar{j}} = \bar{\partial}_j K_i$.

Lemma 3.2. *Under the notions as above, the following properties hold:*

(1) $(D_i\Omega, \bar{\Omega}) = 0$;

(2) $\bar{\partial}_j D_i\Omega = g_{i\bar{j}}\Omega$; *and*

(3) $g_{i\bar{j}} = -(D_i\Omega, D_{\bar{j}}\Omega)/(\Omega, \bar{\Omega})$,

where $1 \leqslant i,\ j \leqslant m$.

Proof. By (3.3) and (3.4), we have

$$(D_i\Omega, \bar{\Omega}) = (\partial_i\Omega + K_i\Omega, \bar{\Omega}) = (\partial_i\Omega, \bar{\Omega}) - \frac{(\partial_i\Omega, \bar{\Omega})}{(\Omega, \bar{\Omega})}(\Omega, \bar{\Omega}) = 0,$$

which proves (1). Property (2) follows from

$$\bar{\partial}_j D_i\Omega = \bar{\partial}_j\partial_i\Omega + (\bar{\partial}_j K_i)\Omega = g_{i\bar{j}}\Omega.$$

Combining the above two equations with (3.2) we have

$$g_{i\bar{j}} = \frac{(\bar{\partial}_j D_i\Omega, \bar{\Omega})}{(\Omega, \bar{\Omega})} = -\frac{(D_i\Omega, \overline{\partial_j\Omega})}{(\Omega, \bar{\Omega})} = -\frac{(D_i\Omega, \overline{D_j\Omega})}{(\Omega, \bar{\Omega})} + \frac{(D_i\Omega, \overline{K_j\Omega})}{(\Omega, \bar{\Omega})} = -\frac{(D_i\Omega, \overline{D_j\Omega})}{(\Omega, \bar{\Omega})}.$$

This finishes the proof. □

From the above lemma, we see that $D_i\Omega$ is the projection of $\partial_i\Omega$ into $H^{n-1,1}$ with respect to the quadratic form $(\cdot,\cdot)$. Now we consider the projection of $\partial_j D_i\Omega$ into $H^{n-2,2}$. In the following we will use Γ_{ij}^k to denote the Christoffel symbol of the Weil–Petersson metric. Let

$$D_j D_i\Omega = \partial_j D_i\Omega - \sum_k \Gamma_{ij}^k D_k\Omega + K_j D_i\Omega. \tag{3.5}$$

Lemma 3.3. *Using the same notations as above, for any* $1 \leqslant i, j, l \leqslant m$, *we have*

(1) $(D_j D_i\Omega, \bar{\Omega}) = 0$;

(2) $(D_j D_i\Omega, \overline{D_l\Omega}) = 0$; *and*

(3) $D_j D_i\Omega = D_i D_j\Omega$.

Proof. A straightforward computation gives

$$(D_j D_i \Omega, \bar{\Omega}) = (\partial_j D_i \Omega, \bar{\Omega}) - \sum_k \Gamma_{ij}^k (D_k \Omega, \bar{\Omega}) + K_j (D_i \Omega, \bar{\Omega}) = \partial_j (D_i \Omega, \bar{\Omega}) = 0,$$

where in the last equality, we used (1) of Lemma 3.2. This proves (1). Using Lemma 3.2, we have

$$\begin{aligned}
&(D_j D_i \Omega, \overline{D_l \Omega}) \\
&\quad = (\partial_j D_i \Omega, \overline{D_l \Omega}) - \sum_k \Gamma_{ij}^k (D_k \Omega, \overline{D_l \Omega}) + K_j (D_i \Omega, \overline{D_l \Omega}) \\
&\quad = \partial_j (-g_{i\bar{l}} (\Omega, \bar{\Omega})) - (D_i \Omega, \overline{\bar{\partial}_j D_l \Omega}) + \sum_k \Gamma_{ij}^k g_{k\bar{l}} (\Omega, \bar{\Omega}) - K_j g_{i\bar{l}} (\Omega, \bar{\Omega}) \\
&\quad = -\partial_j g_{i\bar{l}} (\Omega, \bar{\Omega}) - g_{i\bar{l}} (\partial_j \Omega, \bar{\Omega}) - (D_i \Omega, g_{j\bar{l}} \bar{\Omega}) + \partial_j g_{i\bar{l}} (\Omega, \bar{\Omega}) - g_{i\bar{l}} (K_j \Omega, \bar{\Omega}) \\
&\quad = -g_{i\bar{l}} (D_j \Omega, \bar{\Omega}) - g_{j\bar{l}} (D_i \Omega, \bar{\Omega}) \\
&\quad = 0.
\end{aligned}$$

This proves (2). To prove (3), we see that

$$\begin{aligned}
D_j D_i \Omega &= \partial_j D_i \Omega - \sum_k \Gamma_{ij}^k D_k \Omega + K_j D_i \Omega \\
&= \partial_j \partial_i \Omega + K_i \partial_j \Omega - \sum_k \Gamma_{ij}^k D_k \Omega + K_j \partial_i \Omega + K_j K_i \Omega \\
&\qquad - \frac{(\partial_j \partial_i \Omega, \bar{\Omega})}{(\Omega, \bar{\Omega})} \Omega + \frac{(\partial_i \Omega, \bar{\Omega})(\partial_j \Omega, \bar{\Omega})}{(\Omega, \bar{\Omega})^2} \Omega.
\end{aligned}$$

Thus (3) follows from the fact that the above formula is symmetric with respect to i and j. □

Let $R_{i\bar{j}k\bar{l}}$ be the curvature tensor of $g_{i\bar{j}}$. Then we have the following [**29**].

Theorem 3.4. *Let $(g_{i\bar{j}})_{m\times m}$ be the Weil–Petersson metric and let $D_j D_i \Omega$ be defined as in (3.5). Then the Weil–Petersson metric is Kähler [**25**], and the curvature tensor is*

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}} - \frac{(D_k D_i \Omega, \overline{D_l D_j \Omega})}{(\Omega, \bar{\Omega})} \tag{3.6}$$

for $1 \leqslant i, j, k, l \leqslant m$.

Proof. By definition,

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}_l}. \tag{3.7}$$

From (3) of Lemma 3.2, we know that

$$\frac{\partial g_{i\bar{j}}}{\partial z_k} = -\frac{(\partial_k D_i \Omega, \overline{D_j \Omega})}{(\Omega, \bar{\Omega})} - \frac{(D_i \Omega, \partial_k \overline{D_j \Omega})}{(\Omega, \bar{\Omega})} + \frac{(D_i \Omega, \overline{D_j \Omega})}{(\Omega, \bar{\Omega})^2} (\partial_k \Omega, \bar{\Omega}).$$

By (1) of Lemma 3.2, we get

$$\frac{(D_i\Omega, \partial_k \overline{D_j\Omega})}{(\Omega, \bar{\Omega})} = \frac{(D_i\Omega, \overline{\bar{\partial}_k D_j\Omega})}{(\Omega, \bar{\Omega})} = \frac{(D_i\Omega, g_{k\bar{j}}\bar{\Omega})}{(\Omega, \bar{\Omega})} = 0.$$

Using the definition of K_i, we have

$$\frac{\partial g_{i\bar{j}}}{\partial z_k} = -\frac{(\partial_k D_i\Omega + K_k D_i\Omega, \overline{D_j\Omega})}{(\Omega, \bar{\Omega})}.$$

Let $A_{ij} = \partial_i D_j\Omega + K_i D_j\Omega = D_i D_j\Omega + \Gamma_{ij}^k D_k\Omega$. Then

$$\frac{\partial g_{i\bar{j}}}{\partial z_k} = -\frac{(A_{ki}, \overline{D_j\Omega})}{(\Omega, \bar{\Omega})}. \tag{3.8}$$

Similarly, we have

$$\frac{\partial g_{i\bar{j}}}{\partial \bar{z}_l} = -\frac{(D_i\Omega, \overline{A_{lj}})}{(\Omega, \bar{\Omega})}. \tag{3.9}$$

From (3.8) we have

$$\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} = -\frac{(\bar{\partial}_l A_{ki}, \overline{D_j\Omega})}{(\Omega, \bar{\Omega})} - \frac{(A_{ki}, \overline{\partial_l D_j\Omega})}{(\Omega, \bar{\Omega})} + \frac{(A_{ki}, \overline{D_j\Omega})}{(\Omega, \bar{\Omega})^2}(\Omega, \overline{\partial_l\Omega}). \tag{3.10}$$

We also have

$$\begin{aligned}
\bar{\partial}_l A_{ki} &= \bar{\partial}_l(\partial_k D_i\Omega + K_k D_i\Omega) \\
&= \partial_k(\bar{\partial}_l D_i\Omega) + (\bar{\partial}_l K_k) D_i\Omega + K_k \bar{\partial}_l D_i\Omega \\
&= \partial_k(g_{i\bar{l}}\Omega) + g_{k\bar{l}} D_i\Omega + K_k g_{i\bar{l}}\Omega \\
&= (\partial_k g_{i\bar{l}})\Omega + g_{i\bar{l}} D_k\Omega + g_{k\bar{l}} D_i\Omega
\end{aligned}$$

and

$$\frac{(A_{ki}, \overline{D_j\Omega})}{(\Omega, \bar{\Omega})^2}(\Omega, \overline{\partial_l\Omega}) = -\frac{(A_{ki}, \overline{K_l D_j\Omega})}{(\Omega, \bar{\Omega})}.$$

Thus from (3.10), we have

$$\begin{aligned}
\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} &= -\frac{((\partial_k g_{i\bar{l}})\Omega + g_{i\bar{l}} D_k\Omega + g_{k\bar{l}} D_i\Omega, \overline{D_j\Omega})}{(\Omega, \bar{\Omega})} - \frac{(A_{ki}, \overline{\partial_l D_j\Omega})}{(\Omega, \bar{\Omega})} - \frac{(A_{ki}, \overline{K_l D_j\Omega})}{(\Omega, \bar{\Omega})} \\
&= g_{i\bar{l}} g_{k\bar{j}} + g_{i\bar{j}} g_{k\bar{l}} - \frac{(A_{ki}, \overline{A_{lj}})}{(\Omega, \bar{\Omega})},
\end{aligned} \tag{3.11}$$

by using Lemma 3.2. Combining (3.8), (3.9) and (3.11), and using Lemma 3.3, we have

$$\begin{aligned}
R_{i\bar{j}k\bar{l}} &= g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}} - \frac{(A_{ki}, \overline{A_{lj}})}{(\Omega, \bar{\Omega})} + g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z_k} \frac{(D_p\Omega, \overline{A_{lj}})}{(\Omega, \bar{\Omega})} \\
&= g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}} - \frac{(A_{ik} - \Gamma_{ik}^p D_p\Omega, \overline{A_{lj}})}{(\Omega, \bar{\Omega})} \\
&= g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}} - \frac{(D_k D_i\Omega, \overline{D_l D_j\Omega})}{(\Omega, \bar{\Omega})}.
\end{aligned}$$

This finishes the proof. □

Remark 3.5. For the moduli space of Calabi–Yau threefolds, Strominger [**24**] proved that the curvature tensor is

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}} - \sum_{p,q} \frac{1}{(\Omega, \bar{\Omega})^2} g^{p\bar{q}} F_{ikp}\overline{F_{jlq}},$$

where F_{ikp} is the Yukawa coupling. In the case of Calabi–Yau threefolds, $D_iD_j\Omega \in H^{1,2}$. In fact, $D_iD_j\Omega$ is the orthogonal projection of $\partial_i\partial_j\Omega$ to $H^{1,2}$. Thus

$$(D_iD_k\Omega, \overline{D_jD_l\Omega}) = -\frac{(D_iD_k\Omega, D_p\Omega)(\overline{D_q\Omega}, \overline{D_jD_l\Omega})}{(\Omega, \bar{\Omega})} g^{p\bar{q}}.$$

It is easy to see that $(D_iD_k\Omega, D_p\Omega) = -F_{ikp}$. Thus our theorem is the same as Strominger's in the case of Calabi–Yau threefolds.

Remark 3.6. Theorem 3.4 was proved in [**21**] using the method of [**23**] which is different from ours. In his paper [**26**], Todorov introduced the geodesic coordinates from which it is much easier to get the curvature formula. The current proof was by Wang [**29**] which is purely Hodge theoretic. Such a proof can be generalized to general horizontal slice.

4. Partial Hodge metrics

We use the Ricci curvature of the Weil–Petersson metric to construct a new metric. Let ω_{WP} be the Kähler form of the Weil–Petersson metric and let $\mu > m+1$ be a real number. Let

$$\omega_\mu = \mu\omega_{\mathrm{WP}} + \mathrm{Ric}(\omega_{\mathrm{WP}}). \tag{4.1}$$

By Theorem 3.4, we know that ω_μ is a Kähler metric. We notice here that when the dimension of the Calabi–Yau manifolds is 3 or 4, by choosing suitable μ, the metric ω_μ coincides with the Hodge metric (cf. § 6). For this reason we call ω_μ the 'partial Hodge metric'. It is a metric between the Weil–Petersson metric and the Hodge metric.

