

Quantum layers over surfaces ruled outside a compact set

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In this paper, we prove that quantum layers over a surface which is ruled outside a compact set, nonplanar but asymptotically flat, admit a ground state for the Dirichlet Laplacian. The work here also represents some technical progress toward resolving the conjecture that a quantum layer over any nonplanar, asymptotically flat surface with integrable Gauss curvature must possess ground state. © 2007 American Institute of Physics. [DOI: 10.1063/1.2736518]

I. INTRODUCTION

The spectrum of the Laplacian on manifolds is a classical domain of research within geometric analysis. One of its least developed areas is the spectral analysis on noncompact, noncomplete manifolds. An interesting paper by Duclos *et al.*² demonstrated, under certain geometric conditions, the existence of discrete spectrum for the Dirichlet Laplacian acting on L^2 functions on a particular type of noncompact, noncomplete manifold which they termed the “quantum layer.” The normalized eigenfunctions corresponding to each point in the discrete spectrum are termed “bound states” in quantum physics. The lowest eigenvalue in the discrete spectrum is usually of multiplicity one, and the corresponding normalized eigenfunction is termed the “ground state.” Thus the authors in Ref. 2 proved the existence of a ground state for the Dirichlet Laplacian on certain quantum layers. The main results in their paper were later generalized in Ref. 1. Moreover, in Refs. 5 and 6, Lin and Lu subsequently generalized the same main results to higher dimensions under more general geometric settings. The existence of discrete spectrum for the Laplacian is a nontrivial phenomenon on noncompact manifolds, even when the manifold is complete. Thus the results in Refs. 2, 1, 5, and 6 represent rare instances in which the discrete spectrum is clearly known to exist on noncompact, noncomplete manifolds.

We recall the definition of a quantum layer below, following Ref. 5.

Definition 1: Let $\Sigma \hookrightarrow \mathbb{R}^3$ be an isometrically immersed, complete, noncompact,¹ oriented hypersurface. Let N and A , respectively, be the unit normal vector field and the second fundamental form on Σ , and define the map

$$p: \Sigma \times (-a, a) \rightarrow \mathbb{R}^3 \quad (1)$$

by $(x, u) \mapsto x + uN$, where the number $a > 0$, which is called the depth of the layer, satisfies $a\|A\| < C_o < 1$ on Σ for some constant C_o . We define a quantum layer to be the smooth manifold $\Omega = \Sigma \times (-a, a)$ with the pullback metric $p^*(ds_E^2)$, where ds_E^2 denotes the Euclidean metric in \mathbb{R}^3 .

As one can see from the definition, a quantum layer is just a tubular neighborhood of a surface in \mathbb{R}^3 . The relation between A and the depth a above is simply to ensure that the map (1) is also an immersion. However, the main result in this paper considers an embedding of Σ (which is an

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¹It will be understood in the rest of the paper that the surface over which a quantum layer is built is always complete and noncompact.

extra global condition imposed on the map p) due to topological reasons necessary for our arguments (see Secs. III and IV). The terminology of “quantum” simply eludes to the fact that when suitable geometric conditions are imposed on Ω ,² ground state exists.

On the physics side, the problem above has become increasingly important, due to advances in nanotechnology, quantum waveguides, etc. In particular, the bound state of the Dirichlet Laplacian corresponds to the bound state of an electron trapped in the quantum layer (under hard wall boundary condition). Since this paper is primarily concerned with the geometric aspect of the problem, we refer the reader to Refs. 2 and 1, and the many references cited there for more information on the physical aspect of the problem.

In this paper we will discuss a rather general class of surfaces whose layers admit ground state for the Dirichlet Laplacian (definition is described below). We consider surfaces in \mathbb{R}^3 which are ruled outside a compact subset. A ruled surface is such that for each point on the surface there passes an Euclidean straight line, called a ruling, lying also on the surface, and such that a local collection of such lines and the orthogonal flow lines through them constitute a local coordinate system on the surface. Since this description is local in nature (at least we can allow the situation where the rulings end somewhere on the surface), we can consider surfaces which are potentially not ruled inside the compact subset. More details about ruled surfaces will be given in the next section.

Before we go any further, let us establish a consensus on the functional analysis background of this paper. On a Riemannian manifold (M, g) , the Laplacian Δ is first defined on smooth, compactly supported functions. Then depending on the metric g , one wishes to extend Δ to a self-adjoint operator (also denoted by Δ) on $L^2(M)$. When the manifold is complete without boundary, then the Laplacian has a unique self-adjoint extension. In the case of a manifold with boundary, but compact, a unique self-adjoint extension can also be obtained subject to either Dirichlet or Neumann boundary conditions. For a noncompact, noncomplete manifold such as a quantum layer, we do not have general results concerning the existence and uniqueness of self-adjoint extensions of Δ . However, we can define the Dirichlet Laplacian, also denoted by Δ , via the closure of the quadratic form

$$Q(\phi, \psi) = \int_M \langle \nabla \phi, \nabla \psi \rangle$$

for all smooth, compactly supported functions ϕ and ψ . This is known as the Friedrichs extension of the Laplacian. We refer to Ref. 7 for background on quadratic forms, the spectral theorem, and general definitions for the different spectra. In particular, we are interested in the discrete spectrum (discrete eigenvalues with finite multiplicity) and its complement in the (total) spectrum—the essential spectrum. We will use the notations $\sigma(\Delta)$, $\sigma_{\text{disc}}(\Delta)$, and $\sigma_{\text{ess}}(\Delta)$ to denote the spectrum, discrete spectrum, and essential spectrum, respectively. The idea that we use in this paper to show the existence of discrete spectrum is to show that there is a gap between the bottom of the essential spectrum and the bottom of the total spectrum.

The essential spectrum is stable under compact perturbations in the sense that $\sigma_{\text{ess}}(\Delta + T) = \sigma_{\text{ess}}(\Delta)$ for any compact operator T . Thus one would like to think that compact perturbations of the metric on a manifold amounts to perturbing Δ by a compact operator. This is a useful idealization, although heuristic. However, in light of this heuristic notion, we have the following result, which is not surprising due to the fact that the Dirichlet Laplacian on a layer of depth a over the plane has essential spectrum as its entire spectrum and starts exactly at $(\pi/2a)^2$.

Theorem 1: (see Refs. 2,1, or 5). *Let Ω be a quantum layer over an isometrically immersed surface in \mathbb{R}^3 , if we further assume that $\|\mathbf{A}\| \rightarrow 0$ at infinity, then the bottom of the essential spectrum of the Dirichlet Laplacian is bounded below as*

²The geometric conditions are actually imposed on Σ , and this should be natural.

$$\inf \sigma_{\text{ess}}(\Delta) \geq \left(\frac{\pi}{2a}\right)^2.$$

In fact, it was shown in Ref. 1 that $\inf \sigma_{\text{ess}}(\Delta) = (\pi/2a)^2$ for any quantum layer over an embedded surface with second fundamental form going to zero at infinity (also termed asymptotically flat, or asymptotically planar).

