ANALYSIS AND A LOWER BOUND OF THE FIRST EIGENVALUE OF A COMPACT MANIFOLD

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Dedicated to Professor Guangchang Dong on his 80th birthday

Abstract. We explore the analysis properties of some functions that are often used in construction of barriers and apply the results to give new estimate for the lower bound of either the first non-zero eigenvalue of a closed compact manifold or the Neumann eigenvalues of a compact manifold with boundary, with positive Ricci curvature.

1. Some previous results on lower bounds

Let (M, g) be an n-dimensional compact Riemannian manifold without boundary. It is known that the spectrum of the Laplacian consists of entirely (closed) eigenvalues $\{\lambda_i\}_{i=0}^{\infty}$ that can be arranged in the order

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots .$$

If the manifold has boundary, one considers Dirichlet, Neumann, and mixed eigenvalue with corresponding boundary data and one has an increasing sequence of eigenvalues similar as the above (in Dirichlet case, thereis no λ_0). It is an interesting problem to study the distribution of the eigenvalues since it reveals important relation between geometry of the manifold and analysis. Early work in the field includes Weyl's asymptote formula and Courant's nodal domain theorem [8].

Weyl's asymptote formula sates that

$$N(\lambda) \sim \omega_n (\mathrm{vol} M) \lambda^{n/2}/(2\pi)^n$$

as $\lambda \to +\infty$, where ω_n is the volume of the unit solid ball in $\mathbb{R}^n$, $n(\lambda)$ is the number of eigenvalues less than or equal to λ, counted in multiplicity. Courant's nodal domain theorem sates that the number of nodal domains of k-th eigen function is less than or equal to $k+1$.

A lot of other work has been done, especially for the bounds of first non-zero eigenvalue, and the gap or ratios of the first two eigenvalues. We mention only a few of such research results in this paper, mainly due to the authors' interest.

(i) Some results on the bounds of the first non-zero eigenvalues λ_1 for compact manifolds with positive Ricci curvature.

For a n-dimensional closed Riemannian manifold with positive Ricci curvature curvature, Lichnerowicz [16] proved a lower bound of the first eigenvalue λ_1 of the Laplacian

$$\lambda_1 \geq nK \tag{1}$$

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where $K > 0$ is a constant and $(n-1)K$ is the lower bound of the Ricci curvature. Obata [25] showed that the lower bound is reached if and only if M is isometry to an n-sphere of constant curvature K.

The above lower bound was generalized to the first Neumann eigenvalue for a compact manifold whose boundary has nonnegative second fundamental form with respect to the outward normal by J.F. Escobar [12], and to the first Dirichlet eigenvalue for a compact manifold whose boundary has nonnegative mean curvature with respect to the outward normal by R. Reilly [26].

If the manifold has nonnegative Ricci curvature, Li-Yau gave a different type of lower bound for the first non-zero closed eiganvalue

$$\lambda_1 \geq \frac{\pi^2}{2d^2},$$

where d is the diameter of the manifold. The bound was later improved to optimal one by Zhong and Yang [34],

$$\lambda_1 \geq \frac{\pi^2}{d^2}. \tag{2}$$

The second author [22] proved the same lower bound for the first Neumann eigenvalue of a compact manifold with convex boundary.

There are many estimates in terms of Cheeger constants and other geometric quantities (cf. [8, 27, 24]) and some comparison results [9, 23, 27] under various curvature conditions as well. We would like to mention Faber-Krahn Comparison Theorem [8, 27] here, which states that for any finite disjoint union Ω of regular domains in a complete manifold M and a ball D in a space form, manifold that is complete, and simply-connected and has constant sectional curvature, with the same dimension as that of M, if geometric isoperimetric inequality holds, that is,

$$\mathrm{vol}(\Omega) = \mathrm{vol}(D) \quad \Rightarrow \quad \mathrm{vol}(\partial\Omega) \geq \mathrm{vol}(\partial D),$$

then for the first Dirichlet eigenvalue λ_1, we have

$$\lambda_1(\Omega) \geq \lambda_1(D).$$

In a recent paper [21], the first author gave a similar comparison result, namely, For all Riemannian metrics with the same volume in the same conformal class on a compact surface, the ratio of the the first non-zero eigenvalue and the the minimum of the Gauss curvature has a lower bound that is reached by the metric in the class that has constant Gauss curvature and equal volume.

