

ON THE FUTAKI INVARIANTS OF COMPLETE INTERSECTIONS

ZHIQIN LU

1. Introduction. In 1983, Futaki [2] introduced his invariants that generalize the obstruction of Kazdan-Warner to prescribe Gauss curvature on S^2 . The Futaki invariants are defined for any compact Kähler manifold with positive first Chern class that has nontrivial holomorphic vector fields. Their vanishing is a necessary condition to the existence of Kähler-Einstein metrics on the underlying manifold.

Let M be a compact Kähler manifold with positive first Chern class $c_1(M) > 0$. Choosing an arbitrary positive $(1, 1)$ -form ω in $c_1(M)$ as a Kähler metric on M , we can find a smooth function f on M , determined up to a constant, such that

$$(1.1) \quad \text{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f$$

holds. Let $\mathfrak{b}(M)$ be the Lie algebra of holomorphic vector fields on M . The Futaki invariants are defined as

$$F : \mathfrak{b}(M) \rightarrow \mathbb{C}, \quad F(X) = \int_M X(f) \omega^n.$$

Ding and Tian [1] introduced the Futaki invariants for Fano normal varieties. This is a generalization of Futaki invariants to singular varieties. It also has important application in Kähler-Einstein geometry. The Futaki invariants on singular varieties are related to the stability of Fano manifolds due to the work of Tian [8]. To be more precise, checking the K -stability of a Fano manifold is the same as checking the sign of the real part of the Futaki invariants on the degenerations of the Fano manifold. Because of this, we need an effective way to compute the Futaki invariants on singular varieties.

In this paper, we give a simple formula for the Futaki invariants of Fano complete intersections. The main theorem of this paper is the following.

THEOREM 1.1. *Let M be the $N - s$ dimensional normal Fano variety in CP^N defined by the homogeneous polynomials F_1, \dots, F_s of degree d_1, \dots, d_s , respectively. Let X be a holomorphic vector field on CP^N such that*

$$(1.2) \quad X F_i = \kappa_i F_i, \quad i = 1, \dots, s,$$

Received 17 July 1998. Revision received 3 February 1999.

1991 *Mathematics Subject Classification*. Primary 32F07; Secondary 58G03.

for constants $\kappa_1, \dots, \kappa_s$. Then the Futaki invariant $F(X)$ is

$$(1.3) \quad F(X) = m^{N-s} \prod_{i=1}^s d_i \left(-\sum_{i=1}^s \kappa_i + \frac{m}{N-s+1} \sum_{i=1}^s \frac{\kappa_i}{d_i} \right),$$

where $m = N + 1 - d_1 - \dots - d_s$.

COROLLARY 1.1. *If M is a hypersurface in CP^N , defined by the homogeneous polynomial F of degree d , and if $X F = \kappa F$, then*

$$F(X) = -(N + 1 - d)^{N-1} \frac{(N + 1)(d - 1)}{N} \kappa.$$

In particular, $\operatorname{Re} F(X)$ and $-\operatorname{Re} \kappa$ have the same sign.

The formula is new even in the case when M is a hypersurface or an orbifold. If the zero locus of the holomorphic vector field X on M is a smooth manifold, then using the residue formula of the Atiyah-Bott-Lefschetz type, Futaki was able to develop a method to compute his invariants by the information of the vector field X and the manifold M near the zero locus of the vector field (see [3]). In [1], the authors developed the method to compute the Futaki invariants on 2-dimensional Kähler orbifolds. In [6], the Futaki invariants for toric varieties were calculated.

Acknowledgments. The author thanks G. Tian for his mathematical insights and encouragement during the preparation of this paper. The author also thanks D. Phong, M. Kuranishi, H. Pinkham, L. Borisov, and Z. Wu for the discussions of this topic.

2. Preliminaries. Let $X = (a_{ij})_{\substack{1 \leq i \leq N \\ 1 \leq j \leq N}} \in sl(N + 1, C)$ be a traceless $(N + 1) \times (N + 1)$ matrix. X defines a holomorphic vector field

$$(2.1) \quad X = \sum_{i,j=0}^N a_{ij} z^i \frac{\partial}{\partial z^j}$$

on C^{N+1} and a smooth function

$$(2.2) \quad \theta = -X \log \sum_{i=0}^N |z^i|^2$$

on $C^{N+1} - \{0\}$, where (z^0, \dots, z^N) are the coordinates of C^{N+1} .

Both X and θ descend to a vector field and a smooth function on the projective space CP^N , respectively. Let ω_{FS} be the Fubini-Study metric of CP^N . Then we have the relation

$$(2.3) \quad i(X)\omega_{\text{FS}} = -\frac{\sqrt{-1}}{2\pi} \bar{\partial}\theta.$$

Suppose M is an n -dimensional Q -Fano normal variety in CP^N , and suppose that $\omega = \alpha \omega_{FS} \in c_1(M)$ for a constant $\alpha > 0$. If X is a vector field on CP^N such that the 1-parameter group generated by the real part of X leaves M invariant, we say that X is tangent to M . Suppose (x^1, \dots, x^n) are the local holomorphic coordinates at some smooth point p of M . Equation (2.3) can be written as

$$(2.4) \quad X^i = -\alpha g^{i\bar{j}} \frac{\partial \theta}{\partial \bar{x}^i}, \quad i = 1, \dots, n, \quad X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i},$$

where $(g^{i\bar{j}})$ is the inverse matrix of $(g_{i\bar{j}})$ and $(g_{i\bar{j}})$ is the metric matrix of ω .