In this section, unless otherwise stated, the subscripts $i, j, k, l, p, q, \alpha, \dots$ will range from 1 to m. Define a tensor

$$T_{k\alpha i} = \partial_k D_\alpha D_i\Omega + K_k D_\alpha D_i\Omega - \sum_p \Gamma^p_{\alpha k} D_p D_i \Omega - \sum_p \Gamma^p_{ik} D_\alpha D_p\Omega, \tag{4.2}$$

where $\Gamma^p_{\alpha k}$ is the Christoffel symbol of the Weil–Petersson metric and Ω is a non-zero local holomorphic section of $\underline{F}^n$ as in the previous section. We use $g_{i\bar{j}}$ and $h_{i\bar{j}}$ to denote the metric matrices of the Weil–Petersson metric and the metric ω_μ (for some chosen μ) in local coordinates $(z_1, \dots, z_m)$, respectively, and use $R_{i\bar{j}k\bar{l}}$ and $\tilde{R}_{i\bar{j}k\bar{l}}$ to denote their curvature tensors respectively. We also use $R_{i\bar{j}}$ to denote the Ricci tensor of the Weil–Petersson metric.

Let $D_kD_\alpha D_i\Omega$ be the projection of $T_{k\alpha i}$ into $H^{n-3,3}$ with respect to the quadratic form $(\cdot,\cdot)$ in (3.1). Let $E_{k\alpha i} = T_{k\alpha i} - D_kD_\alpha D_i\Omega$. Then we have the following result.

Lemma 4.1. *Using the same notations as above, we have*

$$T_{k\alpha i} = E_{k\alpha i} + D_k D_\alpha D_i \Omega \in H^{n-2,2} \oplus H^{n-3,3},$$

where $T_{k\alpha i}$ is defined in (4.2).

Proof. By definition of $T_{k\alpha i}$ and the Griffiths transversality,

$$T_{k\alpha i} \in H^{n,0} \oplus H^{n-1,1} \oplus H^{n-2,2} \oplus H^{n-3,3}.$$

Using Lemma 3.3, we have

$$(T_{k\alpha i}, \bar{\Omega}) = (\partial_k D_\alpha D_i \Omega, \bar{\Omega}) = 0.$$

So there is no $H^{n,0}$ components in $T_{k\alpha i}$. On the other hand, $H^{n-1,1}$ is spanned by $D_i\Omega$. Using Lemma 3.3 again, we have

$$(T_{k\alpha i}, \overline{D_j \Omega}) = 0.$$

Thus $T_{k\alpha i}$ has no $H^{n-1,1}$ component and this completes the proof. □

Define the curvature like tensor F by

$$F_{i\bar{j}k\bar{l}} = \frac{(D_k D_i \Omega, \overline{D_l D_j \Omega})}{(\Omega, \bar{\Omega})}. \tag{4.3}$$

Using Lemma 3.3 and the Hodge–Riemann relations we know that the tensor F has all symmetries that a curvature tensor has.

The Strominger formula (Theorem 3.4) can be written as

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}} - F_{i\bar{j}k\bar{l}}. \tag{4.4}$$

The curvature tensor of the partial Hodge metric is as follows.

Theorem 4.2. *The metric ω_μ is Kähler and the curvature tensor of ω_μ is*

$$\begin{aligned}
\tilde{R}_{i\bar{j}k\bar{l}} = {} & (\mu - m - 1)(g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}}) - (\mu - m) F_{i\bar{j}k\bar{l}} + \sum_{\alpha\beta pq} F_{i\bar{q}\alpha\bar{l}} F_{p\bar{j}k\bar{\beta}} g^{\alpha\bar{\beta}} g^{p\bar{q}} \\
& + \sum_{\alpha\beta pq} F_{\alpha\bar{q}k\bar{l}} F_{i\bar{j}p\bar{\beta}} g^{\alpha\bar{\beta}} g^{p\bar{q}} + \sum_{\alpha\beta} \frac{(D_k D_\alpha D_i \Omega, \overline{D_l D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \\
& + \sum_{\alpha\beta} \frac{(E_{k\alpha i}, \overline{E_{l\beta j}})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} - \sum_{\alpha\beta\gamma\tau st} h^{s\bar{t}} \frac{(E_{k\alpha i}, \overline{D_\beta D_t \Omega})}{(\Omega, \bar{\Omega})} \frac{(D_\gamma D_s \Omega, \overline{E_{l\tau j}})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} g^{\gamma\bar{\tau}}.
\end{aligned} \tag{4.5}$$

We will leave the proof of this theorem to the appendix due to its length.

The main theorem of this section is that, for the moduli space of Calabi–Yau fourfolds, with a suitable choice of μ, the partial Hodge metric has the following property.

Theorem 4.3. *Let $n = 4$ and let $\omega_{\rm PH} = (m+2)\omega_{\rm WP} + {\rm Ric}(\omega_{\rm WP})$, then*

(1) *$\omega_{\rm PH}$ is a Kähler metric;*

(2) *the Ricci curvature and the holomorphic sectional curvature of $\omega_{\rm PH}$ are bounded above by the negative constant $-1/(m+4)$; and*

(3) *the holomorphic bisectional curvature of $\omega_{\rm PH}$ is non-positive.*

Proof. By Theorem 4.2 we know that $\omega_{\rm PH}$ is Kähler. From (9.2) we know that

$$h_{i\bar{j}} = g_{i\bar{j}} + F_{i\bar{j}\alpha\bar{\beta}}g^{\alpha\bar{\beta}}. \tag{4.6}$$

Fix a point x_0 in the moduli space. Let $z_1, \ldots, z_m$ be the local holomorphic normal coordinate at x_0 with respect to the Weil–Petersson metric. Then at the point x_0, we have

$$g_{\alpha\bar{\beta}} = \delta_{\alpha\beta}, \qquad \varGamma^{\gamma}_{\alpha\beta} = \frac{\partial g_{\alpha\bar{\beta}}}{\partial z_\gamma} = \frac{\partial g^{\alpha\bar{\beta}}}{\partial z_\gamma} = \frac{\partial g_{\alpha\bar{\beta}}}{\partial \bar{z}_\gamma} = \frac{\partial g^{\alpha\bar{\beta}}}{\partial \bar{z}_\gamma} = 0. \tag{4.7}$$

Replacing $\varOmega$ by $\tilde{\varOmega} = f\varOmega$, where f is a local holomorphic function defined by

$$f(z) = (\varOmega, \bar{\varOmega})^{-1/2}(x_0) - \sum_i \frac{(\partial_i\varOmega, \bar{\varOmega})(x_0)}{((\varOmega, \bar{\varOmega})(x_0))^{3/2}} z_i,$$

we have, at the point x_0,

$$(\partial_k\tilde{\varOmega}, \bar{\tilde{\varOmega}}) = (\tilde{\varOmega}, \overline{\partial_k\tilde{\varOmega}}) = 0 \tag{4.8}$$

for each $k = 1, 2, \ldots, n$ and

$$(\tilde{\varOmega}, \bar{\tilde{\varOmega}}) = 1. \tag{4.9}$$

By abuse of notations, we use $\varOmega$ to replace $\tilde{\varOmega}$ for the rest of this section.

We set $i = j$ and $k = l$. Based on the above notations, from Theorem 4.2 we have

$$\begin{aligned}\tilde{R}_{i\bar{i}k\bar{k}} &= 1 + \delta_{ik} - 2(D_kD_i\varOmega, \overline{D_kD_i\varOmega}) + \sum_{\alpha,\gamma}(D_\alpha D_i\varOmega, \overline{D_kD_\gamma\varOmega})(D_kD_\gamma\varOmega, \overline{D_\alpha D_i\varOmega}) \\ &\quad + \sum_{\alpha,\beta}(D_\alpha D_i\varOmega, \overline{D_\beta D_i\varOmega})(D_\beta D_k\varOmega, \overline{D_\alpha D_k\varOmega}) + \sum_\alpha(D_kD_\alpha D_i\varOmega, \overline{D_kD_\alpha D_i\varOmega}) \\ &\quad + \left(\sum_\alpha(E_{k\alpha i}, \overline{E_{k\alpha i}}) - h^{p\bar{q}}\sum_\alpha(E_{k\alpha i}, \overline{D_\alpha D_q\varOmega})\sum_\beta(D_\beta D_p\varOmega, \overline{E_{k\beta i}})\right).\end{aligned} \tag{4.10}$$

Fix the indices k and i. Let $U_\alpha = h^{p\bar{q}}\sum_\beta(E_{k\beta i}, \overline{D_\beta D_q\varOmega})D_\alpha D_p\varOmega$. Then $U_\alpha \in H^{2,2}$. By the Hodge–Riemann relations we know that, for any α,

$$(E_{k\alpha i} - U_\alpha, \overline{E_{k\alpha i} - U_\alpha}) \geqslant 0. \tag{4.11}$$

By (4.6), we have

$$h_{i\bar{j}} = \delta_{ij} + \sum_{\alpha}(D_\alpha D_i \Omega, \overline{D_\alpha D_j \Omega}). \tag{4.12}$$

Thus

$$\begin{aligned}
&\sum_{\alpha}(U_\alpha, \overline{U_\alpha}) \\
&\quad = \sum_{\alpha}\left(h^{p\bar{q}}\sum_{\beta}(E_{k\beta i}, \overline{D_\beta D_q \Omega})\right)\left(\overline{h^{p_1\overline{q_1}}\sum_{\gamma}(E_{k\gamma i}, \overline{D_\gamma D_{q_1}\Omega})}\right)(D_\alpha D_p \Omega, \overline{D_\alpha D_{p_1}\Omega}) \\
&\quad = \left(h^{p\bar{q}}\sum_{\beta}(E_{k\beta i}, \overline{D_\beta D_q \Omega})\right)\left(h^{q_1\overline{p_1}}\sum_{\gamma}(D_\gamma D_{q_1}\Omega, \overline{E_{k\gamma i}})\right)(h_{p\overline{p_1}} - \delta_{pp_1}) \\
&\quad = h^{q_1\bar{q}}\sum_{\beta}(E_{k\beta i}, \overline{D_\beta D_q \Omega})\sum_{\gamma}(D_\gamma D_{q_1}\Omega, \overline{E_{k\gamma i}}) \\
&\qquad - \sum_{p} h^{p\bar{q}}\sum_{\beta}(E_{k\beta i}, \overline{D_\beta D_q \Omega}) h^{q_1\bar{p}}\sum_{\gamma}(D_\gamma D_{q_1}\Omega, \overline{E_{k\gamma i}}) \\
&\quad = h^{p\bar{q}}\sum_{\beta}(E_{k\beta i}, \overline{D_\beta D_q \Omega})\sum_{\gamma}(D_\gamma D_p \Omega, \overline{E_{k\gamma i}}) - \sum_{p}|h^{p\bar{q}}\sum_{\beta,q}(E_{k\beta i}, \overline{D_\beta D_q \Omega})|^2 \\
&\quad \leqslant h^{p\bar{q}}\sum_{\beta}(E_{k\beta i}, \overline{D_\beta D_q \Omega})\sum_{\gamma}(D_\gamma D_p \Omega, \overline{E_{k\gamma i}}) \\
&\quad = \sum_{\beta}(E_{k\beta i}, \overline{U_\beta}),
\end{aligned} \tag{4.13}$$

and

$$\sum_{\alpha}(U_\alpha, \overline{E_{k\alpha i}}) = h^{p\bar{q}}\sum_{\beta}(E_{k\beta i}, \overline{D_\beta D_q \Omega})\sum_{\alpha}(D_\alpha D_p \Omega, \overline{E_{k\alpha i}}). \tag{4.14}$$

Combining (4.11), (4.13) and (4.14) we have

$$\sum_{\alpha}(E_{k\alpha i}, \overline{E_{k\alpha i}}) - h^{p\bar{q}}\sum_{\alpha}(E_{k\alpha i}, \overline{D_\alpha D_q \Omega})\sum_{\beta}(D_\beta D_p \Omega, \overline{E_{k\beta i}}) \geqslant 0. \tag{4.15}$$

Thus the sum of the last two terms in (4.10) is non-negative.

We shall show that the term

$$\sum_{\alpha}(D_k D_\alpha D_i \Omega, \overline{D_k D_\alpha D_i \Omega})$$

is related to the Yukawa coupling of fourfolds.

Definition 4.4. Using the same notations as above, define a holomorphic section of $\mathrm{Sym}^2\,\underline{F}^4 \otimes (T^*\mathcal{M})^{\otimes 4}$ to be

$$\xi_{ijkl} = (\Omega, \partial_i\partial_j\partial_k\partial_l\Omega). \tag{4.16}$$

We call ξ_{ijkl} the Yukawa coupling for Calabi–Yau fourfolds.

Clearly, ξ_{ijkl} is symmetric with respect to i, j, k, l.