The variational principle says that

$$\inf \sigma(\Delta) = \inf_{\phi \in C_0^\infty(\Omega)} \frac{\int_{\Omega} |\nabla \phi|^2}{\int_{\Omega} |\phi|^2}.$$

Therefore, assuming asymptotic flatness of the surface Σ , to prove that the quantum layer Ω has ground state, it suffices to find a test function ϕ such that

$$\frac{\int_{\Omega} |\nabla \phi|^2}{\int_{\Omega} |\phi|^2} < (\pi/2a)^2. \tag{2}$$

This was achieved first in Ref. 2, and more generally in Refs. 1 and 5 (Theorem 1.1) by assuming L^1 Gauss curvature and nonpositivity of the total Gauss curvature. Thus there is the following conjecture.

Conjecture 1: *Let Σ be a complete, noncompact surface isometrically immersed in \mathbb{R}^3 and asymptotically flat. If Σ is not the plane and the Gaussian curvature K is integrable on Σ , then the Dirichlet Laplacian on any quantum layer over Σ has nonempty discrete spectrum.*

The remaining case of the conjecture above, which is the $\int_{\Sigma} K > 0$ case, was answered partially through special examples of surfaces in both Refs. 2 and 5. In particular, in Ref. 5 (Theorem 1.3) the example of quantum layers over a convex surface (graph of a convex function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ in \mathbb{R}^3) was shown to have ground state (assuming asymptotic flatness). Note that a convex surface has positive Gauss curvature everywhere.

We must emphasize that for the convex surface example, a careful analysis of the (integral of) mean curvature was used in an essential way. Therefore, it has become increasingly clear that in order to answer the conjecture for the $\int_{\Sigma} K > 0$ case, the mean curvature H (or more generally, the second fundamental form) of Σ must always play a central role. In particular, the proof of the Main Theorem in this paper depends on the knowledge of the mean curvature in an essential way.

Our main result is as follows.

Theorem 2: (Main Theorem). *Let Σ be an embedded, nonplanar surface in \mathbb{R}^3 that is ruled outside a compact subset. Then the bottom of the spectrum of the Dirichlet Laplacian on a quantum layer Ω of depth a over Σ has the upper bound*

$$\inf \sigma(\Delta) < \left(\frac{\pi}{2a}\right)^2.$$

Remark 1: *Note that for a surface which is ruled outside a compact set, the Gauss curvature is automatically integrable.³ If the surface is flat outside a compact set, then Theorem 2 overlaps with the Main Theorem in, Ref. 1 or 5. This is because in that case the total Gauss curvature would be nonpositive.⁴ We would also like to point out that our proof for the Main Theorem is*

³See Sec. III.

⁴An easy way to see this is to use Hartman’s formula (see Sec. III).

actually localized in one local ruled coordinate chart with appropriate properities. Therefore, the Main Theorem can, in fact, be stated with weaker conditions by assuming that Σ has one such local coordinate chart. For more details and further comments on this generalization, see Sec. IV.

Combining with Theorem 1, we also obtain the following.

Corollary 1: *The quantum layer above, along with the assumption that the second fundamental form goes to zero at infinity, admits a ground state for the Dirichlet Laplacian.*

At this point we must mention another important result in Ref. 1 (originally appeared in Ref. 2 under a less general setup).

Theorem 3: [Theorem 1(c) of Ref. 1]. *Let Σ be an embedded surface in \mathbb{R}^3 of integrable Gauss curvature, then a quantum layer over Σ satisfies*

$$\inf \sigma(\Delta) < \left(\frac{\pi}{2a} \right)^2$$

provided that $\int_{\Sigma} H^2 = \infty$ and $\int_{\Sigma} |\nabla H|^2 < \infty$.

We will show in this paper that via a result of White,⁸ the following is true.

Proposition 1: *Let Σ be an embedded surface in \mathbb{R}^3 . If $K \in L^1(\Sigma)$ and $\int_{\Sigma} K > 0$, then $\int_{\Sigma} H^2 = \infty$.*

The result above answers affirmatively a conjecture raised by the authors in Ref. 1 [formula (20) in Ref. 1]. We would also like to point out that White's theorem plays an essential role in the proof we give for the Main Theorem.⁵

On the other hand, the condition that $\nabla H \in L^2(\Sigma)$ can be obtained directly for a surface ruled outside a compact set, by following similar computations in the proof of our Main Theorem. However, we would like to point out that the work required to verify this is essentially the same as our method of proof for the Main Theorem. Moreover, by Remark 1 on the generalization of the Main Theorem, it is clear that there are surfaces which do not have $\nabla H \in L^2$ since they are not entirely ruled outside a compact set. But perhaps more importantly, we believe our method of proof (localized in one ruled coordinate chart) illustrates some fundamental techniques which may be employed in the future resolution of Conjecture 1.

We end this section with an overview of the rest of the paper. In Sec. II we discuss the (local) geometry of ruled surfaces. In Sec. III we give relevant information about the topology of non-compact surfaces with integrable Gauss curvature. The discussion there will center around the generalized Gauss-Bonnet Theorem of Hartman.³ The proof of Proposition 1 is also contained in Sec. III. Section IV contains the proof of the Main Theorem.

II. THE GEOMETRY OF RULED SURFACES

Here we discuss some basic geometry about ruled surfaces.

Definition 2: *A nonintersecting surface Σ in \mathbb{R}^3 is called a ruled surface if every point lies in a coordinate chart of the form*

$$x(s, v) = \beta(s) + v \delta(s), \quad (3)$$

where for each s , $\beta(s)$ and $\delta(s)$ are vector fields in \mathbb{R}^3 .

Note that $\beta(s)$ is a curve in \mathbb{R}^3 in the parameter s , and $\delta(s)$ is a vector field along $\beta(s)$. Thus for each s , the straight line formed by varying v in the direction of δ gives the ruling.

It is important to note that the coordinate charts described above are in general only local. From now on, we will call a coordinate chart given by Eq. (3) a ruled coordinate chart. We can always choose β to be unit speed and δ to be a unit vector field. Furthermore, we may reparametrize a ruled coordinate chart so that $\langle \beta', \delta \rangle = 0$.⁶ Then without loss of generality we can assume

⁵See Proposition 2 in Sec. IV.

⁶See Sec. IV for details.

$$|\beta'| = 1,$$

$$|\delta| = 1,$$

$$\langle \beta', \delta \rangle = 0 \tag{4}$$

on each ruled coordinate chart.