We define the divergence of X on M by

$$(2.5) \quad \text{div } X = \frac{\partial X^i}{\partial x^i} + \sum_{i=1}^n X^i \frac{\partial}{\partial x^i} \log \det(g_{i\bar{j}}).$$

The following lemma is the observation on which the whole paper is based.

LEMMA 2.1 [8]. *If M is a normal projective variety, then*

$$(2.6) \quad \text{div } X - \alpha \theta + X(f) = \text{const},$$

where the function f is defined as

$$(2.7) \quad \text{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f.$$

Proof. A straightforward computation yields

$$\frac{\sqrt{-1}}{2\pi} \bar{\partial} \text{div } X = -i(X) \text{Ric}(\omega).$$

By equation (2.7), we see that

$$\frac{\sqrt{-1}}{2\pi} \bar{\partial} \text{div } X = -i(X)\omega - \frac{\sqrt{-1}}{2\pi} i(X) \partial \bar{\partial} f = \frac{\sqrt{-1}}{2\pi} \alpha \bar{\partial} \theta - \frac{\sqrt{-1}}{2\pi} \bar{\partial} X(f).$$

Thus $\text{div } X - \alpha \theta + X(f)$ is a holomorphic function on the normal variety M , which must be a constant. □

COROLLARY 2.1. *The Futaki invariant can be written as*

$$F(X) = \text{const vol}(M) + \alpha \int_M \theta \omega^n.$$

3. An explicit expression of the function f . Suppose M is a complete intersection of CP^N ; that is, M is the zero locus of homogeneous polynomials F_1, \dots, F_s in CP^N with degree d_1, \dots, d_s , respectively, and the dimension of M is $N - s$. By the adjunction formula, the anticanonical bundle of M is

$$K_M^{-1} = (N + 1 - d_1 - \dots - d_s)H,$$

where H is the hyperplane bundle of CP^N . We assume that M is a normal variety. There is a unique function f (up to a constant), defined on the regular part M_{reg} of M , such that if $\omega = (N + 1 - d_1 - \dots - d_s)\omega_{\text{FS}}|_M$, then

$$\text{Ric}(\omega) - \omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} f \quad \text{on } M_{\text{reg}}.$$

In this section, we write out the above function f explicitly. The idea is to trace the proof of the well-known adjunction formula. But here we work on the metric level rather than the cohomological level. This makes the notation a little bit complicated.

We begin by the following general setting: Let V be a Kähler manifold of dimension N , and let V_1 be a submanifold of dimension $N - s$ defined by holomorphic functions $G_1 = G_2 = \dots = G_s = 0$. Suppose U_1 is an open set of V such that

- (1) (x^1, \dots, x^N) is a local holomorphic coordinate system of V ;
- (2) on U_1 , we have

$$\text{rank} \frac{\partial(G_1, \dots, G_s)}{\partial(x^1, \dots, x^s)} = s;$$

- (3) there are holomorphic functions f_1, \dots, f_s on $U_1 \cap V_1$ such that

$$\begin{cases} x^1 = f_1(x^{s+1}, \dots, x^N), \\ x^2 = f_2(x^{s+1}, \dots, x^N), \\ \dots \\ x^s = f_s(x^{s+1}, \dots, x^N). \end{cases}$$

In particular, (x^{s+1}, \dots, x^N) is the local holomorphic coordinate system of $U_1 \cap V_1$.

Suppose $g_1 = \sum_{\substack{s+1 \leq i \leq N \\ s+1 \leq j \leq N}} g_{1i\bar{j}} dx^i \wedge d\bar{x}^j$ is the restriction of the Kähler metric of V on $U_1 \cap V_1$. Define $\det g_1 = \det(g_{1i\bar{j}})_{\substack{s+1 \leq i \leq N \\ s+1 \leq j \leq N}}$. Of course, $\det g_1$ is not a global function on V_1 . In order to study the change of $\det g_1$ with respect to the change of the local holomorphic coordinates, we assume that there is another neighborhood $(U_2, (y^1, \dots, y^N))$ of V such that $U_1 \cap U_2 \cap V_1 \neq \emptyset$. As before, we assume that

$$\text{rank} \frac{\partial(G_1, \dots, G_s)}{\partial(y^1, \dots, y^s)} = s$$

and on $U_2 \cap V_1$, we have

$$\begin{cases} y^1 = g_1(y^{s+1}, \dots, y^N), \\ y^2 = g_2(y^{s+1}, \dots, y^N), \\ \dots\dots \\ y^s = g_s(y^{s+1}, \dots, y^N) \end{cases}$$

for holomorphic functions g_1, \dots, g_s on $U_2 \cap V_1$. We have that (y^{s+1}, \dots, y^N) is the local holomorphic coordinate system of $U_2 \cap V_1$. Let

$$g_2 = \sum_{\substack{s+1 \leq i \leq N \\ s+1 \leq j \leq N}} g_{2i\bar{j}} dy^i \wedge d\bar{y}^j$$

be the restriction of the Kähler metric of V on $U_2 \cap V_1$. Define $\det g_2 = \det (g_{2i\bar{j}})_{\substack{s+1 \leq i \leq N \\ s+1 \leq j \leq N}}$. Then we have the following proposition.