Lemma 4.5. *Using the same notations as above, we have*

$$\xi_{ijkl} = -(D_j D_k D_l \Omega, D_i \Omega) = (D_k D_l \Omega, D_j D_i \Omega). \tag{4.17}$$

Proof. The lemma follows from the definition of $T_{k\alpha i}$ and the first Hodge–Riemann relation. □

Using the above lemma, we have

$$\sum_{\alpha} (D_k D_\alpha D_i \Omega, \overline{D_k D_\alpha D_i \Omega}) = -\sum_{\alpha,l} |\xi_{ik\alpha l}|^2. \tag{4.18}$$

Combining (4.10), (4.15) and (4.18) we have

$$\tilde{R}_{i\bar{i}k\bar{k}} \geqslant 1 + \delta_{ik} - 2(D_k D_i \Omega, \overline{D_k D_i \Omega}) + \sum_{\alpha,\gamma} (D_\alpha D_i \Omega, \overline{D_k D_\gamma \Omega})(D_k D_\gamma \Omega, \overline{D_\alpha D_i \Omega}) + \sum_{\alpha,\beta} (D_\alpha D_i \Omega, \overline{D_\beta D_i \Omega})(D_\beta D_k \Omega, \overline{D_\alpha D_k \Omega}) - \sum_{\alpha,l} |\xi_{ik\alpha l}|^2. \tag{4.19}$$

The quadratic form $(\cdot,\cdot)$ defines an inner product on $H^{2,2}$ by the second Hodge–Riemann relation. Let $\omega_1, \ldots, \omega_N$ be a (real) basis of $H^{2,2}$ such that $(\omega_p, \omega_q) = \delta_{pq}$. Fix the index i. Let $D_i D_\alpha \Omega = \sum_{p=1}^{N} A_{\alpha p} \omega_p$ and let $D_k D_\beta \Omega = \sum_{p=1}^{N} B_{\beta p} \omega_p$. By Lemma 3.3 we have

$$\begin{aligned} &\sum_{\alpha,\beta} (D_\alpha D_i \Omega, \overline{D_\beta D_i \Omega})(D_\beta D_k \Omega, \overline{D_\alpha D_k \Omega}) - \sum_{\alpha\beta} |\xi_{ik\alpha\beta}|^2 \\ &= \sum_{\alpha,\beta} (D_i D_\alpha \Omega, \overline{D_i D_\beta \Omega})(D_k D_\beta \Omega, \overline{D_k D_\alpha \Omega}) - \sum_{\alpha,\beta} (D_k D_\beta \Omega, D_i D_\alpha \Omega)(\overline{D_i D_\beta \Omega}, \overline{D_k D_\alpha \Omega}) \\ &= \sum_{j,l=1}^{N} \sum_{\alpha,\beta} (A_{\alpha j} \overline{A_{\beta j}} B_{\beta l} \overline{B_{\alpha l}} - A_{\alpha j} B_{\beta j} \overline{A_{\beta l} B_{\alpha l}}). \end{aligned} \tag{4.20}$$

Let $u_{jl} = \sum_\alpha A_{\alpha j} \overline{B_{\alpha l}}$. From (4.20) we have

$$\sum_{\alpha,\beta} (D_\alpha D_i \Omega, \overline{D_\beta D_i \Omega})(D_\beta D_k \Omega, \overline{D_\alpha D_k \Omega}) - \sum_{\alpha,\tau} (D_k D_\tau \Omega, D_\alpha D_i \Omega)(\overline{D_k D_\tau \Omega}, \overline{D_\alpha D_i \Omega}) = \sum_{j,l=1}^{N} u_{jl} \overline{u_{jl}} - \sum_{j,l=1}^{N} u_{jl} \overline{u_{lj}} = \sum_{j<l} |u_{jl} - u_{lj}|^2 \geqslant 0. \tag{4.21}$$

Combining (4.19) and (4.21) we have

$$\tilde{R}_{i\bar{i}k\bar{k}} \geqslant 1 + \delta_{ik} - 2(D_k D_i \Omega, \overline{D_k D_i \Omega}) + \sum_{\alpha,\gamma} |(D_\alpha D_i \Omega, \overline{D_k D_\gamma \Omega})|^2. \tag{4.22}$$

If $i \neq k$, then, by (4.22),

$$\tilde{R}_{i\bar{i}k\bar{k}} \geqslant 1 - 2(D_k D_i \Omega, \overline{D_k D_i \Omega}) + |(D_k D_i \Omega, \overline{D_k D_i \Omega})|^2 \geqslant 0. \tag{4.23}$$

This implies the holomorphic bisectional curvature of ω_{PH} is non-positive.

Now we estimate the holomorphic sectional curvature. Let $i = k$. By (4.22) we have

$$\begin{aligned} \tilde{R}_{i\bar{i}i\bar{i}} &\geqslant 2 - 2(D_i D_i \Omega, \overline{D_i D_i \Omega}) + \sum_{\alpha,\beta} |(D_\alpha D_i \Omega, \overline{D_\beta D_i \Omega})|^2 \\ &\geqslant 2 - 2(D_i D_i \Omega, \overline{D_i D_i \Omega}) + \sum_{\alpha} |(D_\alpha D_i \Omega, \overline{D_\alpha D_i \Omega})|^2. \end{aligned} \tag{4.24}$$

By (4.12) we have

$$h_{i\bar{i}} = 1 + \sum_{\alpha} (D_\alpha D_i \Omega, \overline{D_\alpha D_i \Omega}). \tag{4.25}$$

Let $a_\alpha = (D_\alpha D_i \Omega, \overline{D_\alpha D_i \Omega})$ for $\alpha \neq i$ and let $a_i = (D_i D_i \Omega, \overline{D_i D_i \Omega}) - 1$. Clearly, they are real numbers by the Hodge–Riemann relations. From (4.24) and (4.25) we have

$$\tilde{R}_{i\bar{i}i\bar{i}} \geqslant 1 + \sum_{\alpha} a_\alpha^2$$

and

$$h_{i\bar{i}} = 2 + \sum_{\alpha} a_\alpha.$$

Combining the above two inequalities and the following trivial inequality,

$$1 + \sum_{\alpha=1}^{\mathrm{m}} a_\alpha^2 \geqslant \frac{1}{m+4} \left(2 + \sum_{\alpha=1}^{\mathrm{m}} a_\alpha \right)^2,$$

we have

$$\tilde{R}_{i\bar{i}i\bar{i}} \geqslant \frac{1}{m+4} (h_{i\bar{i}})^2. \tag{4.26}$$

This proved the holomorphic sectional curvature of ω_{PH} is bounded above by a negative constant. Clearly, the Ricci curvature is bounded above by the same negative constant since the bisectional curvature is non-positive. □

5. Scalar curvature bounds the sectional curvature

In this section we will prove that the volumes of any subvariety of the moduli space equipped with the Weil–Petersson metric or the Hodge metric (Definition 2.17) are finite. Also, we will show that the Riemannian sectional curvature of the Weil–Petersson metric is finite in the L^1 sense. The key tool we use here is Yau's Schwarz Lemma [**31**]. The following version is proved by Royden [**19**].

Theorem 5.1. *Let M, N be two Kähler manifolds such that M is complete and the Ricci curvature of M is lowerly bounded and the holomorphic sectional curvature of N is upperly bounded by a negative constant. Then there is a constant C, depending only on the lower bound of the Ricci curvature M and the upper bound of the holomorphic sectional curvature of N such that*

$$\omega_N \leqslant C\omega_M.$$

Using the above theorem, we first have the following.

Theorem 5.2. *Let $\mathcal{M}$ be the moduli space of polarized Calabi–Yau n-folds. Then the volume of any subvariety M_1 of $\mathcal{M}$ equipped with the Weil–Petersson metric or the Hodge metric is finite.*

Proof. Since the moduli space is quasi-projective, after desingularization, we can assume that $\mathcal{M} = Y \setminus R$ where Y is a compact Kähler manifold and R is a divisor of normal crossings. From [**9**], we know that there is a complete metric ω_0 on $\mathcal{M}$ such that its volume is finite and its Ricci curvature has a lower bound. Moreover, this metric behaves like the Poincaré metric near R. By [**12**, Theorem 1.2] the holomorphic sectional curvature of the Hodge metric ω_{H} is negative away from zero. Let i be the identity map

$$i : (\mathcal{M}, \omega_0) \to (\mathcal{M}, \omega_{\mathrm{H}}), \tag{5.1}$$

which is holomorphic. By the Schwarz–Yau Lemma [**31**] we have

$$\omega_{\mathrm{H}} = i^*\omega_{\mathrm{H}} \leqslant c\omega_0 \tag{5.2}$$

for some positive constant c. Thus

$$\int_{\mathcal{M}} \omega_{\mathrm{H}}^{\mathrm{m}} \leqslant c_1 \int_{\mathcal{M}} \omega_0^{\mathrm{m}} < +\infty.$$

For any subvariety M_1 of the moduli space $\mathcal{M}$, we restrict the Hodge metric ω_{H} to it. By the Gauss equation, the holomorphic sectional curvature on the smooth part of the subvariety M_1 is negative away from zero. Since M_1 is either compact or quasi-projective, using the same argument for $\mathcal{M}$, we proved the volume with respect to the Hodge metric is finite.

By Corollary 6.5, up to a constant,

$$\omega_{\mathrm{WP}} \leqslant \omega_{\mathrm{H}}. \tag{5.3}$$

So the volume of the Weil–Petersson metric on any subvariety of $\mathcal{M}$ is also finite. This finishes the proof. □

From the above theorem we can bound the L^1 norm of the sectional curvature of the Weil–Petersson metric.

Theorem 5.3. *Let $\mathcal{M}$ be the moduli space of polarized Calabi–Yau n-folds. Then the L^1 norm of the Riemannian sectional curvature of $\mathcal{M}$ equipped with the Weil–Petersson metric is finite.*

Proof. For any point x_0 in the moduli space, let $z_1, \dots, z_m$ be the local normal coordinates at x_0 with respect to the Weil–Petersson metric. Let X and Y be two real unit tangent vectors of $\mathcal{M}$ at x_0. Clearly, there is a constant c, which is independent of P, such that

$$|R(X,Y,X,Y)|^2 \leqslant c|R_{i\bar{j}k\bar{l}}|^2 = c \sum_{i,j,k,l} R_{i\bar{j}k\bar{l}} R_{j\bar{i}l\bar{k}}. \tag{5.4}$$

We make the assumptions (4.7), (4.8) and (4.9) at x_0 like we did in the proof of Theorem 4.3. From the basic fact

$$|(D_k D_i \Omega, \overline{D_l D_j \Omega})|^2 \leqslant \tfrac{1}{2}(|(D_k D_i \Omega, \overline{D_k D_i \Omega})|^2 + |(D_l D_j \Omega, \overline{D_l D_j \Omega})|^2)$$

and the Strominger formula, we have

$$\begin{aligned} |R(X,Y,X,Y)|^2 &\leqslant c \sum_{i,j,k,l} R_{i\bar{j}k\bar{l}} R_{j\bar{i}l\bar{k}} \\ &= c \sum_{i,j,k,l} (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} - (D_k D_i \Omega, \overline{D_l D_j \Omega})) \\ &\qquad\qquad \times (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj} - (D_l D_j \Omega, \overline{D_k D_i \Omega})) \\ &= c\left(2m^2 + 2m - 4\sum_{i,k}(D_k D_i \Omega, \overline{D_k D_i \Omega}) + \sum_{i,j,k,l} |(D_k D_i \Omega, \overline{D_l D_j \Omega})|^2\right) \\ &\leqslant c_1 \left(m + \sum_{i,k}(D_k D_i \Omega, \overline{D_k D_i \Omega})\right)^2 \\ &= c_1 \left(\sum_i h_{i\bar{i}}\right)^2 \end{aligned} \tag{5.5}$$

for some universal constant c_1 only depending on m. Thus from (5.3) we have

$$\int_{\mathcal{M}} |R(X,Y,X,Y)| \omega_{\mathrm{WP}}^{\mathrm{m}} \leqslant \sqrt{c_1} \int_{\mathcal{M}} \sum_{i,j} g^{i\bar{j}} h_{i\bar{j}} \omega_{\mathrm{WP}}^{\mathrm{m}} \leqslant m\sqrt{c_1} \int_{\mathcal{M}} \omega_{\mathrm{H}}^{\mathrm{m}} < +\infty.$$

This proves that the L^1 norm of the Riemannian sectional curvature of the Weil–Petersson is bounded. □

For the rest of this section, we assume that $n = 4$. We will prove that the Riemannian sectional curvature of ω_μ is bounded by the scalar curvature pointwisely up to a constant. The similar result has been proved in [**13**] in the case of Calabi–Yau threefolds.