It is interesting to see how the lower bound of the first non-zero eigenvalue evolve when the Ricci lower bound $(n-1)K > 0$ decays to 0. It is nature to combine lower bound in (1) with one in (2). By Bennet-Myers' Theorem,

$$\frac{\pi^2}{d^2} \geq K$$

Therefore for linear combinations of the two lower bound, the best bound one can expect is

$$\lambda_1 \geq \frac{\pi^2}{d^2} + (n-1)K.$$

For closed, Dirichlet and Neumann boundary data, [30] gave the following estimate

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{1}{4}(n-1)K,$$

where and below d is the diameter of the manifold for closed, and Neumann boundary data, and is the interior diameter for Dirichlet data.

The first author improved the Dirichlet eigenvalue estimate [17] to

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{1}{2}(n-1)K,$$

and the first non-zero closed or the first Neumann eigenvalue estimate to

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{31}{100}(n-1)K$$

in general, and

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \frac{1}{2}(n-1)K$$

when the manifold has a certain symmetry. In the next section of this paper we will give some new results for the closed eigenvalue and for the Neumann eigenvalue that improve upon the above.

(ii) Lower bound for the difference between the first two eigenvalues.

Let us consider the Dirichlet eigenvalues of Laplacian on a bounded convex domain in $\mathbb{R}^n$. The first two eigenvalues λ_1 and λ_2 are distinct, therefore it is interesting to see the gap between two or ratio of the two and to see how it is related to geometry. The well-know Payne-Pólya-Weinberger's conjecture [1, 3] states that λ_2/λ_1 has an upper bound which is reached if and only if Ω is a disk. The conjecture was proved by Ashbaugh and Benguria [2].

Singer, Wong, Yau and Yau [29] showed that

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{4d^2},$$

where and below d is the diameter of the domain. Yu and Zhong [33] later removed the factor $1/4$. The first author [19] proved that global log-convexity holds if the domain in $\mathbb{R}^n$ or $\mathbf{S}^n$ is convex and therefore one has the strict lower bound

$$\lambda_2 - \lambda_1 > \frac{\pi^2}{d^2}$$

Smits [28] gave an alternative derivation of the above result.

Motivated statistical by physics, van den Berg [7] conjectured that the lower bound is $3\pi^2/d^2$. See also Yau [27] and Ashbaugh [1]. Ashbaugh and Benguria [2], Bañuelos and Méndez-Hernández [6], Bañuelos and Kröger [4], and Davis [10] proved the conjecture for some special symmetric domains in $\mathbb{R}^2$ and for some special potential V.

On the other hand, motivated by geometry, Yau [32] gave an interesting estimate

$$\lambda_2 - \lambda_1 \geq \theta\,\frac{\pi^2}{d^2} + 2\big(\cos \pi\sqrt{\theta}\big)^2\,\alpha,$$

where θ is any constant with $0 \leq \theta \leq 1/4$, d is the diameter, and $\alpha > 0$ is the global log-convexity

$$\alpha = \inf_{x\in\Omega}\inf_{\tau\in \mathrm{T}_x\Omega, |\tau|=1}\left[\nabla^2\left(-\ln f\right)\right](\tau,\tau)(x)\,, \tag{3}$$

f is a positive first eigenfunction. By Brascamp and Lieb [11] and the first author[19], $\alpha > 0$. The first author [20] improved the above estimate to

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{d^2} + 0.62\alpha$$

in general and

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{d^2} + \alpha$$

if the domain and the potential has certain symmetry.