PROPOSITION 3.1 (Adjunction formula). *With the above notation, on $V_1 \cap U_1 \cap U_2 \neq \emptyset$, we have*

$$\det(g_1) = \det(g_2) \left| \det \left(\frac{\partial y^j}{\partial x^i} \right)_{\substack{1 \leq i \leq N \\ 1 \leq j \leq N}} \right|^2 \cdot \frac{\left| \det \frac{\partial(G_1, \dots, G_s)}{\partial(y^1, \dots, y^s)} \right|^2}{\left| \det \frac{\partial(G_1, \dots, G_s)}{\partial(x^1, \dots, x^s)} \right|^2}.$$

Proof. Let

$$\begin{cases} \tilde{x}^i = x^i - f_i(x^{s+1}, \dots, x^N), & 1 \leq i \leq s, \\ \tilde{x}^i = x^i, & i > s, \end{cases}$$

and let

$$\begin{cases} \tilde{y}^j = y^j - g_j(y^{s+1}, \dots, y^N), & 1 \leq j \leq s, \\ \tilde{y}^j = y^j, & j > s. \end{cases}$$

Then V_1 is locally defined by $\tilde{x}^i = 0 (1 \leq i \leq s)$ or $\tilde{y}^j = 0 (1 \leq j \leq s)$. In particular, on V_1 , we have

$$(3.1) \quad \frac{\partial \tilde{x}^i}{\partial \tilde{y}^j} = 0, \quad 1 \leq i \leq s, \quad s+1 \leq j \leq N.$$

Before going further, we make the following conventions.

(1) Let $\det(\partial x^i / \partial y^j)_{\substack{1 \leq i \leq N \\ 1 \leq j \leq N}}$ and $\det(\partial \tilde{x}^i / \partial \tilde{y}^j)_{\substack{1 \leq i \leq N \\ 1 \leq j \leq N}}$ denote the restrictions to

V_1 of the Jacobi determinant of the transform $(y^1, \dots, y^N) \rightarrow (x^1, \dots, x^N)$ and $(\tilde{y}^1, \dots, \tilde{y}^N) \rightarrow (\tilde{x}^1, \dots, \tilde{x}^N)$, respectively.

(2) Let $\det(\partial\tilde{x}^i/\partial\tilde{y}^j)_{\substack{1\leq i\leq s \\ 1\leq j\leq s}}$ and $\det(\partial\tilde{x}^i/\partial\tilde{y}^j)_{\substack{s+1\leq i\leq N \\ s+1\leq j\leq N}}$ be the determinants of the submatrices of $(\partial\tilde{x}^i/\partial\tilde{y}^j)_{\substack{1\leq i\leq N \\ 1\leq j\leq N}}$.

(3) Since (x^{s+1}, \dots, x^N) and (y^{s+1}, \dots, y^N) are local coordinates of $U_1 \cap V_1$ and $U_2 \cap V_2$, respectively, there is the transform $(y^{s+1}, \dots, y^N) \rightarrow (x^{s+1}, \dots, x^N)$ by

$$x^i = x^i(g_1(y^{s+1}, \dots, y^N), \dots, g_s(y^{s+1}, \dots, y^N), y^{s+1}, \dots, y^N)$$

for $s+1 \leq i \leq N$, and $\det(\partial x^i/\partial y^j)_{\substack{s+1\leq i\leq N \\ s+1\leq j\leq N}}$ denotes the Jacobi determinant of the above transform.

If $s+1 \leq i, j \leq N$, then

$$\frac{\partial\tilde{x}^i}{\partial\tilde{y}^j} = \frac{\partial x^i(g_1, \dots, g_s, y^{s+1}, \dots, y^N)}{\partial y^j}.$$

Using equation (3.1), we have

$$(3.2) \quad \det\left(\frac{\partial x^i}{\partial y^j}\right)_{\substack{1\leq i\leq N \\ 1\leq j\leq N}} = \det\left(\frac{\partial\tilde{x}^i}{\partial\tilde{y}^j}\right)_{\substack{1\leq i\leq s \\ 1\leq j\leq s}} \cdot \det\left(\frac{\partial x^i}{\partial y^j}\right)_{\substack{s+1\leq i\leq N \\ s+1\leq j\leq N}}.$$

If $1 \leq i, j \leq s$, then

$$\frac{\partial\tilde{x}^i}{\partial\tilde{y}^j} = \sum_{k=1}^s \frac{\partial\tilde{x}^i}{\partial y^k} \cdot \frac{\partial y^k}{\partial\tilde{y}^j} = \frac{\partial x^i}{\partial y^j} - \sum_{k=s+1}^N \frac{\partial f_i}{\partial x^k} \cdot \frac{\partial x^k(g_1, \dots, g_s, y^{s+1}, \dots, y^N)}{\partial y^j}.$$

Thus we have

$$(3.3) \quad \sum_{i=1}^s \frac{\partial G_k}{\partial x^i} \cdot \frac{\partial\tilde{x}^i}{\partial\tilde{y}^j} = \frac{\partial G_k}{\partial y^j}.$$

Proposition 3.1 follows from (3.2) and (3.3). \square

We use Proposition 3.1 in the case of complete intersections of CP^N . Since M is defined by the zero locus of homogeneous functions, we must make some necessary adjustments because homogeneous polynomials are *not* functions on CP^N .