Theorem 5.4. *Let $\mathcal{M}$ be the moduli space of polarized Calabi–Yau fourfolds. Then there are positive constants c_1 and c_2 such that the Riemannian sectional curvature of the partial Hodge metric ω_μ is bounded by $c_1 + c_2|\tilde{R}|$, where $\tilde{R}$ is the scalar curvature of the partial Hodge metric.*

Proof. Fix a point $x_0 \in \mathcal{M}$ and let X and Y be two real tangent vectors at x_0 such that X is perpendicular to Y with respect to ω_μ. Let $\xi = X - \sqrt{-1}JX$ and $\eta = Y - \sqrt{-1}JY$ where J is the complex structure of $\mathcal{M}$. Clearly, ξ is perpendicular to η with respect ω_μ too. We make the assumption (4.7)–(4.9) like we did in the proof of Theorem 4.3 for the local coordinates $(z_1, \ldots, z_m)$ and the local section Ω. We can choose a unitary transformation of the coordinates such that $\xi = a(\partial/\partial z_i)$, $\eta = b(\partial/\partial z_j)$ with $i \neq j$ for some complex numbers a, b and the matrix $h_{i\bar{j}}$ of ω_μ is diagonalized with $h_{i\bar{j}} = \delta_{ij}\lambda_i$. We have

$$\tilde{R}(X, Y, X, Y) = \tfrac{1}{8}(\mathrm{Re}(\tilde{R}(\xi, \bar{\eta}, \xi, \bar{\eta})) - \tilde{R}(\xi, \bar{\xi}, \eta, \bar{\eta})). \tag{5.6}$$

In the following, we will use $\|v\|^2$ to denote the square of the norm of a complex vector with respect to ω_μ. The second term in the right-hand side of the above formula is easy to estimate,

$$\begin{aligned} |\tilde{R}(\xi, \bar{\xi}, \eta, \bar{\eta})| &= |a|^2|b|^2\tilde{R}_{i\bar{i}j\bar{j}} \\ &= \|\xi\|^2\|\eta\|^2\tilde{R}_{i\bar{i}j\bar{j}}\lambda_i^{-1}\lambda_j^{-1} \\ &\leqslant \|\xi\|^2\|\eta\|^2\sum_{i,j}\tilde{R}_{i\bar{i}j\bar{j}}\lambda_i^{-1}\lambda_j^{-1} \\ &= \|\xi\|^2\|\eta\|^2|\tilde{R}|, \end{aligned} \tag{5.7}$$

since $\tilde{R}_{i\bar{i}j\bar{j}} \geqslant 0$ for $1 \leqslant i, j \leqslant m$ by Theorem 4.3. By Theorems 4.2 and 4.3,

$$\begin{aligned} \tilde{R}_{i\bar{j}i\bar{j}} = -2F_{i\bar{j}i\bar{j}} + 2\sum_{\alpha,\beta}F_{i\bar{j}\alpha\bar{\beta}}F_{i\bar{j}\beta\bar{\alpha}} - \sum_{\alpha\beta}\xi_{ii\alpha\beta}\overline{\xi_{jj\alpha\beta}} \\ + \left(\sum_{\alpha}(E_{i\alpha i}, \overline{E_{j\alpha j}}) - \sum_{p,\alpha,\beta}\lambda_p^{-1}(E_{i\alpha i}, \overline{D_\alpha D_p\Omega})(D_\beta D_p\Omega, \overline{E_{j\beta j}})\right). \end{aligned} \tag{5.8}$$

Let G be the vector space spanned by $\{L_i\}$ where $i = 1, \ldots, m$ and $L_i = (E_{i1i}, \ldots, E_{imi})$. We now define a bilinear form $((\cdot, \cdot))$ on G by

$$((L_i, L_j)) = \sum_{\alpha}(E_{i\alpha i}, \overline{E_{j\alpha j}}) - \sum_{p,\alpha,\beta}\lambda_p^{-1}(E_{i\alpha i}, \overline{D_\alpha D_p\Omega})(D_\beta D_p\Omega, \overline{E_{j\beta j}}). \tag{5.9}$$

By (4.15), we know that $((L_i, L_j))$ is a Hermitian semi-inner product on G. So we have the following Cauchy inequality:

$$|((L_i, L_j))| \leqslant \sqrt{((L_i, L_i))((L_j, L_j))}. \tag{5.10}$$

However, by the proof of Theorem 4.3 we know that

$$((L_i, L_i)) = \sum_{\alpha} (E_{i\alpha i}, \overline{E_{i\alpha i}}) - \sum_{p,\alpha,\beta} \lambda_p^{-1} (E_{i\alpha i}, \overline{D_\alpha D_p \Omega})(D_\beta D_p \Omega, \overline{E_{i\beta i}}) \leqslant \tilde{R}_{i\bar{i}i\bar{i}} \leqslant |\tilde{R}|\lambda_i^2. \tag{5.11}$$

Combining (5.9), (5.10) and (5.11) we have

$$\left| \sum_{\alpha} (E_{i\alpha i}, \overline{E_{j\alpha j}}) - \sum_{p,\alpha,\beta} \lambda_p^{-1} (E_{i\alpha i}, \overline{D_\alpha D_p \Omega})(D_\beta D_p \Omega, \overline{E_{j\beta j}}) \right| \leqslant |\tilde{R}|\lambda_i \lambda_j. \tag{5.12}$$

Since $(-1)(\cdot,\cdot)$ is a Hermitian inner product on $H^{1,3}$, from (4.21) we have

$$\sum_{\alpha\beta} \xi_{ii\alpha\beta} \overline{\xi_{jj\alpha\beta}} \leqslant \sum_{\alpha,\beta} |F_{i\bar{i}\alpha\bar{\beta}}|\, |F_{j\bar{j}\alpha\bar{\beta}}|. \tag{5.13}$$

By (4.24) we have

$$\tilde{R}_{i\bar{i}i\bar{i}} \geqslant 2 - 2F_{i\bar{i}i\bar{i}} + \sum_{\alpha,\beta} |F_{i\bar{i}\alpha\bar{\beta}}|^2 \geqslant \frac{1}{2} \sum_{\alpha,\beta} |F_{i\bar{i}\alpha\bar{\beta}}|^2. \tag{5.14}$$

So combining (5.13) and (5.14) we have

$$\sum_{\alpha\beta} \xi_{ii\alpha\beta} \overline{\xi_{jj\alpha\beta}} \leqslant \sqrt{\sum_{\alpha,\beta} |F_{i\bar{i}\alpha\bar{\beta}}|^2 \sum_{\alpha,\beta} |F_{j\bar{j}\alpha\bar{\beta}}|^2} \leqslant 2\sqrt{\tilde{R}_{i\bar{i}i\bar{i}} \tilde{R}_{j\bar{j}j\bar{j}}} \leqslant 2|\tilde{R}|\lambda_i \lambda_j. \tag{5.15}$$

From (4.22) we have

$$\tilde{R}_{i\bar{i}j\bar{j}} \geqslant 1 - 2F_{i\bar{i}j\bar{j}} + \sum_{\alpha,\beta} |F_{i\bar{\beta}\alpha\bar{j}}|^2 \geqslant \sum_{\alpha,\beta} |F_{i\bar{j}\alpha\bar{\beta}}|^2 - |F_{i\bar{i}j\bar{j}}|^2. \tag{5.16}$$

So we have

$$\left| 2 \sum_{\alpha,\beta} F_{i\bar{j}\alpha\bar{\beta}} F_{i\bar{j}\beta\bar{\alpha}} \right| \leqslant \sum_{\alpha,\beta} (|F_{i\bar{j}\alpha\bar{\beta}}|^2 + |F_{i\bar{j}\beta\bar{\alpha}}|^2) = 2 \sum_{\alpha,\beta} |F_{i\bar{j}\alpha\bar{\beta}}|^2 \leqslant 2\tilde{R}_{i\bar{i}j\bar{j}} + 2|F_{i\bar{i}j\bar{j}}|^2. \tag{5.17}$$

From (5.14), since $i \neq j$ we have

$$|F_{i\bar{i}j\bar{j}}| \leqslant \sqrt{2\tilde{R}_{i\bar{i}i\bar{i}}} \tag{5.18}$$

and

$$|F_{i\bar{i}j\bar{j}}| = |F_{j\bar{j}i\bar{i}}| \leqslant \sqrt{2\tilde{R}_{j\bar{j}j\bar{j}}}. \tag{5.19}$$

Combining (5.17), (5.18) and (5.19), we have

$$\left| 2 \sum_{\alpha,\beta} F_{i\bar{j}\alpha\bar{\beta}} F_{i\bar{j}\beta\bar{\alpha}} \right| \leqslant 2\tilde{R}_{i\bar{i}j\bar{j}} + 4\sqrt{\tilde{R}_{i\bar{i}i\bar{i}} \tilde{R}_{j\bar{j}j\bar{j}}} \leqslant |\tilde{R}|\lambda_i \lambda_j + 4\sqrt{\tilde{R}^2 \lambda_i^2 \lambda_j^2} = 5|\tilde{R}|\lambda_i \lambda_j. \tag{5.20}$$

From (5.14) we also have

$$|2F_{i\bar{j}i\bar{j}}| \leqslant 2\sqrt{F_{i\bar{i}i\bar{i}}F_{j\bar{j}j\bar{j}}} \leqslant 2(4\tilde{R}_{i\bar{i}i\bar{i}}\tilde{R}_{j\bar{j}j\bar{j}})^{1/4} \leqslant 2\sqrt{2|\tilde{R}|\lambda_i\lambda_j} \leqslant 1 + 2|\tilde{R}|\lambda_i\lambda_j. \tag{5.21}$$

By the Hodge–Riemann relations we know $\lambda_i = 1 + \sum_\alpha F_{i\bar{i}\alpha\bar{\alpha}} > 1$. Combining (5.8), (5.12), (5.15), (5.20) and (5.21) we have

$$\begin{aligned} |\operatorname{Re}(\tilde{R}(\xi,\bar{\eta},\xi,\bar{\eta}))| &\leqslant |a|^2|b|^2|\tilde{R}_{i\bar{j}i\bar{j}}| \\ &\leqslant |a|^2|b|^2 + 9|\tilde{R}|\,|a|^2|b|^2\lambda_i\lambda_j \\ &\leqslant \|\xi\|^2\|\eta\|^2(1+9|\tilde{R}|). \end{aligned} \tag{5.22}$$

Combining (5.6), (5.7) and (5.22) we have

$$|\tilde{R}(X,Y,X,Y)| \leqslant (\tfrac{1}{8} + \tfrac{9}{8}|\tilde{R}|)\|\xi\|^2\|\eta\|^2 = (\tfrac{1}{2} + \tfrac{9}{8}|\tilde{R}|)\|X\|^2\|Y\|^2. \tag{5.23}$$

This finishes the proof. □

6. Hodge metrics

Let X, Y be finite-dimensional Hermitian vector spaces and let $\langle\cdot,\cdot\rangle_X$, $\langle\cdot,\cdot\rangle_Y$ be the Hermitian inner products of X and Y, respectively. Let $A, B : X \to Y$ be linear operators. Then we can define the natural Hermitian inner product for A, B on the space $\operatorname{Hom}(X,Y)$ as follows. Let $e_1,\ldots,e_n$ be a unitary basis of X. Then define

$$\langle A,B\rangle = \sum_i \langle Ae_i, B\bar{e}_i\rangle. \tag{6.1}$$

Let D be the classifying space defined in Definition 2.6. The complexified tangent bundle $TD\otimes\mathbb{C}$ of D can be realized as the subbundle of

$$TD\otimes\mathbb{C} \subset \bigoplus_{p+q=n} \operatorname{Hom}(H^{p,q}, H/H^{p,q}). \tag{6.2}$$

By Lemma 2.9, $(\sqrt{-1})^{p-q}Q(\cdot,\cdot)$ is the Hermitian inner product of $H^{p,q}$, it naturally induced a Riemannian metric h on $TD\otimes\mathbb{C}$ via the above realization.

Define an almost complex structure J on $TD\otimes\mathbb{C}$ as follows. Let X be a local section of $TD\otimes\mathbb{C}$. Then

$$JX = \begin{cases} \sqrt{-1}X & \text{if } X \in \Gamma\Big(\bigoplus_{p+q=n}\operatorname{Hom}\Big(H^{p,q}, \bigoplus_{r<p} H^{r,s}\Big)\Big), \\ -\sqrt{-1}X & \text{if } X \in \Gamma\Big(\bigoplus_{p+q=n}\operatorname{Hom}\Big(H^{p,q}, \bigoplus_{r>p} H^{r,s}\Big)\Big). \end{cases} \tag{6.3}$$

We have the following result.

Proposition 6.1. *The Riemannian metric h is G and J invariant. Furthermore, J is a G invariant complex structure on D. Thus h defines a Hermitian metric on D.*

Proof. Let $x, y \in D$ and $\xi x = y$ for some $\xi \in G$. Let $e_1^p, \dots, e_{n_p}^p$ be an unitary basis of $H^{p,q}$ at x. Then $\xi e_1^p, \dots, \xi e_{n_p}^p$ will be an unitary basis of $H^{p,q}$ at y by the definition of G. If X is a tangent vector at x, then X induced the tangent vector $\tilde{X} = \xi X \xi^{-1}$ at y. Thus

$$\|\tilde{X}\|^2 = (\sqrt{-1})^{p-q} \sum_p \sum_i Q(\xi X \xi^{-1} \xi e_i^p, \xi X \xi^{-1} \xi e_i^p) = \|X\|^2,$$

which proves the invariance of h with respect to G.

To prove that h is also J invariant, we let X, Y be two holomorphic vectors at x. We just need to prove that $h(X, Y) = 0$. Let $p + q = n$. Suppose X is non-zero restricting to $H^{p,q}$. As above, let $e_1^p, \dots, e_{n_p}^p$ be the unitary basis of $H^{p,q}$. We claim that

$$Q(Xe_i^p, Y\bar{e}_i^p) = 0, \quad 1 \leqslant i \leqslant \dim H^{p,q}.$$

To see this, assume that a component of $Xe_i^p \in H^{r,s}$. Then we have $r < p$. In order that $Q((Xe_i^p)^{r,s}, Y\bar{e}_i^p) \neq 0$, we must have $s < q$. But this is a contradiction because $p + q = r + s = n$.