2. Eigenvalue estimates

In this section, we give some new estimates for the first non-zero closed eigenvalue and for the first Neumann eigenvalue for compact manifolds with positive Ricci curvature. One of important approaches toward to these estimates is to sharpen our analysis tools and have better understanding on the properties of functions that are used to construct barrier functions. Our Theorem 2 is for this purpose, though it itself is interesting in analysis and may have other applications. We get the eigenvalue estimates (Theorem 1) from then the analysis results.

Let us first recall some settings and results in [18]. Let u be an eigenfunction of the first non-zero (closed or Neumann, which applies) eigenvalue λ_1 such that

$$\sup_M u = 1, \quad \inf_M u = -k, \quad \text{and} \quad 0 < k \leq 1.$$

Renormalize u as v such that $\max_M v = 1$ and $\min_M v = -1$ with

$$v = [u - (1-k)/2]/[(1+k)/2].$$

The function v satisfies the following equation

$$\Delta v = -\lambda(v + a) \quad \text{in } M, \tag{4}$$

where

$$a = (1-k)/(1+k). \tag{5}$$

Note that $0 \leq a < 1$. If M has non-empty boundary ∂M, then v satisfies Neumann condition on the boundary,

$$\frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial M, \tag{6}$$

where ν is the the outward normal of ∂M.

Let

$$\alpha = \frac{1}{2}(n-1)K \qquad \text{and} \qquad \delta = \alpha/\lambda_1. \tag{7}$$

By (1) we have

$$\delta \leq \frac{n-1}{2n} \quad \text{and} \quad \lambda_1 \geq \frac{2n}{n-1}\alpha. \tag{8}$$

Our result is the following theorem.

Theorem 1. *If (M, g) is an n-dimensional, compact Riemannian manifold that has an empty or none-empty boundary whose second fundamental form is nonnegative with respect to the outward normal (i.e., weakly convex). Suppose that Ricci curvature* $\mathrm{Ric}(M)$ *has a positive lower bound*

$$\mathrm{Ric}(M) \geq (n-1)K \qquad \text{with constant } K > 0. \tag{9}$$

Then the first nonzero (closed or Neumann, which applies) eigenvalue λ of the Lalacian Δ on M has the following lower bound

$$\lambda \geq \frac{\pi^2}{d^2} + \frac{34}{100}(n-1)K \qquad \textit{for } n \geq 3,$$

where where d is the diameter of M.

Proof. We follow lines in [18].

Since $0 \le a < 1$, either $a = 0$ or $0 < a < 1$.

If $a = 0$, then we apply Theorem B to get the bound with $\mu = 1$,

$$\lambda_1 \ge \frac{\pi^2}{d^2} + \alpha.$$

If $0 < a < 1$, then there are several cases altogether. Let $a_0 = 0.843$.

- (I): $a \ge \frac{\pi^2}{4}\delta$.
- (II): $a < \frac{\pi^2}{4}\delta$.
  - (II-a): $a \ge a_0$.
  - (II-b): $0 < a < a_0$.
    - (II-b-1): $a \ge 1.686\delta$.
    - (II-b-2): $a < 1.686\delta$.

For Case (I): $0 < a < 1$ and $a \ge \frac{\pi^2}{4}\delta$, we apply Theorem A for $\mu = 1$ to get the following lower bound

$$\lambda \ge \frac{\pi^2}{d^2} + \alpha.$$

For Case (II-a): $a_0 \le a < \frac{\pi^2}{4}\delta$, we can apply Theorem A with $\mu = \frac{4}{\pi^2}\frac{a}{\delta}$ since $(\frac{4}{\pi^2}\frac{a}{\delta})\,\delta \le \frac{4}{\pi^2}a$ and $0 < \frac{4}{\pi^2}\frac{a}{\delta} < 1$, and we get

$$\lambda \ge \frac{\pi^2}{d^2} + \frac{4}{\pi^2}\frac{a}{\delta}\alpha = \frac{\pi^2}{d^2} + \frac{4a}{\pi^2}\lambda.$$