Let U_α ($\alpha = 0, \dots, n$) be the standard covering of CP^N , defined by $U_\alpha = \{Z_\alpha \neq 0\}$, where $[Z_0, \dots, Z_N]$ are the homogeneous coordinates of CP^N . Suppose $z_i^\alpha = Z_i/Z_\alpha$ ($i \neq \alpha$) are the standard coordinates on U_α . Let

$$F_i^\alpha(z_0^\alpha, \dots, \hat{z}_\alpha^\alpha, \dots, z_N^\alpha) = \frac{F(Z_0, \dots, Z_N)}{(Z_\alpha)^{d_i}}, \quad 1 \leq i \leq s.$$

For each $\{\alpha, \beta_1, \dots, \beta_s\} \subset \{0, \dots, N\}$, define

$$U_{\alpha\beta_1 \dots \beta_s} = \left\{ p \in U_\alpha \mid \text{rank} \frac{\partial(F_1^\alpha, \dots, F_s^\alpha)}{\partial(z_{\beta_1}^\alpha, \dots, z_{\beta_s}^\alpha)} = s \right\}.$$

Then it is clear that $\cup U_{\alpha\beta_1\dots\beta_s} \supset M_{\text{reg}}$.

At each point $p \in M_{\text{reg}} \cap U_{\alpha\beta_1\dots\beta_s}$, $(z_\beta^\alpha, \beta \notin \{\alpha, \beta_1, \dots, \beta_s\})$ can be used as local coordinate system at p . Let $g_{\alpha\beta_1\dots\beta_s}$ be the corresponding metric matrix, and let $\det g_{\alpha\beta_1\dots\beta_s}$ be its determinant. Define

$$(3.4) \quad \xi_{\alpha\beta_1\dots\beta_s} = \det(g_{\alpha\beta_1\dots\beta_s}) \left| \det \frac{\partial(F_1^\alpha, \dots, F_s^\alpha)}{\partial(z_{\beta_1}^\alpha, \dots, z_{\beta_s}^\alpha)} \right|^2 \left(1 + \sum_{i \neq \alpha} |z_i^\alpha|^2 \right)^{N+1-d_1-\dots-d_s}.$$

Then we have the following lemma.

LEMMA 3.1. $\{\xi_{\alpha\beta_1\dots\beta_s}\}$ defines a global positive function of M_{reg} .

Proof. A straightforward computation shows (cf. [4, p. 146])

$$dz_0^{\alpha_2} \wedge \dots \wedge \widehat{dz_{\alpha_2}^{\alpha_2}} \dots \wedge dz_N^{\alpha_2} = \left(\frac{Z_{\alpha_1}}{Z_{\alpha_2}} \right)^{N+1} dz_0^{\alpha_1} \wedge \dots \wedge \widehat{dz_{\alpha_1}^{\alpha_1}} \dots \wedge dz_N^{\alpha_1}.$$

The lemma follows from Proposition 3.1, (3.4), and the above equation. □

THEOREM 3.1. Let f be the function on M_{reg} , defined by

$$(3.5) \quad f = -\log \xi_{\alpha\beta_1\dots\beta_s} \quad \text{on } U_{\alpha\beta_1\dots\beta_s}.$$

Then

$$\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} f = \text{Ric}(\omega) - \omega,$$

where $\omega = (N + 1 - d_1 - \dots - d_s)\omega_{\text{FS}}|_M$.

4. The trace of the action on F_1, \dots, F_s . Let M be the variety defined in the previous section. The vector field $X = \sum_{i,j=0}^N a_{ij} Z_j (\partial/\partial Z_i)$ naturally acts on F_i by

$$(4.1) \quad X F_i = \sum_{i,j=0}^N a_{ij} Z_j \frac{\partial F_i}{\partial Z_i}, \quad i = 1, \dots, s.$$

Suppose \mathcal{V} is the vector space spanned by F_1, \dots, F_s . Since X is tangent to M , X is an automorphism on \mathcal{V} .

The main result of this section is the following theorem.