It remains to prove that the almost complex structure is integrable. To prove this, we first observe that the same J defines an almost complex structure on $\check{D}$, the compact dual of D. The almost complex structure on $\check{D}$ is defined by the pull back of the complex structure of the flag manifold, which is a complex manifold. □

The main result of this section is the following.

Theorem 6.2. *Let $\mathcal{M}$ be a horizontal slice of the classifying space D coming from the moduli space of the polarized Calabi–Yau manifolds. Then the metric h is the Hodge metric on $\mathcal{M}$. In particular, the Hodge metric is Kähler.*

The assumption in the theorem can be weakened to the case where there is a horizontal slice with the Weil–Petersson metric is defined (see § 8 for details).

Proof. Let D_1 be the Hermitian symmetric space defined by the set of subspaces

$$H^+ = H^{n,0} \oplus H^{n-2,2} \oplus \cdots$$

in H. As in § 2, the natural projection

$$p : D \to D_1$$

is defined by

$$\{F^k\}_{k=1,\dots,n} \mapsto H^+,$$

which is in general not holomorphic. Using the same method as above, we defined the unique complex structure on D_1 by realizing the holomorphic tangent bundle of D_1 as the subbundle of $\mathrm{Hom}(H^+, H/H^+)$.

Let $X \in T^{1,0}\mathcal{M}$ be a holomorphic vector field. Then X is horizontal in the sense that X is a section of the bundle $\mathrm{Hom}(H^{n,0}, H^{n-1,1}) \oplus \mathrm{Hom}(H^{n-1,1}, H^{n-2,2}) \oplus \cdots$. Define X_1 to be an element in the bundle such that

$$X_1 = \begin{cases} X, & \text{restricting to } H^+, \\ 0, & \text{otherwise}, \end{cases}$$

and let $X_2 = X - X_1$. Then we have

$$X = X_1 + X_2.$$

Furthermore, we have the following result.

Lemma 6.3. *According to the complex structure of D_1, the vectors fields X_1 and X_2 are holomorphic and anti-holomorphic, respectively.*

Proof. Let

$$H = H^+ \oplus H^-.$$

Then X_1 is a map $H^+ \mapsto H^-$, which can be identified as a holomorphic vector fields of D_1. X_2 can be identified as a map from H^- to H^+. It is the dual map of $H^+ \to H^-$ under the polarization Q. Thus can be identified as an anti-holomorphic vector field. $\square$

We continue with the proof of Theorem 6.2.

From the above argument, we see that under the invariant Kähler metric of D_1

$$\|X\|^2 = \|X_1\|^2 + \|X_2\|^2. \tag{6.4}$$

If n is an odd number, then D_1 is the Hermitian symmetric space of third kind, that is,

$$D_1 = \mathrm{Sp}(n, \mathbb{R})/U(n).$$

It can be realized as the subset of $n \times n$ complex matrices

$$\{Z \in M_n(\mathbb{C}) \mid I_n - \bar{Z}^{\mathrm{T}} Z > 0,\ Z^{\mathrm{T}} = Z\}.$$

Its invariant Kähler metric can be defined as

$$\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log\det(I_n - \bar{Z}^{\mathrm{T}} Z).$$

If n is an even number, then

$$D_1 = O(m, n, \mathbb{R})/(O(m) \times O(n)).$$

There is a natural inclusion of D_1,

$$D_1 \hookrightarrow D_1' = SU(m, n\mathbb{C})/S(U(m) \times U(n)).$$

D_1' is the Hermitian symmetric space of first kind, which can be realized as the subset of $m \times n$ complex matrices

$$\{Z \in M_{m,n}(\mathbb{C}) \mid I_n - \bar{Z}^{\mathrm{T}} Z > 0\}.$$

The invariant Kähler metric is defined as

$$\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log\det(I_n - \bar{Z}^{\mathrm{T}} Z).$$

The invariant Riemannian metric on D_1 is the pull back of the invariant Hermitian metric on D_1'.

In both cases (of D_1 for n odd and of D_1' for n even), the invariant Kähler metrics are defined using the polarization Q as

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\det Q(\cdot,\cdot).$$

In order to prove the theorem, we just need to prove it at the original point. At the original point of D_1 (respectively, D_1'), we can write the Kähler metric as

$$\frac{\sqrt{-1}}{2\pi}\mathrm{d}Z_{ij}\otimes\mathrm{d}\bar{Z}_{ij}.$$

Let $\{\partial/\partial z_\alpha\}_{\alpha=1,\dots,m}$ be holomorphic vector fields of $\mathcal{M}$. We have

$$\sum_{ij}\frac{\partial Z_{ij}}{\partial z_\alpha}\frac{\partial\bar{Z}_{ij}}{\partial z_\alpha}=0,\quad 1\leqslant\alpha\leqslant m.$$

The reason for the above equality is that each row of the matrix Z_{ij} represents an element in some $H^{p,q}$. By the Griffiths transversality, we have

$$\frac{\partial Z_{ij}}{\partial z_\alpha}\in H^{p-1,q+1},\qquad \frac{\partial\bar{Z}_{ij}}{\partial z_\alpha}\in H^{q-1,p+1}.$$

The inner product of the above two is the same as $Q(\partial Z_{ij}/\partial z_\alpha,\partial\bar{Z}_{ij}/\partial z_\alpha)$, which is zero. Similarly, we have

$$\sum_{ij}\frac{\partial Z_{ij}}{\partial\bar{z}_\alpha}\frac{\partial\bar{Z}_{ij}}{\partial\bar{z}_\alpha}=0.$$

Thus we have

$$\left\|\frac{\partial}{\partial z_\alpha}\right\|^2=\sum_{ij}\left\|\frac{\partial Z_{ij}}{\partial z_\alpha}\right\|^2+\sum_{ij}\left\|\frac{\partial\bar{Z}_{ij}}{\partial z_\alpha}\right\|^2.$$

Comparing the above equation with (6.4), we proved that h is the Hodge metric. Using the result in [**12**], we know that the h is Kähler. □

Let ω_{H} be the Kähler form of the Hodge metric h. Then we have the following result.

Corollary 6.4. *In the case of $n=3$, we have*

$$\omega_{\mathrm{H}}=(m+3)\omega_{\mathrm{WP}}+\mathrm{Ric}(\omega_{\mathrm{WP}}).$$

In the case of $n=4$, we have

$$\omega_{\mathrm{H}}=2(m+2)\omega_{\mathrm{WP}}+2\,\mathrm{Ric}(\omega_{\mathrm{WP}}).$$

Proof. The case $n = 3$ was proved in [**13**] using the result of [**1**]. In the case of $n = 4$, we do not have the similar result as that of [**1**]. However, using the generalized Strominger formula (4.4), we have

$$2(m+2)\omega_{\mathrm{WP}} + 2\,\mathrm{Ric}(\omega_{\mathrm{WP}}) = 2\omega_{\mathrm{WP}} + 2g^{k\bar{l}}F_{i\bar{j}k\bar{l}}\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j,$$

where F is defined in (4.3). Let X be a holomorphic vector on $\mathcal{M}$. Then by the identification (6.2) and the fact that X is horizontal,

$$X \in \mathrm{Hom}(H^{4,0}, H^{3,1}) \oplus \mathrm{Hom}(H^{3,1}, H^{2,2}) \oplus \mathrm{Hom}(H^{2,2}, H^{1,3}) \oplus \mathrm{Hom}(H^{1,3}, H^{0,4}).$$

Using (3) of Lemma 3.2, we know that ω_{WP} gives the part of the metric h restricted on the space $H^{4,0}$. Since $D_i\Omega$ gives a basis of the space $H^{3,1}$, by (6.1), h restricts to $H^{3,1}$, which gives

$$\|X\|^2_{H^{3,1}} = \sum_{ij} g^{i\bar{j}}(X(D_i\Omega), \overline{X(D_j\Omega)})(\Omega, \bar{\Omega})^{-1}.$$

In particular, if $X = \partial/\partial z_\alpha$, then

$$\|X\|^2_{H^{3,1}} = g^{i\bar{j}}F_{i\bar{j}\alpha\bar{\alpha}}.$$

To compute the norm of X on $H^{1,3}$ and $H^{0,4}$, we use the duality as follows. Let $w_1, \ldots, w_N$ be a (real) orthonormal basis of $H^{2,2}$. Then we have

$$\|X\|^2_{H^{2,2}} = \sum_{\alpha=1}^{N}(X(e_\alpha), \overline{X(e_\alpha)}) = \sum_{ij}\sum_{\alpha=1}^{N} g^{i\bar{j}}(\Omega, \bar{\Omega})^{-1}(X(e_\alpha), D_i\Omega)(\overline{D_j\Omega}, \overline{X(e_\alpha)}).$$

Since the polarization is invariant infinitesimally under X, the above is equal to

$$\begin{aligned}\|X\|^2_{H^{2,2}} &= \sum_{ij}\sum_{\alpha=1}^{N} g^{i\bar{j}}(\Omega, \bar{\Omega})^{-1}(e_\alpha, D_\alpha D_i\Omega)(\overline{D_\alpha D_j\Omega}, e_\alpha)\\ &= \sum_{ij} g^{i\bar{j}}(\Omega, \bar{\Omega})^{-1}(D_\alpha D_i\Omega, \overline{D_\alpha D_j\Omega}).\end{aligned}$$

Thus the norm restricted to $H^{2,2}$ is the same as that on $H^{3,1}$. Using the same method, we can prove that the norm of X on $H^{1,3}$ is given by the Weil–Petersson metric. The corollary follows from

$$\|X\|^2_h = \|X\|^2_{H^{4,0}} + \|X\|^2_{H^{3,1}} + \|X\|^2_{H^{2,2}} + \|X\|^2_{H^{1,3}}.$$

□

Corollary 6.5 (cf. [12]). *Up to a constant, the Weil–Petersson metric and its Ricci curvature are less than or equal to the Hodge metric.*

Proof. This is an easy consequence of Theorem 3.4. □

Remark 6.6. It was a well-known theorem of [**7**] that the holomorphic sectional curvature with respect to the Hermitian connection at the horizontal direction is negative away from zero. In the previous section, we give an explicit formula proving that the holomorphic bisectional curvature on the horizontal slice is non-positive and the holomorphic sectional curvature is negative away from zero in the case of Calabi–Yau fourfolds. Since the Hodge metric is Kähler, the connection is also the Levi–Civita connection.

Remark 6.7. One of the most confusing parts of the theory of the Hodge metric is that the projection in Definition 2.11 is, in general, not holomorphic. This is of course true if D_1 is not a Hermitian symmetric space. Even if D_1 is a Hermitian symmetric space, the projection is Definition 2.11 is in general not holomorphic. However, in this case, there is a unique complex structure on D that will make the projection holomorphic and thus make the manifold D homogeneous Kähler. D is in general not homogeneous Kähler, thus the invariant Hermitian metric cannot be a Kähler metric.

Take a closer look of the above phenomena*. Let $D = G/V$ as in § 2. Consider the isotropy representation of the compact group V in $T_0(D)$, the tangent space of D at the original point. If V is the maximal compact subgroup of G, then the representation is irreducible and thus there is only one invariant almost complex structure. In general the group representation is not irreducible. Thus there are 2^N different almost complex structures on D where N is the number of irreducible components of the representation.

7. The curvature of the Hodge metric in dimension 1

In this section, we prove that in the one-dimensional case, the curvature of the Hodge metric is bounded near the boundary points with infinite Hodge distance. We will consider the n-dimensional case in the next paper [**14**].

Our starting point is the relation between the completeness of the metrics and the limiting Hodge structures. Such a relation was first drawn by Wang. In his paper [**28**], among the other results, Wang proved the following.

Theorem 7.1. *Let Δ^* be the one-dimensional parameter space of a family of polarized Calabi–Yau manifolds. Then the necessary and sufficient condition for the Weil–Petersson metric to be complete is $NA_0 \neq 0$, where N is the nilpotent operator in (7.1) of Δ^* and A_0 is defined in (7.2).*

As above, let Δ^* be the one-dimensional parameter space of a family of polarized Calabi–Yau manifolds. Let Ω be the section of the first Hodge bundle F^n. Then by the Nilpotent Orbit Theorem of Schmid (Theorem 2.19), after a possible base change, we have

$$\Omega = \exp\left(\frac{\sqrt{-1}}{2\pi} N \log \frac{1}{z}\right) A(z), \tag{7.1}$$

where N is the nilpotent operator, $N^{n+1} = 0$ for n the dimension of the Calabi–Yau manifolds, and

$$A(z) = A_0 + A_1 z + \cdots \tag{7.2}$$

* This was pointed out to the authors by Professor A. Todorov.

is a vector-valued convergent power series with the convergent radius $\delta > 0$ (see § 2 for details). Let

$$f_{k,l}(z) = z^k \left(\log \frac{1}{z}\right)^l$$

for any $k, l \geqslant 0$. Then we can write Ω as the convergent series

$$\Omega = \sum_{k,l} A_{k,l} z^k \left(\log \frac{1}{z}\right)^l = \sum_{k,l} A_{k,l} f_{k,l}. \tag{7.3}$$

Define $\deg f_{k,l} = k - l/(n+1)$. Then we have the following lemma.