Thus

$$\lambda \ge \frac{1}{1 - \frac{4a}{\pi^2}}\frac{\pi^2}{d^2} \ge \frac{1}{1 - \frac{4a_0}{\pi^2}}\frac{\pi^2}{d^2}.$$

For Case (II-b-1): $0 < a < a_0$, $a < \frac{\pi^2}{4}\delta$ and $a \ge 1.686\delta$, we apply Theorem A with with $\mu = \frac{4}{\pi^2}\frac{a}{\delta}$ since $(\frac{4}{\pi^2}\frac{a}{\delta})\,\delta \le \frac{4}{\pi^2}a$ and $0 < \frac{4}{\pi^2}\frac{a}{\delta} < 1$. Then

$$\lambda \ge \frac{\pi^2}{d^2} + \frac{4}{\pi^2}\frac{a}{\delta}\alpha \ge \frac{\pi^2}{d^2} + \frac{4}{\pi^2}1.686\alpha > \frac{\pi^2}{d^2} + \frac{34}{50}\alpha,$$

which is what we want to prove.

For the remaining Case (II-b-2): $0 < a < a_0$, $a < \frac{\pi^2}{4}\delta$ and $a < 1.686\delta$, we define a function z by

$$z(t) = 1 + c\eta(t) + (\delta - \sigma c^2)\xi(t) \qquad \text{on } [-\sin^{-1}\frac{1}{b}, \sin^{-1}\frac{1}{b}],$$

where

$$\sigma = \frac{\tau}{\left([\frac{3}{2} - \frac{\pi^2}{8} - (\frac{\pi^2}{32} - \frac{1}{6})1.686]\frac{1}{0.843} - \frac{(\frac{8}{3\pi} - \frac{\pi}{4})^2}{[-1+(12-\pi^2)\frac{1}{1.686}]}\right)c} \tag{10}$$

and

$$\tau = \frac{2}{3\pi^2}\left(\frac{12-\pi^2}{2(4-\pi)} + \frac{2(4-\pi)}{12-\pi^2} - 2\right) \approx 0.003158975652 \tag{11}$$

As showed in [18], if we define a function Z by

$$Z(t) = \max_{x\in M, t=\sin^{-1}(v(x)/b)} \frac{|\nabla v|^2}{b^2 - v^2}\Big/\lambda_1$$

then Li-Yau's estimate implies that

(12) $$Z(t) \leq 1 + a \qquad \text{on } [-\sin^{-1}(1/b), \sin^{-1}(1/b)].$$

Let $\bar{I} = [-\sin^{-1}\frac{1}{b}, \sin^{-1}\frac{1}{b}]$. We now show that

(13) $$Z(t) \leq z(t) \qquad \text{on } \bar{I}.$$

If (13) is not true, then there exists a constant $P > 0$ and t_0 such that

$$Pc^2 = \frac{Z(t_0) - z(t_0)}{-\xi(t_0)} = \max_{t \in [-\sin^{-1}\frac{1}{b}, \sin^{-1}\frac{1}{b}]} \frac{Z(t) - z(t)}{-\xi(t)}.$$

Let $w(t) = z(t) - Pc^2\xi(t) = 1 + c\eta(t) + m\xi(t)$, where $m = \delta - \sigma c^2 - Pc^2$. Then

$$Z(t) \leq w(t) \quad \text{on } \bar{I} \qquad \text{and} \qquad Z(t_0) = w(t_0).$$

By Lemma 3, $w(t_0) > 0$. So w satisfies (25) in Theorem 4 in [18],

$$0 \leq -2(\sigma + P)c^2 \cos^2 t_0 - \frac{w'(t_0)}{4w(t_0)} \cos t_0 \left(\frac{8c}{\pi} \cos t + 4mt \cos t \right).$$