By Eq. (4), product rule also gives $\langle \delta', \delta \rangle = 0$. Therefore, we have $\langle x_s, x_v \rangle = \langle \beta', \delta \rangle + v \langle \delta', \delta \rangle = 0$, i.e., we have local orthogonal coordinate systems on a ruled surface Σ . Then letting $X = x_v$ and $Y = x_s/|x_s|$, we get local orthonormal frames on Σ . Denote by $f(v)$ and $h(t)$ the integral curves of X and Y , respectively, where $h(0)$ is on the curve $\beta(s)$. Let N denote the oriented unit normal on Σ , then the Gauss curvature on Σ is

$$K = \left\langle \frac{d}{dv} N \circ f, X \right\rangle \left\langle \frac{d}{dt} N \circ h, Y \right\rangle - \left\langle \frac{d}{dv} N \circ f, Y \right\rangle^2. \tag{5}$$

Since the ruling $f(v)$ is really a line segment, $f'' = 0$, so by product rule we see that

$$\left\langle \frac{d}{dv} N \circ f, X \right\rangle = - \langle N, f'' \rangle = 0.$$

Thus $K \leq 0$ on a ruled surface by Eq. (5). A surface Σ is called a *developable surface* if it is a ruled surface such that its normal is parallel in \mathbb{R}^3 along any of its generators, i.e., $(d/dv)N \circ f = 0$. Then again by Eq. (5) we see that $K \equiv 0$ on a developable surface. In fact, the concept of a developable surface is inseparable from the zero Gauss curvature condition.

Theorem 4: (Massey’s Theorem). [Corollary to Theorem 5 (Ref. 4)] *A complete, connected surface in \mathbb{R}^3 is a developable surface if and only if $K \equiv 0$.*⁷

Next we compute the mean curvature on a ruled surface Σ . Let \times denote the cross-product operation in \mathbb{R}^3 . Moreover, since $|x_v| = |\delta| = 1$ we have $|x_s \times x_v| = |x_s|$. The mean curvature on Σ is computed as

$$H = \left\langle - \frac{d}{dv} N \circ f, X \right\rangle + \left\langle - \frac{d}{dt} N \circ h, Y \right\rangle = \frac{\langle x_s \times x_v, x_{ss} \rangle}{|x_s \times x_v|^3}. \tag{6}$$

III. THE TOPOLOGY OF NONCOMPACT SURFACES

The famous Gauss-Bonnet Theorem asserts that the total Gauss curvature of a compact two-dimensional manifold without boundary is a constant multiple of its Euler characteristic. If a surface is complete, noncompact, but has integrable Gauss curvature, then the total Gauss curvature is no longer completely topological. In 1957, Huber showed that a complete, noncompact surface Σ of integrable curvature is conformally equivalent to a compact Riemann surface with finitely many punctures. In particular, Huber also showed that in this case

$$\int_{\Sigma} K \leq 2\pi\chi(\Sigma), \tag{7}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ . The finitely many punctures correspond to the ends of Σ . Let us denote the ends by $\{E_1, \dots, E_k\}$, and define the corresponding isoperimetric constants,

⁷It is an elementary exercise to show that a ruled surface is developable if and only if it has zero Gauss curvature. The more general theorem of Massey, on the other hand, is quite nontrivial.

$$\lambda_i = \lim_{r \rightarrow \infty} \frac{\text{area}(B(r) \cap E_i)}{\pi r^2},$$

relative to any fixed point $p \in \Sigma$ with respect to which the geodesic ball $B(r)$ of radius r is taken. The ends contribute to the deficit in Eq. (7) via the following formula.

Theorem 5: [Hartman (Ref. 3)]. *Let Σ be a complete, noncompact surface with integrable Gauss curvature. Then*

$$\frac{1}{2\pi} \int_{\Sigma} K = \chi(\Sigma) - \sum_{i=1}^k \lambda_i, \quad (8)$$

where $\chi(\Sigma)$ is the Euler characteristic of the surface.

The Euler characteristic is defined by $\chi(\Sigma) = \sum (-1)^i b_i$, where b_i is the i th Betti number. Now, we always assume implicitly that manifolds are connected, hence path connected. Thus for a connected, noncompact, embedded surface, $\chi(\Sigma) = 1 - b_1$, and in the case that Σ also has integrable Gauss curvature we see from Hartman's formula that, in fact,

$$\int_{\Sigma} K \leq 2\pi. \quad (9)$$

Moreover, if we assume that $\int_{\Sigma} K > 0$, then by Eq. (8) we must have $b_1 = 0$ as well. This means $\chi(\Sigma) = 1$, which via the uniformization theorem for surfaces implies that Σ is conformally equivalent to \mathbb{R}^2 . Therefore, we see that positive total Gauss curvature surfaces are topologically very simple (it is just \mathbb{R}^2).⁸ Ironically, the analysis which is hopeful toward deducing the existence of ground state seems destined to be significantly less straightforward than the resolved nonpositive total Gaussian curvature case. Note that interestingly, by Hartman's formula we can start with any surface and add sufficiently many handles until the total Gauss curvature becomes nonpositive, and subsequently obtains a ground state.⁹

The assumption of integrable Gauss curvature is actually flexible in Hartman's theorem above—in the sense that if we know the surface Σ is of finite topological type (i.e., finite Betti numbers, and hence finite number of ends) and the asymptotic volume ratios λ_i are also finite, then $K \in L^1(\Sigma)$ because $K \leq 0$ outside a compact set. In view of this fact, we have the following.

Corollary 2: *A surface that is ruled outside a compact set has integrable Gauss curvature.* □

At last we state the following result of White, which depends on the way in which a surface sits in \mathbb{R}^3 .

Theorem 6: [White (Ref. 8)]. *Let Σ be a surface immersed in \mathbb{R}^3 . If $\int_{\Sigma} |A|^2 < \infty$, then K is integrable and $\int_{\Sigma} K = 4\pi n$ for some $n \in \mathbb{Z}$.*

We end this section by giving the proof of Proposition 1 via White's result.

Proof of Proposition 1: The assumption of connectedness and noncompactness of Σ implies that we must have Euler characteristic $\chi(\Sigma) \leq 1$. Then by Hartman's formula and our hypothesis of positive total Gauss curvature, we must have

$$0 < \int_{\Sigma} K \leq 2\pi. \quad (10)$$

Now, if $H \in L^2(\Sigma)$ then by the elementary formula $H^2 = |A|^2 + 2K$ and the assumption $K \in L^1(\Sigma)$, we must then have $\int_{\Sigma} |A|^2 < \infty$, which by Theorem 6 means that $\int_{\Sigma} K = 4\pi n$ for some $n \in \mathbb{Z}$. However, this is in contradiction with Eq. (10). ■

⁸This was also pointed out in Ref. 1.

⁹This was first discussed in the subsequent paper (Refs. 1 and 2).