Lemma 7.2. *The convergence of (7.3) is in the C^∞ sense. Furthermore, we have*

$$\left\| \Omega - \sum_{\deg f_{k,l} \leqslant \mu} A_{k,l} f_{k,l} \right\|_{C^s} \leqslant C r^{k_0 - s} \left(\log \frac{1}{r}\right)^{l_0}, \tag{7.4}$$

where $r = |z|$, k_0, l_0 are the unique pair of non-negative integers such that $l_0 \leqslant n$, $k_0 - l_0/(n+1) > \mu$ and for any pair of integers k', l' with $k' - l'/(n+1) > \mu$ we have $k' - l'/(n+1) \geqslant k_0 - l_0/(n+1)$. C is a constant depending only on k_0, l_0, μ and Ω.

Proof. From (7.2), we have $|A_k| \leqslant (\frac{1}{2}\delta)^{-k}$ and thus $|A_{k,l}| \leqslant (\frac{1}{4}\delta)^{-k-1}$ for small δ and large k. Thus we know that

$$\sum_{k,l} |A_{k,l} f_{k,l}| \leqslant \sum (\tfrac{1}{4}\delta)^{-k-1} r^k \left(\log \frac{1}{r}\right)^l < +\infty,$$

and thus the convergence in (7.3) is uniform for $r < \frac{1}{4}\delta$. To prove that the convergence is C^s for any $s \geqslant 1$, we observe that

$$f'_{k,l} = k f_{k-1,l} - l f_{k-1,l-1}.$$

Thus we have

$$\sum_{k,l} |A_{k,l} k f_{k-1,l}| + |A_{k,l} l f_{k-1,l-1}| \leqslant \sum_{k,l} (\tfrac{1}{4}\delta)^{-k-1} r^{k-1} \left(\log \frac{1}{r}\right)^l (k+n) < +\infty \tag{7.5}$$

for $r < \frac{1}{4}\delta$. Thus the convergence is C^1. Using mathematical induction, the convergence is in fact in the C^k sense. To get the quantitative result (7.4), we just observe that $(f_{k,l})^{(s)}$ is a linear combination of $f_{k-s,l}, \ldots, f_{k-s,l-s}$ with the coefficients not more than $(2(|k| + |l|))^s$. An inequality like (7.5) gives the requires estimates. □

Having finished the convergence of the series, we prove the following.

Theorem 7.3. *Assume the moduli space $\mathcal{M}$ of polarized Calabi–Yau threefolds is one dimensional. If Δ^* is a holomorphic chart of $\mathcal{M}$ such that Δ^* is complete at 0 with respect to the Hodge metric, then the Gauss curvature of the Hodge metric is bounded*.*

* The referee pointed out that the result is also true for partial Hodge metric.

To prove the theorem, we first assume that $NA_0 \neq 0$. Under this assumption, we have the following.

Lemma 7.4. *If $NA_0 \neq 0$, then the expression*

$$\left(\exp\left(\frac{\sqrt{-1}}{2\pi}N\log\frac{1}{z}\right)A_0, \overline{\exp\left(\frac{\sqrt{-1}}{2\pi}N\log\frac{1}{z}\right)A_0}\right)$$

is a non-constant polynomial of $\log(1/r)$, *where* $r = |z|$.

Proof. By the definition of the operator N, we know that N is an element of the Lie algebra of the Lie group G_R. Thus we know that the above expression is equal to

$$\left(\exp\left(\frac{\sqrt{-1}}{2\pi}N\log\frac{1}{r^2}\right)A_0, \overline{A_0}\right).$$

If the above expression is a constant, we then would have

$$(N^l A_0, \overline{A_0}) = 0 \tag{7.6}$$

for any positive integer l. Thus we would have

$$\left(\partial_z \exp\left(\frac{\sqrt{-1}}{2\pi}N\log\frac{1}{z}\right)A_0, \overline{\exp\left(\frac{\sqrt{-1}}{2\pi}N\log\frac{1}{z}\right)A_0}\right) = 0.$$

Since, by the assumption, $\partial_z \exp((\sqrt{-1}/2\pi)N\log(1/z))A_0 \neq 0$, the Nilpotent Orbit Theorem implies that

$$\left(\partial_z \exp\left(\frac{\sqrt{-1}}{2\pi}N\log\frac{1}{z}\right)A_0, \overline{\partial_z \exp\left(\frac{\sqrt{-1}}{2\pi}N\log\frac{1}{z}\right)A_0}\right) < 0,$$

which is a contradiction. □

In what follows we use l to denote the degree of the polynomial in the above lemma.

Corollary 7.5. *If $NA_0 \neq 0$, then*

$$r^{s+2}\left(\log\frac{1}{r}\right)^3 \left\| \omega_{\mathrm{WP}} - \tfrac{1}{4}l\frac{1}{r^2(\log(1/r))^2}\mathrm{d}z\wedge \mathrm{d}\bar{z}\right\|_{C^s} \leqslant C,$$

for any integer $s \geqslant 0$, where C is a constant depending only on s, n and the convergence radius δ.

Proof. For any mononomials of the form $z^t\bar{z}^s(\log(1/r))^l$, with integers t, s, l, we define the degree of it to be $t+s-l/(n+1)$. We write

$$(\Omega, \bar{\Omega}) = c\left(\log\frac{1}{r}\right)^l + R_0\left(\log\frac{1}{r}\right) + \tilde{R}\left(z, \bar{z}, \log\frac{1}{r}\right),$$

where $c(\log(1/z))^l$ is the highest-order term of the polynomial in Lemma 7.4, R_0 is the polynomial of $\log(1/r)$ of degree less than or equal to $l-1$ and $\tilde{R}(z,\bar{z},\log(1/r))$ contains the terms with degree at least positive. From Lemma 7.2, the above series converges in the sense of C^∞. The corollary follows from the fact that

$$\omega_{\mathrm{WP}} - \tfrac{1}{4}l\frac{1}{r^2(\log(1/r))^2}\mathrm{d}z\wedge\mathrm{d}\bar{z} = -\partial\bar{\partial}\log\frac{(\Omega,\bar{\Omega})}{(\log(1/r))^l}.$$

□

Now we assume that $NA_0 = 0$. We normalize $(\Omega,\bar{\Omega})$ such that $(A_0,\overline{A_0}) = 1$. Then we have the following expansion,

$$\log(\Omega,\bar{\Omega}) = P + \bar{P} + f(z,\bar{z})\left(\log\frac{1}{r}\right)^l + R(z,\bar{z}), \tag{7.7}$$

where $f(z,\bar{z}) \not\equiv 0$ is a homogeneous polynomial of degree $2k$*, and P is a polynomial of z of degree less than or equal to $2k-1$ but no less than 1 and $R(z,\bar{z})$ is a series of mononomials of degree great than $2k - l/(n+1)$. In the expansion, we allow that $l = 0$. But if $l = 0$, we assume that $f(z,\bar{z})$ is not of the form of $c(z^{2k}+\bar{z}^{2k})$, otherwise, we can include $f(z,\bar{z})$ in $P+\bar{P}$. By Lemma 7.2, the expansion is convergent in the C^∞ sense. We have the following observation.

Lemma 7.6. *If $l \geqslant 1$, then there are no z^{2k} or $\bar{z}^{2k}$ terms in the polynomial $f(z,\bar{z})$. In particular,*

$$\partial_z\bar{\partial}_z f(z,\bar{z}) \not\equiv 0.$$

Proof. From (7.7), we have the following expansion

$$(\Omega,\bar{\Omega}) = 1 + P + \bar{P} + f(z,\bar{z})\left(\log\frac{1}{r}\right)^l + \cdots,$$

where the terms in '$\cdots$' are the terms of degree at least $2k-(l-1)/(n+1)$, or the terms without $\log(1/r)$. If there is a non-zero z^{2k} term in $f(z,\bar{z})$, we must have

$$(N^l A_{2k}, \bar{A}_0) \neq 0,$$

which is not possible because of the assumption $NA_0 = 0$. Thus there are no z^{2k} or $\bar{z}^{2k}$ terms. Since f is not identically zero. This implies that

$$\partial_z\bar{\partial}_z f(z,\bar{z}) \not\equiv 0.$$

□

* We shall prove that the degree of the polynomial is actually an even number.

By (7.7) and the C^∞ convergence, we have

$$\lambda = -\partial_z \bar{\partial}_z f(z,\bar{z}) \left(\log \frac{1}{r}\right)^l + R\left(z, \bar{z}, \log \frac{1}{r}\right), \tag{7.8}$$

where $\lambda \, \mathrm{d}z \otimes \mathrm{d}\bar{z}$ defines the Weil–Petersson metric and where $R(z, \bar{z}, \log(1/r))$ contains terms of degree no less than $2k - 2 - (l-1)/(n+1)$. Since $\lambda > 0$, we must have

$$-\partial_z \bar{\partial}_z f(z, \bar{z}) \geqslant 0.$$

Thus $2k$ is an even number, otherwise the integral of the above expression along the unit circle would be zero, contradicting to Lemma 7.6. So k is actually an integer.

Lemma 7.7. *Using the same notations as above, we have*

$$f(z, \bar{z}) = cr^{2k},$$

for some constant c.

Proof. By Corollary 6.5, we have, up to a constant

$$h \geqslant -\partial_z \bar{\partial}_z \log \lambda,$$

where $h \mathrm{d}z \otimes \mathrm{d}\bar{z}$ defines the Hodge metric. By the Schwartz–Yau Lemma, we have

$$-\partial_z \bar{\partial}_z \log \lambda \leqslant h \leqslant \frac{1}{r^2 (\log(1/r))^2} \tag{7.9}$$

up to a constant. However, at a point where $\partial_z \bar{\partial}_z f \neq 0$, we have

$$\log \lambda = \log(-\partial_z \bar{\partial}_z f(z, \bar{z})) + l \log\left(\log \frac{1}{r}\right) + \log\left(1 + \frac{R(z, \bar{z}, \log(1/r))}{-\partial_z \bar{\partial}_z f(z, \bar{z})(\log(1/r))^l}\right).$$

Using the same method as in the proof of Corollary 7.5, we have

$$\partial_z \bar{\partial}_z \log\left(1 + \frac{R(z, \bar{z}, \log(1/r))}{-\partial_z \bar{\partial}_z f(z, \bar{z})(\log(1/r))^l}\right) = O\left(\frac{1}{r^2 (\log(1/r))^3}\right).$$

Using (7.9), we have

$$\partial_z \bar{\partial}_z \log(-\partial_z \bar{\partial}_z f(z, \bar{z})) \equiv 0,$$

otherwise it could have been of the order r^{-2}, which is a contradiction to (7.9). An elementary argument using Lemma 7.6 shows the $f(z, \bar{z})$ must be of the form stated in the lemma. □

Proof of Theorem 7.3. First we compute the scalar curvature of the Hodge metric. We use the same notation as in the previous sections. Let λ be the Weil–Petersson metric and let h be the Hodge metric. Let

$$\begin{aligned} K &= -\log(\Omega, \bar{\Omega}), \\ \Gamma_{11}^1 &= \frac{\partial \log \lambda}{\partial z} = \lambda^{-1}\partial_z \lambda, \\ K_1 &= -\partial_z \log(\Omega, \bar{\Omega}), \\ F_{111} &= (\Omega, \partial_z\partial_z\partial_z \Omega), \\ F_{1111} &= \partial_1 F_{111} - 3\Gamma_{11}^1 F_{111} + 2K_1 F_{111}, \\ A &= \lambda^{-2} e^{2K} |F_{111}|^2. \end{aligned}$$

Let $R_{1\bar{1}1\bar{1}}$ be the curvature of the Weil–Petersson metric and let $\tilde{R}_{1\bar{1}1\bar{1}}$ be the curvature of the Hodge metric.