THEOREM 4.1. Let κ be the trace of the automorphism of X on \mathcal{V} . Then

$$\text{div } X + X(f) - (N + 1 - d_1 \dots - d_s)\theta = -\kappa.$$

Proof. We adopt all notation from the last section. Consider a smooth point p of M_{reg} in $U_{\alpha\beta_1\dots\beta_s}$. From (2.2), the function θ in the local coordinates $\{z_j^\alpha, j \neq \alpha\}$ is

$$(4.2) \quad \theta = -\tilde{X} \log \left(1 + \sum_{i \neq \alpha} |z_i^\alpha|^2 \right) - a_{\alpha\alpha} - \sum_{j \neq \alpha} a_{\alpha j} z_j^\alpha.$$

By (2.1), let

$$(4.3) \quad \tilde{X} = \sum_{i \neq \alpha} \left(\left(\sum_{j \neq \alpha} a_{ij} z_j^\alpha - a_{\alpha j} z_j^\alpha z_i^\alpha \right) + (a_{i\alpha} - a_{\alpha\alpha} z_i^\alpha) \right) \frac{\partial}{\partial z_i^\alpha}.$$

Let the holomorphic vector field X on M be written as

$$(4.4) \quad X = \sum_{i \notin \{\alpha, \beta_1, \dots, \beta_s\}} X^i \frac{\partial}{\partial z_i^\alpha}.$$

If $i : M \rightarrow CP^N$ is the embedding, then $i_*X = \tilde{X}$.

By (3.4), (3.5), (4.2), (4.4), and the definition of $\text{div } X$ in (2.5), we see that

$$(4.5) \quad \begin{aligned} & \text{div } X + X(f) - (N + 1 - d_1 - \dots - d_s)\theta \\ &= \sum_{i \notin \{\alpha, \beta_1, \dots, \beta_s\}} \frac{\partial X^i}{\partial z_i^\alpha} - X \log \det \frac{\partial (F_1^\alpha, \dots, F_s^\alpha)}{\partial (z_{\beta_1}^\alpha, \dots, z_{\beta_s}^\alpha)} \\ & \quad + (N + 1 - d_1 - \dots - d_s) \left(a_{\alpha\alpha} + \sum_{j \neq \alpha} a_{\alpha j} z_j^\alpha \right). \end{aligned}$$

Before going on, we need a general elementary lemma. To begin, we use the general setting in Section 3. In addition, we let X be a holomorphic vector field of V such that X is tangent to V_1 . In what follows, we temporarily distinguish the X on V and the X on V_1 . So let us denote the X on V to be \tilde{X} . In the local coordinates,

$$\tilde{X} = \sum_{i=1}^N \tilde{X}^i \frac{\partial}{\partial x^i}.$$

Then X on V_1 can be written as

$$X = \sum_{i=s+1}^N X^i \frac{\partial}{\partial x^i},$$

from which we have the chain rule

$$(4.6) \quad X^i = \tilde{X}^i(f_1, \dots, f_s, X^{s+1}, \dots, X^N), \quad s + 1 \leq i \leq N.$$

If $i : V_1 \rightarrow V$ is the embedding, then $i_*X = \tilde{X}$.

We have the following elementary lemma.

LEMMA 4.1. *Let*

$$A = \frac{\partial(G_1, \dots, G_s)}{\partial(x^1, \dots, x^s)},$$

let $A_{ij} = (\partial G_i / \partial x^j)$ for $i, j = 1, \dots, s$, and let (A^{ij}) be the inverse matrix of A .

Then, on V_1 , we have

$$\sum_{i=s+1}^N \frac{\partial X^i}{\partial x^i} - \tilde{X} \log \det A = \sum_{i=1}^N \frac{\partial \tilde{X}^i}{\partial x^i} - \sum_{i,j=1}^s A^{ji} \frac{\partial}{\partial x^j} \tilde{X} G_i.$$

Proof. By definition,

$$\tilde{X} \log \det A = \sum_{k=1}^N \sum_{i,j=1}^s A^{ji} \tilde{X}^k \frac{\partial^2 G_i}{\partial x^j \partial x^k}.$$

We can write the above equation as

$$(4.7) \quad \tilde{X} \log \det A = \sum_{i,j=1}^s \left(A^{ji} \frac{\partial}{\partial x^j} \tilde{X} G_i - \sum_{k=1}^N A^{ji} \frac{\partial \tilde{X}^k}{\partial x^j} \frac{\partial G_i}{\partial x^k} \right).$$

By the implicit differentiation, we see that, on V_1 ,

$$(4.8) \quad \frac{\partial f_j}{\partial x^k} = - \sum_{i=1}^s A^{ji} \frac{\partial G_i}{\partial x^k}, \quad j = 1, \dots, s, \quad k = s+1, \dots, N.$$

Using (4.7) and (4.8), we get

$$\tilde{X} \log \det A = \sum_{i,j=1}^s A^{ji} \frac{\partial}{\partial x^j} \tilde{X} G_i + \sum_{k=s+1}^N \sum_{j=1}^s \frac{\partial \tilde{X}^k}{\partial x^j} \frac{\partial f_j}{\partial x^k} - \sum_{i=1}^s \frac{\partial \tilde{X}^k}{\partial x^i}.$$

The lemma follows from the above identity and the fact that

$$\sum_{i=s+1}^N \frac{\partial X^i}{\partial x^i} = \sum_{i=s+1}^N \frac{\partial \tilde{X}^i}{\partial x^i} + \sum_{i=s+1}^N \sum_{j=1}^s \frac{\partial \tilde{X}^i}{\partial x^j} \cdot \frac{\partial f_j}{\partial x^i}. \quad \square$$