IV. PROOF OF MAIN RESULT

We assume that the surface Σ is ruled outside $\overline{B(R_0)}$, where we have suppressed the reference point x_0 with respect to which the geodesic distance R_0 is measured. From now on if the center of a geodesic ball of certain radius is suppressed, it is implicit that the center is the point x_0 . Thus each point in $\Sigma \setminus \overline{B(R_0)}$ is contained in a local coordinate chart given by $x(s, v) = \beta(s) + v\delta(s)$ for a curve β in \mathbb{R}^3 and a nonzero outward-pointing vector field δ along β .

Now we briefly describe how to make $\langle \beta', \delta \rangle = 0$ for each coordinate chart. Let $x(s, v) = \beta(s) + v\delta(s)$ be a ruled coordinate chart with $|\beta'| = |\delta| = 1$. Consider the curve $\gamma(s) = \beta(s) + t(s)\delta(s)$, where using the unit length condition on δ and requiring that $\langle \gamma', \delta \rangle = 0$, we see that necessarily

$$t(s) = - \int_0^s \langle \beta'(u), \delta(u) \rangle du.$$

Finally, by reparametrizing the curve γ to have unit speed, the reparametrization $y(s, w) = \gamma(s) + w\delta(s)$ is the desired one for which Eq. (4) is satisfied.¹⁰

We would like to have a finite cover of $\Sigma \setminus \overline{B(R_0)}$ with ruled coordinate charts satisfying property (4). Let $R_1 > R_0$. For any point $p \in \partial B(R_1)$, we want a local ruled coordinate chart y above (satisfying Eq. (4)). However, we must pay attention to the possibility that in the reparametrization described above, $t(s)$ may be so negative for some s that $\gamma(s)$ is no longer on the surface Σ . Such a possibility requires consideration since our surface is not entirely ruled. Thus for the original chart x (before the reparametrization), if we take $\epsilon > 0$ small enough then it will ensure that γ (and hence the image of the new chart y) stays on Σ . It is clear that we can always make $x(0, 0) = p = y(0, 0)$. Then since $\partial B(R_1)$ is compact, there is a finite collection of such coordinate charts covering $\Sigma \setminus \overline{B(R_0)}$.

By Eq. (6) and using property (4), the mean curvature has the expression

$$H = \frac{\langle x_s \times x_v, x_{ss} \rangle}{|x_s \times x_v|^3} = \frac{\langle \beta' \times \delta, \beta'' \rangle + v(\langle \delta' \times \delta, \beta'' \rangle + \langle \beta' \times \delta, \delta'' \rangle) + v^2 \langle \delta' \times \delta, \delta'' \rangle}{(1 + 2v\langle \beta', \delta' \rangle + v^2|\delta'|^2)^{3/2}}. \tag{12}$$

Let us denote the numerator of H above by P , and the term inside the 3/2 power in the denominator by Q . Note that P and Q are polynomials in v , with coefficient functions in s . We will simplify matters by using the notations

$$A = 2\langle \beta', \delta' \rangle, \quad B = |\delta'|^2, \quad D = \langle \beta' \times \delta, \beta'' \rangle,$$

$$E = \langle \delta' \times \delta, \beta'' \rangle + \langle \beta' \times \delta, \delta'' \rangle, \quad J = \langle \delta' \times \delta, \delta'' \rangle.$$

Note that if $\int_{\Sigma} K > 0$ and Σ is minimal ($H \equiv 0$) on $\Sigma \setminus \overline{B(R_0)}$, then $H \in L^2(\Sigma)$ and this contradicts Proposition 1. Thus, if Σ is not minimal outside some compact set, there must be a point $p \in \Sigma \setminus \overline{B(R_0)}$ such that $H(p) \neq 0$. Moreover, by Eq. (12) if we assume that v is large enough, $H \neq 0$ along the ruling line thorough p . Switching the orientation of Σ if necessarily, we can then take ϵ small enough and v_0 large enough for all $s \in (-\epsilon, \epsilon)$ such that $H > 0$ for all $v \geq v_0$ on $x((-\epsilon, \epsilon) \times (v_0, \infty))$. This assumption on the positive sign of H will be made implicitly throughout the remaining arguments in this paper.

Lemma 1: *Let $t > 0$ and $\alpha, \beta \in (-\epsilon, \epsilon)$ such that $\alpha < \beta$. Suppose that $\deg(P)$ and $\deg(Q)$ remain unchanged on (α, β) . Then the integral*

$$\int_{\alpha}^{\beta} \int_t^{t+t_0} H |x_s \times x_v| dv ds = o(t_0),$$

¹⁰Note that reparametrizing γ to have unit speed does not affect the length of δ and its orthogonality with γ .

for large enough t if and only if $\deg(P) < \deg(Q)$ for all $s \in (\alpha, \beta)$, where \deg denotes the degree of the polynomial.

Proof: First we suppose $\deg(P) < \deg(Q)$, which consists of only two cases: $(\deg(P), \deg(Q)) = (1, 2)$, and $(0, 2)$. For the first case, since $[\alpha, \beta]$ is compact we can choose constants $C_1, C_2 > 0$ such that for large enough v ,

$$P < C_1 v \quad \text{and} \quad Q > C_2 v^2,$$

for all $s \in (-\varepsilon, \varepsilon)$. Then by Eq. (12), since $|x_s \times x_v| = \sqrt{1 + Av + Bv^2}$, we see that

$$\int_{\alpha}^{\beta} \int_t^{t+t_0} H|x_s \times x_v| dv ds = \int_{\alpha}^{\beta} \int_t^{t+t_0} \frac{P}{Q} dv ds < C \log\left(\frac{t+t_0}{t}\right), \tag{13}$$

for some constant $C > 0$, hence $o(t_0)$. For the second case, similarly we can find constants $C_1, C_2 > 0$ such that for large enough v ,

$$P < C_1 \quad \text{and} \quad Q > C_2 v^2,$$

for all $s \in (\alpha, \beta)$. Then we see that for large enough t ,

$$\int_{\alpha}^{\beta} \int_t^{t+t_0} H|x_s \times x_v| dv ds < C\left(\frac{1}{t} - \frac{1}{t+t_0}\right), \tag{14}$$

which is clearly $o(t_0)$.

Next suppose $\deg(P) \geq \deg(Q)$, which consists of three cases: $(\deg(P), \deg(Q)) = (2, 2)$, $(0, 0)$, and $(1, 0)$. In any of these cases, we can long divide (for each s) and get

$$\frac{P}{Q} = \tilde{P} + \frac{R}{Q}, \tag{15}$$

where $\deg(\tilde{P}) = \deg(P) - \deg(Q)$, and $\deg(R) < \deg(Q)$. For large enough t , the integral over $(\alpha, \beta) \times (t, t+t_0)$ of the second term in Eq. (15) is $o(t_0)$ by applying the previous argument. However, integrating the first term in Eq. (15) we see that it becomes a polynomial in t_0 of degree at least 1. To be more precise, for large enough t we have

$$\int_{\alpha}^{\beta} \int_t^{t+t_0} H|x_s \times x_v| dv ds = f(t_0) + o(t_0),$$

where $f(t_0)$ is a polynomial of degree either 1 [corresponding to the $(2, 2)$ and $(0, 0)$ cases listed above] or 2 [corresponding to the $(1, 0)$ case] with coefficients in t . Hence it cannot be $o(t_0)$. ■

Lemma 1 is an important lemma to be used below. In particular, we will need the integral of H over a sectorlike region in $\Sigma \setminus B(R_0)$ to grow at least linearly in t_0 (i.e., in the ruling direction). In view of this, we shall need to prove the following result.