Since $\mathcal{M}$ is the moduli space of polarized Calabi–Yau threefolds and $\mathcal{M}$ is one dimensional, from the Strominger formula we have

$$R_{1\bar{1}1\bar{1}} = 2\lambda^2 - \lambda^{-1} e^{2K} |F_{111}|^2.$$

So the Ricci curvature of the Weil–Petersson metric is

$$\mathrm{Ric}(\lambda) = -\partial_z\partial_{\bar{z}} \log \lambda = -\lambda^{-1} R_{1\bar{1}1\bar{1}} = -2\lambda + \lambda^{-2} e^{2K} |F_{111}|^2 = -2\lambda + A.$$

This implies that

$$h = (m+3)\lambda + \mathrm{Ric}(\lambda) = 4\lambda + (-2\lambda + A) = 2\lambda + A = \lambda(2 + \lambda^{-3} e^{2K} |F_{111}|^2).$$

So we have

$$\begin{aligned} \partial_z h &= \partial_z \lambda (2 + \lambda^{-3} e^{2K} |F_{111}|^2) \\ &\quad + \lambda[-3\lambda^{-4}\partial_z \lambda e^{2K} |F_{111}|^2 - 2\lambda^{-3}(\Omega, \bar{\Omega})^{-3}(\partial_z \Omega, \bar{\Omega})|F_{111}|^2 + \lambda^{-3} e^{2K} \partial_z F_{111} \overline{F_{111}}] \\ &= h\lambda^{-1}\partial_z \lambda + \lambda^{-2} e^{2K} \overline{F_{111}}(-3\Gamma_{11}^1 F_{111} + 2K_1 F_{111} + \partial_z F_{111}) \\ &= h\Gamma_{11}^1 + \lambda^{-2} e^{2K} \overline{F_{111}} F_{1111}. \end{aligned} \tag{7.10}$$

Similarly, we have

$$\partial_{\bar{z}} h = h\overline{\Gamma_{11}^1} + \lambda^{-2} e^{2K} F_{111} \overline{F_{1111}}.$$

So the curvature of the Hodge metric is

$$\begin{aligned} \tilde{R}_{1\bar{1}1\bar{1}} &= \partial_z \partial_{\bar{z}} h - h^{-1}\partial_z h \partial_{\bar{z}} h \\ &= \partial_{\bar{z}}(h\Gamma_{11}^1 + \lambda^{-2} e^{2K} \overline{F_{111}} F_{1111}) - h^{-1}\partial_z h \partial_{\bar{z}} h \\ &= \partial_{\bar{z}} h \Gamma_{11}^1 + h\partial_{\bar{z}}\Gamma_{11}^1 - 2\lambda^{-3}\partial_{\bar{z}}\lambda e^{2K} \overline{F_{111}} F_{1111} \\ &\qquad - 2\lambda^{-2}(\Omega, \bar{\Omega})^{-3}(\Omega, \overline{\partial_z \Omega})\overline{F_{111}} F_{1111} + \lambda^{-2} e^{2K} \overline{\partial_z F_{111}} F_{1111} \\ &\qquad\qquad + \lambda^{-2} e^{2K} \overline{F_{111}} \partial_{\bar{z}} F_{1111} - h^{-1}\partial_z h \partial_{\bar{z}} h \\ &= (h\overline{\Gamma_{11}^1} + \lambda^{-2} e^{2K} F_{111} \overline{F_{1111}})\Gamma_{11}^1 + h\partial_{\bar{z}}\Gamma_{11}^1 - 3\lambda^{-2}\Gamma_{11}^1 e^{2K} \overline{F_{111}} F_{1111} \\ &\qquad + \lambda^{-2}\Gamma_{11}^1 e^{2K} \overline{F_{111}} F_{1111} + 2\lambda^{-2} e^{2K} K_1 \overline{F_{111}} F_{1111} + \lambda^{-2} e^{2K} \overline{\partial_z F_{111}} F_{1111} \\ &\qquad\qquad + \lambda^{-2} e^{2K} \overline{F_{111}} \partial_{\bar{z}}(\partial_z F_{111} - 3\Gamma_{11}^1 F_{111} + 2K_1 F_{111}) - h^{-1}\partial_z h \partial_{\bar{z}} h. \end{aligned} \tag{7.11}$$

Using $\partial_{\bar z}\Gamma^1_{11} = \partial_z\partial_{\bar z}\log\lambda = -\operatorname{Ric}(\lambda) = 2\lambda - A$, from the above formula we have

$$\begin{aligned}
\tilde R_{1\bar11\bar1} &= h|\Gamma^1_{11}|^2 + \Gamma^1_{11}\lambda^{-2}\mathrm{e}^{2K}F_{111}\overline{F_{1111}} + (2\lambda + A)(2\lambda - A) + \lambda^{-2}\mathrm{e}^{2K}|F_{1111}|^2 \\
&\quad + \overline{\Gamma^1_{11}}\lambda^{-2}\mathrm{e}^{2K}\overline{F_{111}}F_{1111} + \lambda^{-2}\mathrm{e}^{2K}\overline{F_{111}}(-3(2\lambda - A) + 2\lambda)F_{111} \\
&\quad - h^{-1}(h\Gamma^1_{11} + \lambda^{-2}\mathrm{e}^{2K}\overline{F_{111}}F_{1111})(h\overline{\Gamma^1_{11}} + \lambda^{-2}\mathrm{e}^{2K}F_{111}\overline{F_{1111}}) \\
&= 4\lambda^2 - A^2 + \lambda^{-2}\mathrm{e}^{2K}|F_{1111}|^2 + A(3A - 4\lambda) - h^{-1}\lambda^{-4}e^{4K}|F_{111}|^2|F_{1111}|^2 \\
&= 4\lambda^2 - 4\lambda A + 2A^2 + \lambda^{-2}\mathrm{e}^{2K}|F_{1111}|^2(1 - h^{-1}A) \\
&= 4\lambda^2 - 4\lambda A + 2A^2 + \lambda^{-2}\mathrm{e}^{2K}|F_{1111}|^2(2\lambda h^{-1}) \\
&= 4\lambda^2 - 4\lambda A + 2A^2 + 2\lambda^{-1}\mathrm{e}^{2K}|F_{1111}|^2h^{-1}.
\end{aligned} \tag{7.12}$$

The scalar curvature of the Hodge metric is given by

$$\begin{aligned}
\rho &= -h^{-2}\tilde R_{1\bar11\bar1} \\
&= -\frac{4\lambda^2 - 4\lambda A + 2A^2}{(2\lambda + A)^2} - \frac{2\lambda^{-1}\mathrm{e}^{2K}|F_{1111}|^2}{(2\lambda + A)^2} \\
&= -\frac{4 - 4\mathrm{e}^{2K}\lambda^{-3}|F_{111}|^2 + 2\mathrm{e}^{4K}\lambda^{-6}|F_{111}|^4}{(2 + \mathrm{e}^{2K}\lambda^{-3}|F_{111}|^2)^2} - \frac{2\mathrm{e}^{2K}\lambda^{-4}|F_{1111}|^2}{(2 + \mathrm{e}^{2K}\lambda^{-3}|F_{111}|^2)^3}.
\end{aligned} \tag{7.13}$$

Apparently, the first term on the right-hand side of (7.13) is bounded. Thus in order to prove the theorem, we just need to bound the second term of the right-hand side of (7.13).

Case 1 ($NA_0 \neq 0$). In this case, by Corollary 7.5, we have

$$\lambda \sim \frac{1}{r^2(\log(1/r))^2}. \tag{7.14}$$

For the Yukawa coupling F_{111}, we always have $F_{111} = O(1/r^3)$. If $|F_{111}| = O(1/r^2)$, then $|F_{1111}| = O(1/r^3)$. Thus

$$2\mathrm{e}^{2K}\lambda^{-4}|F_{1111}|^2 \to 0,$$

and is bounded. If $F_{111} \sim 1/z^3$, then we have the following asymptotic computations:

$$\begin{aligned}
\partial_1 F_{111} &\sim \frac{-3}{z^4}, \\
\Gamma^1_{11}F_{111} &\sim \frac{-1}{z^4}, \\
|K_1F_{111}| &\leqslant C\frac{1}{r^4\log(1/r)}.
\end{aligned}$$

Thus we have

$$|F_{1111}| \leqslant C\frac{1}{r^4\log(1/r)}. \tag{7.15}$$

Using the fact that $F_{111} \sim 1/z^3$, we have

$$e^{2K}\lambda^{-3} \sim r^6. \tag{7.16}$$

Using (7.14), (7.15) and (7.16), we proved that in this case the curvature is bounded.

Case 2 ($NA_0 = 0$). In this case, by (7.8) and Lemma 7.7,

$$\lambda = -ck^2 r^{2(k-1)}\left(\log\frac{1}{r}\right)^l + R\left(z,\bar{z},\log\frac{1}{r}\right),$$

where $R(z,\bar{z},\log(1/r))$ contains terms of order at least $2(k-1)-(l-1)/(n+1)$. We claim that $l \geqslant 1$, otherwise, by the above equation, we would have that the Hodge metric, as the linear combination of the Weil–Petersson metric and its Ricci curvature, satisfying

$$h \leqslant \frac{(\log(1/r))^s}{r},$$

for some positive integer s, and thus is incomplete. A straightforward computation gives

$$e^{2K}\lambda^{-2}|F_{111}|^2 \sim \frac{1}{r^2(\log(1/r))^2}.$$

This implies that

$$F_{111} \sim z^{2k-3},$$

and by using the same argument as we did in Case 1, we have

$$|F_{1111}| \leqslant Cr^{2k-4}.$$

Thus we have

$$\frac{2e^{2K}\lambda^{-4}|F_{1111}|^2}{(2+e^{2K}\lambda^{-3}|F_{111}|^2)^3} \leqslant \frac{2e^{-4K}\lambda^5|F_{1111}|^2}{|F_{111}|^6},$$

and it is bounded. □

8. The Weil–Petersson geometry

By a classical result of Wolpert [**30**], the curvature of the Weil–Petersson metric on Teichmüller space is non-positive. However, the curvature of the Weil–Petersson metric on the moduli space of Calabi–Yau manifolds does not have such a good property*. The bad curvature property makes it difficult to do geometric analysis on the moduli space. In order to overcome this difficulty, in [**12**, **13**], the first author introduced a new Kähler metric called Hodge metric. On one side, the holomorphic bisectional curvature of the Hodge metric is non-positive, on the other side, up to a constant, the Weil–Petersson metric is smaller than the Hodge metric. Thus one can use the Hodge metric to do the

* In fact, physicists found that the curvature of the Weil–Petersson metric on certain moduli space can either be positive or negative [**2**, p. 65].

similar geometric analysis as that on Teichmüller space and then translate the results back in the language of the Weil–Petersson metric.

In the proof of the non-positivity of the curvature of Hodge metrics (cf. [**12**, **13**] and Theorem 4.3), we do not need the assumption that the manifold is the moduli space of Calabi–Yau manifolds. All we need is the fact that the manifold is a horizontal slice and there is a Weil–Petersson metric on it. In fact, the existence of the Weil–Petersson metric gives severe restrictions on the variation of the Hodge structures. These kinds of restrictions have not been studied comprehensively.

Lemmas 7.6 and 7.7 are good examples of how the existence of the Weil–Petersson metric affects the variation of the Hodge structures at infinity of the horizontal slices. In fact, using the notations in § 7, Lemma 7.7 implies the following.

Proposition 8.1. *Let k, l be defined in (7.7). Then if $l \geqslant 1$, we have*

$$(N^l A_p, \bar{A}_q) = 0$$

for any $p + q = 2k$ but $p \neq q$, where the vectors A_p are defined in (7.2).

Besides the case $p = 0$, it is rather difficult to prove the above result without using the Schwarz–Yau inequality. We believe that there are more properties of this kind. Because of this, we defined the following concept of the Weil–Petersson geometry and would like to study the properties in a systematic way.

Definition 8.2. The Weil–Petersson geometry contains a Kähler orbifold M with the orbifold metric ω_{WP} such that the following hold.

(1) Let $\tilde{M}$ be the universal covering space of M. Then there is a natural immersion $\tilde{M} \to D$ from M to the classifying space D (cf. [**6**]) such that M is a horizontal slice of D. In this way, we can also endowed the Hodge bundles $F^1, \dots, F^n$ to M where F^n is a line bundle.

(2) ω_{WP} is the curvature of the bundle F^n. It is positive-definite and thus defines a Kähler metric in M and is called the Weil–Petersson metric.

(3) M is quasi-projective and F^n is an ample line bundle of M. The compactification is called Viehweg compactification [**27**, p. 21, Theorem 1.13]. The Hodge bundles $F^1, \dots, F^n$ extend to the compactification $\bar{M}$ of M*.

(4) After passing to a finite covering and after desingularization, in a neighbourhood of the infinity, M can be written as

$$\Delta^{n-k} \times (\Delta^*)^k,$$

where Δ is the unit disk and Δ^* is the punctured unit disk. Let Ω be a local section of F^n in the neighbourhood, then locally, Ω can be (multi-valuedly) written as

$$\Omega = \exp\left(\sqrt{-1}\left(N_1 \log \frac{1}{z_1} + \cdots + N_k \log \frac{1}{z_k}\right)\right) A(z_1, \dots, z_n),$$

* This follows from Schmid's Nilpotent Orbit Theorem [**20**].

where $N_1, \ldots, N_k$ are nilpotent operators and A is a vector valued holomorphic function of $z_1, \ldots, z_n$.

Remark 8.3. The first property of above is basically the Griffiths transversality [**6**]. The second property is a theorem of Tian [**25**]. The third one is the compactification theorem of Viehweg [**27**] and the fourth property is the Nilpotent Orbit Theorem of Schmid [**20**].

The theorems in this paper are true for abstract Weil–Petersson geometry defined above. A further study if the Weil–Petersson geometry will be the project of future study. In particular, we wish to define a natural metric which is a modification of the Hodge metric at infinity similar to that of McMullen's [**15**] in the case of Teichmüller space. It would be interesting if we can do so in the category of the Weil–Petersson Geometry.

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9. Appendix

In this appendix we prove Theorem 4.2. As before, the subscripts $i, j, \ldots$ all range from 1 to m, unless otherwise noted.

Proof of Theorem 4.2. By definition, $\omega_\mu = \mu\omega_{\mathrm{WP}} + \mathrm{Ric}(\omega_{\mathrm{WP}})$, since the Weil–Petersson metric is Kähler, we know ω_μ is d-closed. From the Strominger formula (4.4), we know the Ricci tensor of the Weil–Petersson metric is

$$R_{i\bar{j}} = -(m+1)g_{i\bar{j}} + g^{k\bar{l}}F_{i\bar{j}k\bar{l}}. \tag{9.1}$$

Thus we have

$$h_{i\bar{j}} = \lambda g_{i\bar{j}} + g^{\alpha\bar{\beta}}F_{i\bar{j}\alpha\bar{\beta}}, \tag{9.2}$$

where $\lambda = \mu - m - 1$. Thus $\omega_\mu > 0$ which implies ω_μ is Kähler.