Go back to the proof of the theorem. Let

$$A = \frac{\partial(F_1^\alpha, \dots, F_s^\alpha)}{\partial(z_{\beta_1}^\alpha, \dots, z_{\beta_s}^\alpha)},$$

and temporarily denote \tilde{X} to be the vector field X on $U_\alpha = \{Z_\alpha \neq 0\}$. The representation of \tilde{X} is in (4.3). Obviously,

$$X \log \det \frac{\partial(F_1^\alpha, \dots, F_s^\alpha)}{\partial(z_{\beta_1}^\alpha, \dots, z_{\beta_s}^\alpha)} = \tilde{X} \log \det \frac{\partial(F_1^\alpha, \dots, F_s^\alpha)}{\partial(z_{\beta_1}^\alpha, \dots, z_{\beta_s}^\alpha)}.$$

Using Lemma 4.1, (4.5) becomes

$$\begin{aligned}
 & \operatorname{div} X + X(f) - (N+1 - d_1 - \dots - d_s)\theta \\
 (4.9) \quad &= \sum_{i \neq \alpha} \frac{\partial \tilde{X}^i}{\partial z_i^\alpha} - \sum_{i=1}^s \sum_{j \in \{\beta_1, \dots, \beta_s\}} A^{ji} \frac{\partial}{\partial z_j^\alpha} \tilde{X} F_i^\alpha \\
 &+ (N+1 - d_1 - \dots - d_s) \left(a_{\alpha\alpha} + \sum_{j \neq \alpha} a_{\alpha j} z_j^\alpha \right).
 \end{aligned}$$

Since $\sum a_{ii} = 0$, a simple calculation gives

$$(4.10) \quad \sum_{i \neq \alpha} \frac{\partial \tilde{X}^i}{\partial z_i^\alpha} = -(N+1) \left(a_{\alpha\alpha} + \sum_{j \neq \alpha} a_{\alpha j} z_j^\alpha \right).$$

Recall the definition of $X F_i$ in (4.1). We see that for $i = 1, \dots, s$,

$$\tilde{X} F_i^\alpha = X \frac{F_i}{Z_\alpha^{d_i}} = \frac{X F_i}{Z_\alpha^{d_i}} - d_i \frac{F_i}{Z_\alpha^{d_i}} \left(a_{\alpha\alpha} + \sum_{j \neq \alpha} a_{\alpha j} z_j^\alpha \right).$$

Thus on M_{reg} ,

$$\begin{aligned}
 (4.11) \quad & \sum_{i=1}^s \sum_{j \in \{\beta_1, \dots, \beta_s\}} A^{ji} \frac{\partial}{\partial z_j^\alpha} \tilde{X} F_i^\alpha \\
 &= \sum_{i=1}^s \sum_{j \in \{\beta_1, \dots, \beta_s\}} A^{ji} Z_\alpha \left(\frac{(\partial/\partial Z_j) X F_i}{Z_\alpha^{d_i}} - d_i \frac{(\partial F_i/\partial Z_j)}{Z_\alpha^{d_i}} \left(a_{\alpha\alpha} + \sum_{j \neq \alpha} a_{\alpha j} z_j^\alpha \right) \right) \\
 &= \kappa - \left(\sum_{i=1}^s d_i \right) \left(a_{\alpha\alpha} + \sum_{j \neq \alpha} z_j^\alpha \right),
 \end{aligned}$$

where we use the fact that, on M_{reg} , $F_1 = \dots = F_s = 0$ in the second identity. Theorem 4.1 follows from (4.9), (4.10), and (4.11). \square

5. The computation of the θ invariants. Let M be the complete intersection defined in Section 3. Let $M_0 = CP^N$, and let $M_k = N_1 \cap \dots \cap N_k$ ($k = 1, \dots, s$). Then $M_s = M$. We assume that $X F_i = \kappa_i F_i$, $i = 1, \dots, s$.

Let $[Z_0, \dots, Z_N]$ be the homogeneous coordinates of CP^N . Define

$$\xi_i = \frac{|F_i|^2}{\sum_{i=0}^N (|Z_i|^2)^{d_i}}, \quad i = 1, \dots, s.$$

Then ξ_i 's are global smooth functions on CP^N .

In this section, we compute the θ -invariant $\int_M \theta \omega^n$, where for simplifying the notation, we assume that $\omega = \omega_{\text{FS}}$ is the Fubini-Study metric of the CP^N . The key result is the following lemma.