We used (22), (23), (25) and (26) to get the above inequality. Thus

(14) $$\sigma + P \leq -\frac{w'(t_0)}{2c^2 w(t_0)} \left(\frac{2c}{\pi} + mt \right) = -\frac{\eta'(t_0)}{\pi w(t_0)} \left(1 + \frac{m\xi'(t_0)}{c\eta'(t_0)} \right) \left(1 + \frac{\pi m}{2c} t_0 \right).$$

The righthand side is not positive for $t_0 \geq 0$, by Lemmas 5 and 6. Thus $t_0 < 0$, and

$$\begin{aligned}
&-\left(1 + \frac{m\xi'(t_0)}{c\eta'(t_0)}\right)\left(1 + \frac{\pi m}{2c} t_0\right) \\
&= \frac{2\xi'(t_0)}{\pi t_0 \eta'(t_0)} \left(\frac{\pi t_0 \eta'(t_0)}{2\xi'(t_0)} + \frac{\pi m}{2c} t_0 \right) \left(-1 - \frac{\pi m}{2c} t_0 \right) \\
&\leq \frac{1}{4} \frac{2\xi'(t_0)}{\pi t_0 \eta'(t_0)} \left(\frac{\pi t_0 \eta'(t_0)}{2\xi'(t_0)} - 1 \right)^2 \\
&= \frac{1}{4} \left(\frac{2\xi'(t_0)}{\pi t_0 \eta'(t_0)} + (\frac{2\xi'(t_0)}{\pi t_0 \eta'(t_0)})^{-1} - 2 \right).
\end{aligned}$$

By Lemmas 5 and 6 and 2, we have $2(3 - \frac{\pi^2}{4}) \leq \frac{\xi'(t)}{t} \leq \frac{4}{3}$ and $2(\frac{4}{\pi} - 1) \leq \eta'(t) \leq \frac{8}{3\pi}$.

$$\frac{\pi}{2} = \frac{\xi'(t)}{t\eta'(t)}\Big|_{\pi/2} \leq \frac{\xi'(t)}{t\eta'(t)} \leq \lim_{t \to 0} \frac{\xi'(t)}{t\eta'(t)} = \frac{\xi''(0)}{\eta(0)} = \frac{\pi(12 - \pi^2)}{4(4 - \pi)} \approx 1.94920$$

So

$$1 \leq \frac{2\xi'(t_0)}{\pi t_0 \eta'(t_0)} \leq \frac{12 - \pi^2}{2(4 - \pi)} \approx 1.240900.$$

Note that the function $f(t) = t + \frac{1}{t} - 2$ achieves it maximum on $[A, B]$ not containing 0 at an endpoint. Therefore

$$\left| -\left(1 + \frac{m\xi'(t_0)}{c\eta'(t_0)}\right)\left(1 + \frac{\pi m}{2c} t_0\right) \right| \leq \frac{1}{4} \left(\frac{12 - \pi^2}{2(4 - \pi)} + \frac{2(4 - \pi)}{12 - \pi^2} - 2 \right) \approx 0.01169169.$$

By (14), $|\eta'| \leq 8/(3\pi)$ and the above estimate,

(15) $$\sigma + P \leq \frac{\tau}{w(t_0)}.$$

$$\tau = \frac{2}{3\pi^2} \left(\frac{12 - \pi^2}{2(4 - \pi)} + \frac{2(4 - \pi)}{12 - \pi^2} - 2 \right) \approx 0.003158975652$$

On the other hand, by Lemma 3,

(16)
$$z(t_0) \geq \left([\frac{3}{2} - \frac{\pi^2}{8} - (\frac{\pi^2}{32} - \frac{1}{6})1.686]\frac{1}{0.843} - \frac{(\frac{8}{3\pi} - \frac{\pi}{4})^2}{[-1 + (12 - \pi^2)\frac{1}{1.686}]} \right) c = \frac{\tau}{\sigma} > 0.$$

Since $-P\xi(t_0) \geq 0$, we have $w(t_0) \geq z(t_0)$. This fact, (15) and (16) imply that for $P > 0$

$$\sigma + P < \sigma,$$

which is impossible.