Proposition 2: *Let Σ be a surface embedded in \mathbb{R}^3 that is ruled, but nonflat outside $\overline{B(R_0)}$ for some $R_0 > 0$. Furthermore, assume $\int_{\Sigma} K > 0$ and that Σ is not minimal outside $B(R_0)$. Then there exist $p \in \Sigma \setminus B(R_0)$, $H(p) \neq 0$, and a ruled coordinate chart $x: (-\varepsilon, \varepsilon) \times (b, \infty) \rightarrow \Sigma$ satisfying Eq. (4) such that $p = x(0, 0)$ and $\deg(P) \geq \deg(Q)$ at $s = 0 \in (-\varepsilon, \varepsilon)$.*

Before we proceed with the proof of Proposition 2, let us remark on its nature. Any point $p \in \Sigma \setminus B(R_0)$ such that $H(p) \neq 0$ lies in a ruled coordinate chart satisfying Eq. (4), and we can always reparametrize the chart so that $x(0, 0) = p$ by translation in the v direction. Moreover, the degrees of P and Q certainly do not change under translation in v . Therefore, the proposition is really a statement about the ruled coordinate charts covering $\Sigma \setminus B(R_0)$.

Proof of Proposition 2: Suppose for a contradiction, we assume that at any point $p \in \Sigma \setminus B(R_0)$ where $H(p) \neq 0$, with $p = x(0, 0)$ for some ruled coordinate chart $x: (-\varepsilon, \varepsilon) \times (b, \infty) \rightarrow \Sigma$, we have $\deg(P) < \deg(Q)$ at $s = 0$. By continuity, we may assume that ε is small enough so

that $H(s, 0) \neq 0$ for all $s \in (-\varepsilon, \varepsilon)$. For such a chart x , we claim that, in fact, $\deg(P) < \deg(Q)$ for all $s \in (-\varepsilon, \varepsilon)$. To see this, suppose $\deg(P) \geq \deg(Q)$ at some $s_0 \in (-\varepsilon, \varepsilon)$. Then we can make the reparametrization

$$\beta(s) \rightarrow \beta(s + s_0) \quad \text{and} \quad \delta(s) \rightarrow \delta(s + s_0)$$

to obtain a new ruled coordinate chart \tilde{x} with $\tilde{x}(0, 0) = x(s_0, 0)$, contained in the original chart x . Note that property (4) is preserved for \tilde{x} , and moreover the coefficients A, B, D, E, J are also invariant under the reparametrization above. Thus $\deg(\tilde{P}) = \deg(P)$ and $\deg(\tilde{Q}) = \deg(Q)$ at all points shared by the charts \tilde{x} and x . This is a contradiction since we still have $H \neq 0$ at $\tilde{x}(0, 0)$ but $\deg(\tilde{P}) \geq \deg(\tilde{Q})$ at $\tilde{x}(0, 0)$, thus proving the claim.

Now, assume we are in a ruled coordinate chart $x: (-\varepsilon, \varepsilon) \times (b, \infty) \rightarrow \Sigma$ such that $\deg(P) < \deg(Q)$ for all $s \in (-\varepsilon, \varepsilon)$. By our notations before, we observe that

$$\int_0^t H^2|_{x_s \times x_v} dv = \int_0^t \frac{P^2}{Q^{5/2}} dv. \tag{16}$$

First consider all $s \in (-\varepsilon, \varepsilon)$ such that $(\deg(P), \deg(Q)) = (1, 2)$. Since the coefficients of P and Q are functions of s bounded on $(-\varepsilon, \varepsilon)$, we can choose $c > 0$ such that $Q > 1$, $Q^2 > Cv^4$ for some constant $C > 0$, and $P^2 < Cv^2$ for some constant $C > 0$, for all $v > c$ and for all such s . Then for any such s , we have

$$\int_c^t \frac{P^2}{Q^{5/2}} dv \leq \int_c^t \frac{P^2}{Q^2} dv < C \int_c^t \frac{1}{v^2} dv = C \left(\frac{1}{c} - \frac{1}{t} \right), \tag{17}$$

which converges as $t \rightarrow \infty$. Similarly, for any $s \in (-\varepsilon, \varepsilon)$ at which $(\deg(P), \deg(Q)) = (0, 2)$ we have

$$\int_c^t \frac{P^2}{Q^{5/2}} dv \leq P^2 \int_c^t \frac{1}{Q^2} dv < CP^2 \int_c^t \frac{1}{v^4} dv = CP^2 \left(\frac{1}{c^3} - \frac{1}{t^3} \right), \tag{18}$$

which again converges as $t \rightarrow \infty$. Combining Eqs. (17) and (18), we see that

$$\int_{-\varepsilon}^{\varepsilon} \int_0^{\infty} H^2|_{x_s \times x_v} dv ds < \infty \tag{19}$$

on such a chart. Then via Eq. (19) and by our earlier remark on the existence of a finite cover of $\Sigma \setminus \overline{B(R_0)}$ by ruled coordinate charts, it would mean that $H \in L^2(\Sigma)$. By our hypothesis of embedded Σ and positive total Gauss curvature, we have a contradiction to Proposition 1. This completes the proof of the proposition. ■

Remark 2: Although this is not necessary for the proof of Proposition 2, it is in good spirit to check that $\deg(P) \geq \deg(Q)$ at $p = x(0, 0)$ does not render H^2 unintegrable over $(-\varepsilon, \varepsilon) \times (0, \infty)$ for a small enough $\varepsilon > 0$ where $\deg(P)$ and $\deg(Q)$ are unchanged. As in the previous case of $\deg(P) < \deg(Q)$, we can narrow down to three cases. The first two consist of $\deg(P) = 1, 0$ coupled with $\deg(Q) = 0$. It is clear in these two cases that

$$\int_{-\varepsilon}^{\varepsilon} \int_0^t H^2|_{x_s \times x_v} dv ds$$

does not converge as $t \rightarrow \infty$. For the last case of $\deg(P) = \deg(Q) = 2$, one can verify that there exists $c, k > 0$ such that

$$\int_c^t \frac{P^2}{Q^{5/2}} dv \geq k \int_c^t \frac{1}{v} dv = k(\log t - \log c) \rightarrow \infty \tag{20}$$

as $t \rightarrow \infty$.