LEMMA 5.1. *For $k = 2, \dots, s$, we have*

$$(5.1) \quad \int_{M_k} (\theta + \omega)^{N-k+1} = d_k \int_{M_{k-1}} (\theta + \omega)^{N-k+2} + \kappa_k d_1 \cdots d_{k-1},$$

and in addition, we have

$$(5.2) \quad \int_{M_1} (\theta + \omega)^N = \kappa_1.$$

Proof. We have the following identities for $k = 1, \dots, s$:

$$(5.3) \quad \begin{aligned} \bar{\partial} \left(\frac{\sqrt{-1}}{2\pi} \partial \log \xi_k \wedge \theta \omega^{N-k} \right) + i(X) \left(\frac{\partial \log \xi_k}{N-k+1} \omega^{N-k+1} \right) \\ = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \xi_k \wedge \theta \omega^{N-k} + \frac{1}{N-k+1} X \log \xi_k \wedge \omega^{N-k+1}. \end{aligned}$$

Integration against M_{k-1} gives

$$\int_{M_{k-1}} \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \xi_k \wedge \theta \omega^{N-k} = \frac{1}{N-k+1} \int_{M_{k-1}} X \log \xi_k \wedge \omega^{N-k+1}.$$

Since, for $k = 1, \dots, s$,

$$\begin{aligned} \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \xi_k &= [N_k] - d_k \omega, \\ X \log \xi_k &= \kappa_k + d_k \theta, \end{aligned}$$

where $[N_k]$ is the divisor of the zero locus of F_k , we have

$$\int_{M_k} \theta \omega^{N-k} - d_k \int_{M_{k-1}} \theta \omega^{N-k+1} = \frac{1}{N-k+1} \int_{M_{k-1}} (\kappa_k + d_k \theta) \omega^{N-k+1}.$$

Thus,

$$\int_{M_k} \theta \omega^{N-k} = \frac{N-k+2}{N-k+1} d_k \int_{M_{k-1}} \theta \omega^{N-k+1} + \frac{\kappa_k d_1 \cdots d_{k-1}}{N-k+1}.$$

So (5.1) is proved. To prove (5.2), let us first see that by (5.3),

$$\int_{M_1} \theta \omega^{N-1} = \frac{N+1}{N} d_1 \int_{CP^N} \theta \omega^N + \frac{\kappa_1}{N}.$$

Thus (5.2) follows from the simple fact that

$$\int_{CP^N} \theta \omega^N = 0. \quad \square$$

Equation (5.1) can be rewritten as

$$\frac{1}{d_1 \cdots d_k} \int_{M_k} (\theta + \omega)^{N-k+1} = \frac{1}{d_1 \cdots d_{k-1}} \int_{M_{k-1}} (\theta + \omega)^{N-k+2} + \frac{\kappa_k}{d_k}$$

for $k = 2, \dots, s$. Thus, we have the following theorem.

THEOREM 5.1. *With the notation as above, we have*

$$\int_M \theta \omega^{N-s} = \frac{d_1 \cdots d_s}{N-s+1} \sum_{k=1}^s \frac{\kappa_k}{d_k}.$$

Now we prove the main theorem of this paper.

Proof of Theorem 1.1. Theorem 1.1 follows from Theorem 4.1, Theorem 5.1, and the fact that

$$\omega = (N + 1 - d_1 - \cdots - d_s) \omega_{\text{FS}} \in c_1(M). \quad \square$$

6. Examples. In this section, we use our formula to compute some examples given by Ding-Tian [1], Jeffres [5], and Wu [9]. Recall that the Futaki invariants defined in [1] and [5] differ from ours by a factor 3 in the case of surfaces. So, in what follows, the Futaki invariant $F(X)$ is actually three times the Futaki invariant in the previous sections.

COROLLARY 6.1. *With the notation as in Theorem 1.1, if M is the cubic surface in CP^3 , then the Futaki invariants are*

$$F(X) = -8\kappa.$$

COROLLARY 6.2. *With the notation as in Theorem 1.1, if M is the variety of the intersection of two quadratic polynomials in CP^4 , then*

$$F(X) = -10(\kappa_1 + \kappa_2).$$

The first four examples are due to Ding and Tian [1].

Example 1. Let $X_f \subset CP^3$ be the zero locus of a cubic polynomial f . Put $f = z_0 z_1^2 + z_2 z_3 (z_2 - z_3)$, where z_0, z_1, z_2, z_3 are homogeneous coordinates of CP^3 . The X_f has a unique quotient singularity at $p_0 = [1, 0, 0, 0]$. This singularity is of the form C^2/Γ , where Γ is the dihedral subgroup in $SU(2)$ of type D_4 . One can check that X_f is a Kähler orbifold with $c_1(X) > 0$. Let X be the holomorphic vector field whose real part generates the 1-parameter subgroup $\{\text{diag}(1, e^{3t}, e^{2t}, e^{2t})\}_{t \in R}$ in $SL(4, C)$. Then X restricts to a holomorphic vector field on X_f and has five zeros $[1, 0, 0, 0]$, $[0, 1, 0, 0]$, $[0, 0, 1, 0]$, $[0, 0, 0, 1]$, and $[0, 0, 1, 1]$.