Therefore we have the estimate (13). Now we proceed as in the proof of Theorem 8 in [18]. We get the following

$$\lambda d^2 \geq \frac{\pi^3}{\pi[1 - (\delta - \sigma c^2)]}.$$

Since $\delta - \sigma c^2 > 0.68\delta$ by Lemma 3, we have

$$\lambda \geq \frac{1}{[1 - (\delta - \sigma c^2)]} \frac{\pi^2}{d^2} \geq \frac{1}{[1 - 0.68\delta]} \frac{\pi^2}{d^2}$$

and

$$\lambda \geq \frac{\pi^2}{d^2} + 0.68\alpha > \frac{\pi^2}{d^2} + \frac{34}{50}\alpha.$$

□

3. Analysis results

In [18], the first author studied the function

(17)
$$\xi(t) = \frac{\cos^2 t + 2t \sin t \cos t + t^2 - \frac{\pi^2}{4}}{\cos^2 t} \qquad \text{on} \quad [-\frac{\pi}{2}, \frac{\pi}{2}],$$

and the Zhong-Yang's function

(18)
$$\eta(t) = \frac{\frac{4}{\pi}t + \frac{4}{\pi}\cos t \sin t - 2\sin t}{\cos^2 t} \qquad \text{on} \quad [-\frac{\pi}{2}, \frac{\pi}{2}].$$

The two functions have some properties for eigenvalue estimates. For convenience, a few the properties studied in [18] are record in appendix. We have the following new result for ratio $r(t) = \frac{\xi'(t)}{\eta'(t)}$.

Theorem 2. *Let ξ and η be two functions defined in (17) and in (18) and let*

(19)
$$r(t) = \frac{\xi'(t)}{\eta'(t)}.$$

Then we have

$$r'' < 0 \qquad \text{in } (0, \pi/2).$$

Proof. We let

$$A = \frac{1}{2} p \cos t,$$
$$B = \frac{3}{2} p' \cos t - \frac{5}{2} p \sin t,$$
$$C = p' \sin t + 2p \cos t - 3.$$

Then

$$Ar''' + Br'' + Cr' = -4\cos t$$

and

$$Ar^{(4)} + (A' + B)r''' + (B' + C)r'' = 4\sin t - C'r'.$$

Suppose that at $t_0 \in (0, \pi/2)$ r'' achieves it non positive maximum on $[0, \pi/2]$. Then at t_0 we have

$$r''' = 0, \quad \text{and} \quad r^(4) \le 0.$$

Thus at t_0

$$(B' + C)r'' \ge 4\sin t - C'r' > -C'r'.$$

If $B' + C < 0$ and $C' < 0$, then by Lemma 7 there is a contradiction to the above inequality.

By (25), $p = \eta'$ satisfies the following equations in $(-\frac{\pi}{2}, \frac{\pi}{2})$

$$\frac{1}{2}p'' \cos t - 2p' \sin t - 2p \cos t = -1, \tag{20}$$

and we have

$$\frac{1}{2}p'' \cos t = -1 + 2(p' \sin t + p \cos t) \le -1 + 2\sqrt{p^2 + (p')^2}.$$

Since $|p| \le 8/(3\pi)$ and $|p'| \le 1/2$ we have

$$|\frac{1}{2}p'' \cos t| < 1.$$

Thus by Lemma 6

$$B' + C = \frac{3}{2}p'' \cos t - 3p' \sin t - \frac{1}{2}p \cos t - 3 < 0.$$

By Lemma 4,

$$C' = -\frac{4\sin t}{\cos^5 t}\left(2\cos^3 t - 3\pi \cos^2 t - 24\cos t - 6\frac{t}{\sin t} - 18t \sin t + 12\pi\right).$$

To prove $C' < 0$, we need only prove that

$$f(t) = 2\cos^3 t - 3\pi \cos^2 t - 24\cos t - 6\frac{t}{\sin t} - 18t \sin t + 12\pi > 0$$