Corollary 3: *With the same assumptions on Σ as in Proposition 2, there exist a point $p \in \Sigma \setminus \overline{B(R_0)}$, $H(p) \neq 0$, and a ruled coordinate chart $x: (-\varepsilon, \varepsilon) \times (b, \infty) \rightarrow \Sigma$ satisfying Eq. (4) such that $\deg(P) \geq \deg(Q)$ for all $s \in (-\varepsilon, \varepsilon)$. Moreover, we can choose the chart so that $\deg(P)$ and $\deg(Q)$ are fixed for all $s \in (-\varepsilon, \varepsilon)$.*

Proof: By Proposition 2, there exist at least one point $p \in \Sigma \setminus \overline{B(R_0)}$, $H(p) \neq 0$, and a ruled coordinate chart $x: (-\varepsilon, \varepsilon) \rightarrow \Sigma$ such that $x(0, 0) = p$ and $\deg(P) \geq \deg(Q)$ at $s = 0$. Suppose at such a point $p = x(0, 0)$, the first assertion of the corollary is false. Then by the smoothness of the coefficients of P and Q in s , there must exist an $\bar{\varepsilon} > 0$, $\bar{\varepsilon} \leq \varepsilon$, such that $\deg(P) \geq \deg(Q)$ at $s = 0$ and $\deg(P) < \deg(Q)$ for all $s \in (-\bar{\varepsilon}, \bar{\varepsilon}) \setminus \{0\}$. Thus the ruling lines that pass through these points must comprise a set of discrete lines, and hence of measure zero. Then applying the integration of H^2 argument in Proposition 2, we would get the same contradiction as we did there.

Next we argue that we can fix $\deg(P)$ and $\deg(Q)$ in a small enough interval of s . Observe that due to the smoothness of the coefficient functions in s , the degrees of P and Q cannot decrease in an arbitrarily small neighborhood of s , but it can certainly increase. Now, if $(\deg(P), \deg(Q)) = (2, 2)$ at $s = 0$, then, since this is the case of the largest possible degrees, we can certainly find an $\varepsilon > 0$ small enough so that the degrees remain 2 for all $s \in (-\varepsilon, \varepsilon)$. If we are in the $(1, 0)$ case at $s = 0$, then either the degrees remain as $(1, 0)$ in a small interval about $s = 0$, or there exists an s_0 near s at which the degrees increase to $(2, 2)$ and to which we can apply the preceding argument upon reparametrization of the chart. For the last case of $(0, 0)$, if the degrees increase near $s = 0$ then we simply apply the arguments in the previous two cases. \square

Proof of Theorem 2: Now, the surface is not totally geodesic (nonplanar) by our hypothesis. Therefore, if $\int_{\Sigma} K \leq 0$, the conclusion of the theorem follows as a special case of previous results. For the remainder of the proof we will implicitly assume that $\int_{\Sigma} K > 0$ (although not used in the proof), which also implies that the surface cannot be minimal nor flat outside the compact subset.

By the variational principle, it suffices to find a test function $\phi \in W_0^{1,2}(\Omega)$ such that

$$Q(\phi, \phi) = \int_{\Omega} |\nabla \phi|^2 - \left(\frac{\pi}{2a}\right)^2 \int_{\Omega} |\phi|^2 < 0. \tag{21}$$

We define a test function (family of test functions) of the form¹¹

$$\phi_{\sigma} = \chi \psi + \sigma \chi_1 j, \tag{22}$$

where σ is some nonzero number to be determined, $\chi = \cos(\pi/2a)u$, $\chi_1 = u \cos(\pi/2a)u$, and ψ and j are defined below.

Since on Σ we have integrable Gauss curvature, it implies that Σ is parabolic,¹² and hence for any $R_1 > 0$ there exists an $R_2 > R_1$ for which

$$\int_{\Sigma} |\nabla \psi_{R_1, R_2}|^2 < \frac{\varepsilon_0}{2},$$

where ψ_{R_1, R_2} is the unique solution to the boundary value problem

$$\Delta \psi = 0 \quad \text{on } B(R_2) \setminus B(R_1),$$

$$\psi|_{B_p(R_1)} \equiv 1,$$

¹¹This form of test function first appeared in Ref. 2, and was also used extensively in Ref. 5. However, the argument we will give here to establish Eq. (21) is different in an essential way.

¹²For a proof of this, see Ref. 5.

$$\psi|_{\Sigma \setminus B_p(R_2)} \equiv 0. \tag{23}$$

We will let $R_1 > R_0$. Then for $R_1 < R_2 < R_3 < R_4$, we let

$$\psi = \psi_{R_3, R_4} - \psi_{R_1, R_2}.$$

We want j to be a $W^{1,2}$ function on Σ with support in $\{\psi \equiv 1\}$ and $j \leq 1$. Before defining j precisely, we proceed with some preliminary estimates. By our choices of χ and χ_1 , the fact that $K \leq 0$ on $\Sigma \setminus B(R_0)$, and the requirement that $\psi \leq 1$ and $\text{supp } j \subset \{\psi \equiv 1\}$, we get

$$Q(\phi_\sigma, \phi_\sigma) = Q(\chi\psi, \chi\psi) + 2\sigma Q(\chi\psi, \chi_1 j) + \sigma^2 Q(\chi_1 j, \chi_1 j) \leq C_1 \int_\Sigma |\nabla \psi|^2 + \sigma a \int_\Sigma jH + \sigma^2 C_2 \|j\|_{W^{1,2}}^2, \tag{24}$$

for $C_1, C_2 > 0$ depending only on the geometry of Ω . We will choose j so that $\|j\|_{W^{1,2}} \neq 0$. Then viewing the right-hand side of the inequality in Eq. (24) as a quadratic polynomial in the variable σ , it will be negative for some σ if and only if its discriminant is positive, which is equivalent to the condition

$$\frac{\left(\int_\Sigma jH\right)^2}{\|j\|_{W^{1,2}}^2} > C_1 \int_\Sigma |\nabla \psi|^2, \tag{25}$$

where we absorbed all the geometric constants into a single constant C_1 . Now, by our choice of ψ along with the parabolicity of Σ , for $R_1 > R_0$ fixed we can choose R_2 and then $R_3 < R_4$ big enough so that

$$\int_\Sigma |\nabla \psi|^2 = \int_\Sigma |\nabla \psi_{R_1, R_2}|^2 + \int_\Sigma |\nabla \psi_{R_3, R_4}|^2 < \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0. \tag{26}$$

Observe that what is essential is the choice of R_2 , as $R_3 < R_4$ can always be chosen after R_2 so that Eq. (26) holds. By our requirement on j , the choice of R_2 may affect the ratio on the left-hand side of Eq. (25). In view of this consideration, we seek a (family of) j such that

$$\frac{\left(\int_\Sigma jH\right)^2}{\|j\|_{W^{1,2}}^2} \geq C, \tag{27}$$

for a constant C independent of R_2 , as long as R_2 is large enough. Inequality (27) is a sufficient condition for Eq. (25) since we can then choose R_2 large enough for a small enough ε_0 satisfying Eq. (26) and

$$C_1 \varepsilon_0 < C.$$

In a nutshell, the proof will be complete if we construct a (family of) j so that Eq. (27) holds for some constant C independent of R_2 , for R_2 large enough.