We use three methods to compute the Futaki invariants. The first method is the original method in [1]. We can rewrite the function f near $[1, 0, 0, 0]$ in the standard

form

$$f = z_1^2 - z_3(z_2^2 - 4z_3^2).$$

In [7], we see that there are standard coverings $C^2 \rightarrow C^2/\Gamma$ by

$$\begin{cases} z_1 = uv(u^4 - v^4), \\ z_2 = u^4 + v^4, \\ z_3 = u^2v^2. \end{cases}$$

If we assume that on the $(u - v)$ -plane, $X = au(\partial/\partial u) + bv(\partial/\partial v)$, then we have $a = b = (1/2)$. Since the order of the group D_4 is 8, a computation using the formula in [1, p. 324] shows

$$F(X) = \frac{1}{8} \cdot \frac{1^3}{1/4} + \frac{(-2)^3}{1} + 3 \frac{(-1)^3}{-2} = -6.$$

Our second method is a trick that can be generalized to give another proof of the main theorem of this paper in the case of hypersurfaces. Suppose that in $X = au(\partial/\partial u) + bv(\partial/\partial v)$, we do not know what a and b are. By using the Bott residue formula, we see that

$$\frac{1}{8} \cdot \frac{(a+b)}{ab} + \frac{(-2)}{1} + 3 \frac{(-1)}{-2} = 0$$

and

$$\frac{1}{8} \cdot \frac{(a+b)^2}{ab} + \frac{(-2)^2}{1} + 3 \frac{(-1)^2}{-2} = 3.$$

Thus we solved $a = b = (1/2)$. The Futaki invariant is obtained.

The last method is to use Corollary 6.1, which gives $F(x) = -6$.

Example 2. Let $f = z_0z_1^2 + z_1z_2^2 + z_3^3$, and let $X = \{\text{diag}(1, e^{6t}, e^{3t}, e^{4t})\}$. Then Corollary 6.1 gives $F(X) = -18$.

Example 3. Let $f = z_0(z_1^2 + z_2^2) + z_3^2z_1$, and let $X = \text{diag}\{(1, e^{2t}, e^{2t}, e^t)\}$. Then Corollary 6.1 gives $F(X) = -2$.

Example 4. Let $f = z_0(z_1^2 + z_2^2) + z_3^3$, and let $X = \text{diag}\{(1, e^{3t}, e^{3t}, e^{2t})\}$. Then Corollary 6.1 gives $F(X) = 0$.

The following examples are given by Jeffres [5]. Let $[z_0, z_1, w, x, z]$ be the general point in CP^4 .

Example 5. Let

$$\begin{cases} f = z_0z_1 + w^2 + x^2, \\ g = z_1l(w, x) + z^2, \end{cases}$$

where $l(w, x)$ is a linear function of w, x . Let $X = \text{diag}\{1, e^{2t}, e^t, e^t, e^{3/2t}\}$. Then $\kappa_1 = -1/5$ and $\kappa_2 = 4/5$. Using Corollary 6.2, $F(X) = -6$.

Example 6. Let

$$\begin{cases} f = z_0 z_1 + z^2, \\ g = z_1^2 + wx, \end{cases}$$

and let $X = \text{diag}\{1, e^{2t}, e^{2t}, e^{2t}, e^t\}$. Then $\kappa_1 = -4/5$ and $\kappa_2 = 6/5$. So, using Corollary 6.2, the Futaki invariant $F(X) = -4$.

The following examples are given by Wu [9].

Example 7. Let $M_0 \subset CP^4$ be the zero locus defined by

$$f = z_0 z_1^2 + z_1 z_2^2 + z_3^3 + z_4^3 = 0,$$

and let $X = \text{diag}(1, e^{6t}, e^{3t}, e^{4t}, e^{4t})$. Then $\kappa = 9/5$ and by Corollary 1.1, $F(X) = -36$.

Example 8. Let $M_0 \subset CP^3$ be the zero locus defined by

$$f = z_3(z_1^2 - z_0 z_2) + z_2^3 = 0,$$

and let $X = \text{diag}(1, e^t, e^{2t}, e^{4t})$. Then $\kappa = 3/4$ and by Corollary 1.1, $F(X) = -2$.

REFERENCES

- [1] W. DING AND G. TIAN, *Kähler-Einstein metrics and the generalized Futaki invariant*, Invent. Math. **110** (1992), 315–335.
- [2] A. FUTAKI, *An obstruction to the existence of Einstein Kähler metrics*, Invent. Math. **73** (1983), 437–443.
- [3] ———, *Kähler-Einstein metrics and integral invariants*, Lecture Notes in Math. **1314**, Springer-Verlag, Berlin, 1988.
- [4] P. GRIFFITHS AND J. HARRIS, *Principles of Algebraic Geometry*, Pure Appl. Math., Wiley, New York, 1978.
- [5] T. JEFFRES, *Singular set of some Kähler orbifolds*, Trans. Amer. Math. Soc. **349** (1997), 1961–1971.
- [6] T. MABUCHI, *Einstein-Kähler forms, Futaki invariants and convex geometry on toric Fano varieties*, Osaka J. Math. **24** (1987), 705–737.
- [7] H. PINKHAM, “Singularités de Klein, I” in *Séminaire sur les singularités des surfaces (Centre de Mathématiques de l’École Polytechnique, Palaiseau, 1976–1977)*, Lecture Notes in Math. **777**, Springer-Verlag, Berlin, 1980, 1–9.
- [8] G. TIAN, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. **130** (1997), 1–37.
- [9] Z. WU, *Kähler-Einstein metrics and K-stability*, Ph.D. thesis, Columbia University, New York, 1998.

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027, USA;
lu@cpw.math.columbia.edu