Since $f(\pi/2) = 0$ we need only show

$$f'(t) = 6\cos t(-\cos t \sin t + \pi \sin t - \frac{\cos t}{\sin t} + \frac{t}{\sin^2 t} - 3t) = \frac{6\cos t}{\sin^2 t} g(t) < 0,$$

or equivalently

$$g(t) < 0$$

in $(0, \pi/2)$, where

$$g(t) = t - \cos t \sin t - \cos t \sin^3 t + (\pi \sin t - 3t)\sin^2 t.$$

Now

$$g'(t) = h(t) \cos t \sin t,$$

where

$$h(t) = -4\cos t \sin t + 3\pi \sin t - 6t$$

The function

$$h'(t) = -8\cos^2 t + 3\pi \cos t - 2$$

has at most two zeros on $[0, \pi/2]$. Therefore the function h has at most three zeros on $[0, \pi/2]$. Since 0 and $\pi/2$ are two zeros of h, h has at most one zero in

$(0, \pi/2)$. Since $h'(0) = 3\pi - 10 < 0$ and $h'(\pi/2) = -2 < 0$, h has the unique zero $t_1 \in (0, \pi/2)$. Therefore

$$g'(t) < 0 \text{ in } (0, t_1) \quad \text{and} \quad g'(t) > 0 \text{ in } (t_1, \pi/2)$$

Since $g(0) = 0 = g(\pi/2)$, we have $g(t) < 0$ in $(0, \pi/2)$. □

Lemma 3. *If $a < 1.686\delta$ and $0 < a < 0.843$ then*

$$\begin{aligned} z(t) &= 1 + c\eta(t) + \delta\xi(t) \\ &\geq \left([\frac{3}{2} - \frac{\pi^2}{8} - (\frac{\pi^2}{32} - \frac{1}{6})1.686]\frac{1}{0.843} - \frac{(\frac{8}{3\pi} - \frac{\pi}{4})^2}{[-1 + (12 - \pi^2)\frac{1}{1.686}]} \right) c > 0, \end{aligned}$$

for $t \in [-\pi/2, \pi/2]$ and

$$\delta - \sigma c^2 \approx 0.68880\delta > 0.68\delta,$$

where $c = a/b$ and $b > 1$ is any constant and σ is the constant in (10).

Proof. By Lemmas 7, Lemma 5 and 6, the function z on $[-\pi/2, \pi/2]$ has a unique critical point $t_1 \in (-\pi/2, 0)$ if $0 < a < \frac{\pi^2}{4}\delta$ and z is decreasing on $[-\pi/2, t_1]$ and increasing on $[t_1, \pi/2]$. Therefore

$$\min_{[-\pi/2, \pi/2]} z = \min_{[-\pi/2, 0]} z = z(t_1).$$

So we need only consider the restricted function $z|_{[-\pi/2,0]}$ for the minimum.

Now first consider the Taylor expansion of ξ at 0 for $t \in [-\pi/2, 0]$. By Lemma 5, $\xi(0) = -\frac{\pi^2}{4} + 1$, $\xi'(0) = 0$ and $\xi''(0) = 2(3 - \frac{\pi^2}{4})$ and $\xi'''(t) < 0$ on $(-\pi/2, 0)$.

Thus

$$\begin{aligned} \xi(t) &= \xi(0) + \xi'(0) + \frac{\xi''(0)}{2!}t^2 + \frac{\xi'''(t_2)}{2!}t^3 \\ &\geq \xi(0) + \xi'(0) + \frac{\xi''(0)}{2!}t^2 \\ &= -(\frac{\pi^2}{4} - 1) + (3 - \frac{\pi^2}{4})t^2, \end{aligned}$$

where t_2 is a constant in $(t, 0)$. Similarly, using the data $\eta(-\pi/2) = -1$, $\eta'(-\pi/2) = \frac{8}{3\pi}$ and $\eta'''(t) > 0$ on $(-\pi/2, 0)$ (actually on $[-\pi/2, \pi/2]$), and the Taylor expansion of η at $-\pi/2$, we have for $t \in [-\pi/2, 0]$,