By Corollary 3, let $p \in \Sigma \setminus B(R_0)$ be a point such that $H(p) \neq 0$ and consider a ruled coordinate chart $x: (-\varepsilon, \varepsilon) \times (b, \infty) \rightarrow \Sigma$ satisfying Eq. (4) such that $p = x(0, 0)$, $\deg(P) \geq \deg(Q)$, with $\deg(P)$ and $\deg(Q)$ fixed for all $s = 0 \in (-\varepsilon, \varepsilon)$. Moreover, we let $R_1 = \text{dist}(x_0, p)$. For any $R_2 > R_1$ and $t_0 > 1$, let

$$\Gamma = \{(s, v) \in \mathbb{R}^2 \mid -\varepsilon \leq s \leq \varepsilon, v_0 \leq v \leq v_0 + t_0\}$$

such that $x(\Gamma) \subset B_p(R_3) \setminus B_p(R_2)$. We define j by $j = j_1(s)j_2(v)$, with cutoff functions

$$j_1(s) = \begin{cases} 1, & \alpha - \varepsilon < s < \varepsilon - \alpha \\ 0, & |s| \geq \varepsilon, \end{cases} \tag{28}$$

and

$$j_2(v) = \begin{cases} 1, & v_0 + \alpha < v < v_0 + t_0 - \alpha \\ 0, & v \leq v_0, v \geq v_0 + t_0, \end{cases} \tag{29}$$

where $\alpha > 0$ is a fixed small number, $|j'_1(s)| \leq 1/\alpha$, and $|j'_2(v)| \leq 1/\alpha$.

Now, by the definition of j above and Eq. (12), we have

$$\int_{\Sigma} jH > \int_{\{j=1\}} H = \int_{\alpha-\varepsilon}^{\varepsilon-\alpha} \int_{v_0+\alpha}^{v_0+t_0-\alpha} \frac{P}{Q} dv ds.$$

By our choice of the ruled coordinate chart x above, Lemma 1 implies

$$\int_{\Sigma} jH > f(t_0) + o(t_0), \tag{30}$$

for large enough v_0 , where $f(t_0)$ is a polynomial in t_0 of degree $n=1$ or 2 and has coefficients in v_0 .

Next, we wish to give an upper bound estimate for $\|j\|_{W^{1,2}}$. First, we see that

$$\|j\|_{W^{1,2}}^2 = \int_{\Sigma} j^2 + \int_{\Sigma} |\nabla j|^2 \leq (1 + \|\nabla j\|_{\infty}^2) \text{vol}(x(\Gamma)). \tag{31}$$

By the metric on Σ given by the chart (3), we see that

$$|\nabla j|^2 = j_2^2 G^{ss} \left| \frac{\partial j_1}{\partial s} \right|^2 + 2j_1 j_2 G^{sv} \frac{\partial j_1}{\partial s} \frac{\partial j_2}{\partial v} + j_1^2 G^{vv} \left| \frac{\partial j_2}{\partial v} \right|^2 \leq \left(\frac{1}{1 + 2v\langle \beta', \delta' \rangle + v^2|\delta'|^2} \right) \frac{1}{\alpha^2} + \frac{1}{\alpha^2}, \tag{32}$$

where G^{ij} is the inverse to the metric tensor G_{ij} with respect to coordinates. If the polynomial $1 + 2v\langle \beta', \delta' \rangle + v^2|\delta'|^2$ has degree 2 for all $s \in (-\varepsilon, \varepsilon)$, then since its discriminant

$$4\langle \beta', \delta' \rangle^2 - 4|\delta'|^2 \leq 4|\beta'|^2|\delta'|^2 - 4|\delta'|^2 = 0,$$

it is always positive for v large enough. The other possibility is that it is identically equal to 1. In any case, we can choose a v_0 large enough so that for all $v \geq v_0$

$$|\nabla j|^2 < C_3, \tag{33}$$

for all $s \in (-\varepsilon, \varepsilon)$, for some constant $C_3 > 0$.

Next, we will estimate the volume growth of $x(\Gamma)$. The volume form in the ruled coordinate system is

$$d\Sigma = \sqrt{1 + Av + Bv^2} ds dv.$$

There are two possibilities at $s=0$, either $B=0$ or $B \neq 0$. In the latter case, we can certainly assume that ε is small enough so that $B \neq 0$ for all $s \in (-\varepsilon, \varepsilon)$. If $B=0$ at $s=0$, then by a similar argument as in Corollary 3, we can assume that $B=0$ for all $s \in (-\varepsilon, \varepsilon)$.

Suppose $B \neq 0$ for all $s \in (-\varepsilon, \varepsilon)$. If $A^2 - 4B < 0$ at $s=0$ we can always take ε small enough so that $A^2 - 4B < 0$ for all $s \in (-\varepsilon, \varepsilon)$. Assuming so, we have

$$\begin{aligned}
 \text{vol}(x(\Gamma)) &= \int_{-\varepsilon}^{\varepsilon} \int_{v_0}^{v_0+t_0} \sqrt{1 + Av + Bv^2} dv ds = \int_{-\varepsilon}^{\varepsilon} \int_{v_0}^{v_0+t_0} \sqrt{B \left(v + \frac{A}{2B} \right)^2 + 1 - \frac{A^2}{4B}} dv ds \\
 &= \int_{-\varepsilon}^{\varepsilon} \int_{v_0+(A/2B)}^{v_0+t_0+(A/2B)} \sqrt{Bx^2 + 1 - \frac{A^2}{4B}} dx ds \\
 &= \int_{-\varepsilon}^{\varepsilon} \sqrt{\frac{4B - A^2}{4B}} \int_{v_0+(A/2B)}^{v_0+t_0+(A/2B)} \sqrt{\frac{4B^2}{4B - A^2} x^2 + 1} dx ds \\
 &= \int_{-\varepsilon}^{\varepsilon} \frac{4B - A^2}{4B^{3/2}} \int_{\sqrt{[4B^2/(4B-A^2)](v_0+(A/2B))}}^{\sqrt{[4B^2/(4B-A^2)](v_0+t_0+(A/2B))}} \sqrt{y^2 + 1} dy ds \\
 &\leq \int_{-\varepsilon}^{\varepsilon} \frac{4B - A^2}{4B^{3/2}} \int_{\sqrt{[4B^2/(4B-A^2)](v_0+(A/2B))}}^{\sqrt{[4B^2/(4B-A^2)](v_0+t_0+(A/2B))}} (y + 1) dy ds \\
 &= t_0 v_0 \int_{-\varepsilon}^{\varepsilon} \sqrt{B} ds + t_0^2 \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{B}}{2} ds + t_0 \int_{-\varepsilon}^{\varepsilon} \left(\frac{A}{2\sqrt{B}} + \sqrt{\frac{4B - A^2}{4B}} \right) ds. \tag{34}
 \end{aligned}$$