Usually, choosing a normal coordinate system will simplify the computation greatly. However, in the following computation, the use of general coordinates will make the computation easier.

To simplify the computation we first calculate $\bar{\partial}_l D_\alpha D_i \Omega$. Using Remark 3.1 and Lemma 3.2 we have

$$\begin{aligned}
\bar{\partial}_l D_\alpha D_i \Omega &= \bar{\partial}_l(\partial_\alpha D_i \Omega + K_\alpha D_i \Omega - \Gamma^\gamma_{i\alpha} D_\gamma \Omega) \\
&= \partial_\alpha(\bar{\partial}_l D_i \Omega) + (\bar{\partial}_l K_\alpha) D_i \Omega + K_\alpha \bar{\partial}_l D_i \Omega - (\bar{\partial}_l \Gamma^\gamma_{i\alpha}) D_\gamma \Omega - \Gamma^\gamma_{i\alpha} \bar{\partial}_l D_\gamma \Omega \\
&= \partial_\alpha(g_{i\bar{l}} \Omega) + g_{\alpha\bar{l}} D_i \Omega + K_\alpha g_{i\bar{l}} \Omega - R_{i\bar{\tau}\alpha\bar{l}} g^{\gamma\bar{\tau}} D_\gamma \Omega - g^{\gamma\bar{\tau}} \frac{\partial g_{i\bar{\tau}}}{\partial z_\alpha} g_{\gamma\bar{l}} \Omega \\
&= g_{i\bar{l}} D_\alpha \Omega + g_{\alpha\bar{l}} D_i \Omega - R_{i\bar{\tau}\alpha\bar{l}} g^{\gamma\bar{\tau}} D_\gamma \Omega \\
&= F_{i\bar{\tau}\alpha\bar{l}} g^{\gamma\bar{\tau}} D_\gamma \Omega.
\end{aligned} \tag{9.3}$$

Similarly, we have

$$\partial_k \overline{D_\beta D_j \Omega} = \overline{\bar{\partial}_k D_\beta D_j \Omega} = F_{p\bar{\beta}k\bar{j}} g^{p\bar{q}} \overline{D_q \Omega} \tag{9.4}$$

since F is a curvature like tensor. Now because Ω is holomorphic, we have $\partial_k \bar{\Omega} = \bar{\partial}_l \Omega = 0$. Using Lemma 3.3, equation (9.4) and the Hodge–Riemann relations, we know that

$$(D_\alpha D_i \Omega, \partial_k \overline{D_\beta D_j \Omega}) = F_{p\bar{\beta}k\bar{j}} g^{p\bar{q}} (D_\alpha D_i \Omega, \overline{D_q \Omega}) = 0,$$

which implies

$$\begin{aligned}
\frac{\partial h_{i\bar{j}}}{\partial z_k} &= \lambda \frac{\partial g_{i\bar{j}}}{\partial z_k} + \frac{(\partial_k D_\alpha D_i \Omega, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \frac{(D_\alpha D_i \Omega, \partial_k \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \\
&\quad - \frac{(D_\alpha D_i \Omega, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})^2} g^{\alpha\bar{\beta}} (\partial_k \Omega, \bar{\Omega}) + \frac{(D_\alpha D_i \Omega, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} \frac{\partial g^{\alpha\bar{\beta}}}{\partial z_k} \\
&= \lambda \frac{\partial g_{i\bar{j}}}{\partial z_k} + \frac{(\partial_k D_\alpha D_i \Omega, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \\
&\quad + \frac{(K_k D_\alpha D_i \Omega, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} - \frac{(\Gamma^p_{\alpha k} D_p D_i \Omega, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \\
&= \lambda \frac{\partial g_{i\bar{j}}}{\partial z_k} + \frac{(T_{k\alpha i}, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \Gamma^p_{ik} \frac{(D_p D_\alpha \Omega, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \\
&= \lambda \frac{\partial g_{i\bar{j}}}{\partial z_k} + \frac{(T_{k\alpha i}, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \Gamma^p_{ik} (h_{p\bar{j}} - \lambda g_{p\bar{j}}) \\
&= \frac{(T_{k\alpha i}, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \Gamma^p_{ik} h_{p\bar{j}}.
\end{aligned} \tag{9.5}$$

Similarly, we have

$$\frac{\partial h_{i\bar{j}}}{\partial \bar{z}_l} = \frac{(D_\alpha D_i \Omega, \overline{T_{l\beta j}})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \overline{\Gamma^q_{jl}} h_{i\bar{q}}. \tag{9.6}$$

From (9.5) and (9.6), we have

$$
\begin{aligned}
\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} &= \frac{(\bar{\partial}_l T_{k\alpha i}, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \frac{(T_{k\alpha i}, \overline{\partial_l D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \\
&\quad - \frac{(T_{k\alpha i}\Omega, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})^2} g^{\alpha\bar{\beta}} (\Omega, \overline{\partial_l \Omega}) + \frac{(T_{k\alpha i}, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} \frac{\partial g^{\alpha\bar{\beta}}}{\partial \bar{z}_l} \\
&\quad + (\bar{\partial}_l \Gamma^p_{ik}) h_{p\bar{j}} + \Gamma^p_{ik} \frac{\partial h_{p\bar{j}}}{\partial \bar{z}_l} \\
&= \frac{(\bar{\partial}_l T_{k\alpha i}, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \frac{(T_{k\alpha i}, \overline{\partial_l D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \\
&\quad + \frac{(T_{k\alpha i}, \overline{K_l D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} - \frac{(T_{k\alpha i}, \overline{\Gamma^q_{\beta l} D_q D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \\
&\quad + R_{i\bar{q}k\bar{l}} g^{p\bar{q}} h_{p\bar{j}} + \Gamma^p_{ik} \left(\frac{(D_\alpha D_p \Omega, \overline{T_{l\beta j}})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \overline{\Gamma^q_{jl}} h_{p\bar{q}} \right) \\
&= \frac{(\bar{\partial}_l T_{k\alpha i}, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \frac{(T_{k\alpha i}, \overline{T_{l\beta j}})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \overline{\Gamma^q_{jl}} \frac{(T_{k\alpha i}, \overline{D_q D_\beta \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \\
&\quad + R_{i\bar{q}k\bar{l}} g^{p\bar{q}} h_{p\bar{j}} + \Gamma^p_{ik} \left(\frac{(D_\alpha D_p \Omega, \overline{T_{l\beta j}})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \overline{\Gamma^q_{jl}} h_{p\bar{q}} \right). \qquad (9.7)
\end{aligned}
$$

Since

$$
\tilde{R}_{i\bar{j}k\bar{l}} = \frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} - h^{s\bar{t}} \frac{\partial h_{i\bar{t}}}{\partial z_k} \frac{\partial h_{s\bar{j}}}{\partial \bar{z}_l},
$$

by (9.7), (9.5) and (9.6), we have

$$
\begin{aligned}
\tilde{R}_{i\bar{j}k\bar{l}} &= \frac{(\bar{\partial}_l T_{k\alpha i}, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \frac{(T_{k\alpha i}, \overline{T_{l\beta j}})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \overline{\Gamma^q_{jl}} \frac{(T_{k\alpha i}, \overline{D_q D_\beta \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \\
&\quad + R_{i\bar{q}k\bar{l}} g^{p\bar{q}} h_{p\bar{j}} + \Gamma^p_{ik} \left(\frac{(D_\alpha D_p \Omega, \overline{T_{l\beta j}})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \overline{\Gamma^q_{jl}} h_{p\bar{q}} \right) \\
&\quad - h^{s\bar{t}} \left(\frac{(T_{k\alpha i}, \overline{D_\beta D_t \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \Gamma^p_{ik} h_{p\bar{t}} \right) \left(\frac{(D_\gamma D_s \Omega, \overline{T_{l\tau j}})}{(\Omega, \bar{\Omega})} g^{\gamma\bar{\tau}} + \overline{\Gamma^q_{jl}} h_{s\bar{q}} \right) \\
&= \frac{(\bar{\partial}_l T_{k\alpha i}, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \frac{(T_{k\alpha i}, \overline{T_{l\beta j}})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \overline{\Gamma^q_{jl}} \frac{(T_{k\alpha i}, \overline{D_q D_\beta \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \\
&\quad + R_{i\bar{q}k\bar{l}} g^{p\bar{q}} h_{p\bar{j}} + \Gamma^p_{ik} \frac{(D_\alpha D_p \Omega, \overline{T_{l\beta j}})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \Gamma^p_{ik} \overline{\Gamma^q_{jl}} h_{p\bar{q}} \\
&\quad - h^{s\bar{t}} \frac{(T_{k\alpha i}, \overline{D_\beta D_t \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \frac{(D_\gamma D_s \Omega, \overline{T_{l\tau j}})}{(\Omega, \bar{\Omega})} g^{\gamma\bar{\tau}} - h^{s\bar{t}} \Gamma^p_{ik} h_{p\bar{t}} \overline{\Gamma^q_{jl}} h_{s\bar{q}} \\
&\quad - h^{s\bar{t}} \frac{(T_{k\alpha i}, \overline{D_\beta D_t \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \overline{\Gamma^q_{jl}} h_{s\bar{q}} - h^{s\bar{t}} \frac{(D_\gamma D_s \Omega, \overline{T_{l\tau j}})}{(\Omega, \bar{\Omega})} g^{\gamma\bar{\tau}} \Gamma^p_{ik} h_{p\bar{t}}
\end{aligned}
$$

$$= \frac{(T_{k\alpha i}, \overline{T_{l\beta j}})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} - h^{s\bar{t}} \frac{(T_{k\alpha i}, \overline{D_\beta D_t \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \frac{(D_\gamma D_s \Omega, \overline{T_{l\tau j}})}{(\Omega, \bar{\Omega})} g^{\gamma\bar{\tau}} + \frac{(\bar{\partial}_l T_{k\alpha i}, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + R_{i\bar{q}k\bar{l}} g^{p\bar{q}} h_{p\bar{j}}. \tag{9.8}$$

Using (9.3), Lemmas 3.2 and 3.3 and the Hodge–Riemann relations, we have

$$\begin{aligned} &\frac{(\bar{\partial}_l T_{k\alpha i}, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \\ &= \frac{(\partial_k \bar{\partial}_l D_\alpha D_i \Omega, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \frac{((\bar{\partial}_l K_k) D_\alpha D_i \Omega, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \\ &\quad - \frac{((\bar{\partial}_l \Gamma^p_{\alpha k}) D_p D_i \Omega, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} - \frac{((\bar{\partial}_l \Gamma^p_{ik}) D_p D_\alpha \Omega, \overline{D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \\ &= F_{\alpha\bar{l}i\bar{q}} F_{p\bar{\beta}k\bar{j}} g^{\alpha\bar{\beta}} g^{p\bar{q}} + F_{\alpha\bar{\beta}i\bar{j}} g^{\alpha\bar{\beta}} g_{k\bar{l}} - R_{\alpha\bar{q}k\bar{l}} F_{p\bar{\beta}i\bar{j}} g^{\alpha\bar{\beta}} g^{p\bar{q}} - R_{i\bar{q}k\bar{l}} F_{p\bar{\beta}\alpha\bar{j}} g^{\alpha\bar{\beta}} g^{p\bar{q}}. \end{aligned} \tag{9.9}$$

By (9.2) and the Strominger formula (4.4), the above expression is

$$F_{i\bar{q}\alpha\bar{l}} F_{p\bar{j}k\bar{\beta}} g^{\alpha\bar{\beta}} g^{p\bar{q}} + F_{\alpha\bar{q}k\bar{l}} F_{i\bar{j}p\bar{\beta}} g^{\alpha\bar{\beta}} g^{p\bar{q}} + \lambda(g_{i\bar{j}} g_{k\bar{l}} + g_{i\bar{l}} g_{k\bar{j}}) - (\lambda+1) F_{i\bar{j}k\bar{l}}. \tag{9.10}$$

Using the Hodge–Riemann relations we have

$$\frac{(T_{k\alpha i}, \overline{T_{l\beta j}})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} = \frac{(D_k D_\alpha D_i \Omega, \overline{D_l D_\beta D_j \Omega})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} + \frac{(E_{k\alpha i}, \overline{E_{l\beta j}})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} \tag{9.11}$$

and

$$h^{s\bar{t}} \frac{(T_{k\alpha i}, \overline{D_\beta D_t \Omega})}{(\Omega, \bar{\Omega})} \frac{(D_\gamma D_s \Omega, \overline{T_{l\tau j}})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} g^{\gamma\bar{\tau}} = h^{s\bar{t}} \frac{(E_{k\alpha i}, \overline{D_\beta D_t \Omega})}{(\Omega, \bar{\Omega})} \frac{(D_\gamma D_s \Omega, \overline{E_{l\tau j}})}{(\Omega, \bar{\Omega})} g^{\alpha\bar{\beta}} g^{\gamma\bar{\tau}}. \tag{9.12}$$

Theorem 4.2 follows from (9.10), (9.11) and (9.12). □

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