On the other hand, if $B \neq 0$ for all $s \in (-\varepsilon, \varepsilon)$ and $A^2 - 4B = 0$ at $s = 0$, using the same argument in Corollary 3 we can assume that $A^2 - 4B = 0$ for all $s \in (-\varepsilon, \varepsilon)$. Assuming so, we have

$$\begin{aligned}
 \text{vol}(x(\Gamma)) &= \int_{-\varepsilon}^{\varepsilon} \int_{v_0}^{v_0+t_0} \sqrt{1 + Av + Bv^2} dv ds = \int_{-\varepsilon}^{\varepsilon} \int_{v_0}^{v_0+t_0} \sqrt{B} \left(v + \frac{A}{2B} \right) dv ds \\
 &= t_0 v_0 \int_{-\varepsilon}^{\varepsilon} \sqrt{B} ds + t_0^2 \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{B}}{2} ds + t_0 \int_{-\varepsilon}^{\varepsilon} \frac{A}{2\sqrt{B}} ds. \tag{35}
 \end{aligned}$$

For brevity, we will use the following notations:

$$\begin{aligned}
 C_4 &= \int_{-\varepsilon}^{\varepsilon} \sqrt{B} ds, \quad C_5 = \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{B}}{2} ds, \\
 C_6 &= \int_{-\varepsilon}^{\varepsilon} \left(\frac{A}{2\sqrt{B}} + \sqrt{\frac{4B - A^2}{4B}} \right) ds \quad \text{or} \quad \int_{-\varepsilon}^{\varepsilon} \frac{A}{2\sqrt{B}} ds.
 \end{aligned}$$

If $B = 0$ for all $s \in (-\varepsilon, \varepsilon)$, then

$$\text{vol}(x(\Gamma)) = 2\varepsilon t_0. \tag{36}$$

Now, by Eq. (30), with any fixed $t_0 > 0$ there exists a v_0 large enough so that

$$\int_{\Sigma} jH > C_7 t_0^n, \tag{37}$$

for some constant $C_7 > 0$ (which depends on v_0), with $n = 1$ for cases (34) or (35) and $n = 1$ or 2 for case (36) corresponding to $\text{deg}(P) = 0$ or 1 .

Renaming the square of C_7 as itself, by Eqs. (31), (33), and (37) with a choice of a large enough v_0 , we see that in either case (34) or case (35),

$$\frac{\left(\int_{\Sigma} jH \right)^2}{\|j\|_{W^{1,2}}^2} > \frac{C_7 t_0^2}{C_3(t_0 v_0 C_4 + t_0^2 C_5 + t_0 C_6)}. \tag{38}$$

Note that once v_0 is fixed, the constants $C_3, C_4, C_5, C_6,$ and C_7 depend only on the metric along the s -parameter curve $\beta(s) : (-\varepsilon, \varepsilon) \rightarrow \Sigma$, which is fixed from the start. Then the right-hand side of

Eq. (38) either converges to C_7/C_5 (when $n=1$) or goes to infinity (when $n=2$), as we take $t_0 \rightarrow \infty$ (by letting $R_3 \rightarrow \infty$). Therefore, for a large enough t_0 , there must be a constant $C > 0$ such that

$$\frac{\left(\int_{\Sigma} jH\right)^2}{\|j\|_{W^{1,2}}^2} > C. \quad (39)$$

For the case of Eq. (36), the denominator of the right-hand side of Eq. (38) will always be linear in t_0 , while the numerator is either quadratic or to the fourth power in t_0 , hence Eq. (39) is readily achieved. The proof is now complete. \square

An immediate consequence of Theorem 2 is the following result, which follows via Massey's Theorem.

Corollary 4: *Let Σ be an embedded, nonplanar surface in \mathbb{R}^3 with zero Gauss curvature outside a compact subset. Then for a layer Ω over Σ , we have*

$$\inf \sigma(\Delta) < \left(\frac{\pi}{2a}\right)^2.$$

Moreover, if the second fundamental form $A \rightarrow 0$ at infinity on Σ then a ground state exists.

Remark 3: *As we can see from the proof, all the required analysis is done on a single ruled coordinate chart satisfying the properties in Corollary 3. In particular, the test function which establishes Eq. (21) is supported in this single local coordinate chart. Then the work amounts to studying the ratio of the growth of the integral of the mean curvature to the volume growth on this sector. Therefore, we only have to assume that the surface contains one ruled coordinate chart outside a compact set satisfying the properties in Corollary 3, and where the chart may not necessarily cover the complement of the compact set. Hence the Main Theorem can indeed be generalized. On the other hand, for a surface that is entirely ruled outside a compact set, in order to obtain a ruled coordinate chart of Corollary 3 we used, in Proposition 2, the fact that outside the compact set the surface can be covered by finitely many ruled coordinate charts. The point we want to make is that a surface being ruled outside a compact set is one of the conditions guaranteeing the existence of a ruled coordinate chart of Corollary 3, and while other examples of surfaces containing such a ruled coordinate chart are plenty, they would generically have to be constructed.*

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¹Carron, G., Exner, P., and Krejčířik, D., "Topologically nontrivial quantum layers," J. Math. Phys. **45**, 774 (2004).

²Duclos, P., Exner, P., and Krejčířik, D., "Bound states in curved quantum layers," Commun. Math. Phys. **223**, 13 (2001).

³Hartman, P., "Geodesic parallel coordinates in the large," Am. J. Math. **86**, 705 (1964).

⁴Hicks, N. J., *Notes on Differential Geometry*, Van Nostrand Mathematical Studies No. 3 (D. Van Nostrand Co., Inc., Princeton, NJ, 1965).

⁵Lin, C. and Lu, Z., "Existence of bounded states for layers built over hypersurfaces in R^{n+1} ," J. Funct. Anal. **244**(1), 1–25 (2007).

⁶Lin, C. and Lu, Z., "On the discrete spectrum of generalized quantum tubes," Commun. Partial Differ. Equ. **31**, 1529 (2006).

⁷Reed, M. and Simon, B., *Methods of Modern Mathematical Physics. I*, 2nd ed. (Academic, New York, 1980).

⁸White, B., "Complete surfaces of finite total curvature," J. Diff. Geom. **26**, 315 (1987).