$$\begin{aligned} \eta(t) &= \eta(-\frac{\pi}{2}) + \eta'(-\frac{\pi}{2})(t + \frac{\pi}{2}) + \frac{\eta''(-\frac{\pi}{2})}{2!}(t + \frac{\pi}{2})^2 + \frac{\eta''(t_3)}{3!}(t + \frac{\pi}{2})^3 \\ &\geq \eta(-\frac{\pi}{2}) + \eta'(-\frac{\pi}{2})(t + \frac{\pi}{2}) + \frac{\eta''(-\frac{\pi}{2})}{2!}(t + \frac{\pi}{2})^2 \\ &= -1 + \frac{8}{3\pi}(t + \frac{\pi}{2}) - \frac{1}{4}(t + \frac{\pi}{2})^2 \\ &= -(\frac{\pi^2}{16} - \frac{1}{3}) + (\frac{8}{3\pi} - \frac{\pi}{4})t - \frac{1}{4}t^2, \end{aligned}$$

where t_3 is some constant in $(-\pi/2, t)$. Therefore on $[-\pi/2, 0]$,

$$\begin{aligned} z(t) &= 1 + c\eta(t) + \delta\xi(t) \\ &\geq 1 - (\frac{\pi^2}{16} - \frac{1}{3})c - (\frac{\pi^2}{4} - 1)\delta + (\frac{8}{3\pi} - \frac{\pi}{4})ct + [-\frac{1}{4}c + (3 - \frac{\pi^2}{4})\delta]t^2 \end{aligned}$$

Let $\nu = 1.686$ and $a_0 = 0.843$. That $a \le \nu\delta$ implies $c = a/b < \nu\delta$, where $b > 1$ is a constant. Using conditions (8) $\delta \le \frac{n-1}{2n} < \frac{1}{2}$ and $a \le a_0$, we get

$$\begin{aligned}
&1 - (\frac{\pi^2}{16} - \frac{1}{3})c - (\frac{\pi^2}{4} - 1)\delta \\
\ge\ & 1 - (\frac{\pi^2}{16} - \frac{1}{3})\nu\delta - (\frac{\pi^2}{4} - 1)\delta \\
\ge\ & \frac{3}{2} - \frac{\pi^2}{8} - (\frac{\pi^2}{32} - \frac{1}{6})\nu \\
>\ & \left(\frac{3}{2} - \frac{\pi^2}{8} - (\frac{\pi^2}{32} - \frac{1}{6})\nu\right)\frac{1}{a_0}c
\end{aligned}$$

and

$$\begin{aligned}
&1 + c\eta(t) + \delta\xi(t) \\
\ge\ & \left(\frac{3}{2} - \frac{\pi^2}{8} - (\frac{\pi^2}{32} - \frac{1}{6})\nu\right)\frac{1}{a_0}c + (\frac{8}{3\pi} - \frac{\pi}{4})ct + [-\frac{1}{4}c + (3 - \frac{\pi^2}{4})\frac{1}{\nu}c]t^2 \\
=\ & \left([\frac{3}{2} - \frac{\pi^2}{8} - (\frac{\pi^2}{32} - \frac{1}{6})\nu]\frac{1}{a_0} + (\frac{8}{3\pi} - \frac{\pi}{4})t + [-\frac{1}{4} + (3 - \frac{\pi^2}{4})\frac{1}{\nu}]t^2\right)c \\
\ge\ & \left([\frac{3}{2} - \frac{\pi^2}{8} - (\frac{\pi^2}{32} - \frac{1}{6})\nu]\frac{1}{a_0} - \frac{(\frac{8}{3\pi} - \frac{\pi}{4})^2}{4[-\frac{1}{4} + (3 - \frac{\pi^2}{4})\frac{1}{\nu}]}\right)c \\
\ge\ & \left([\frac{3}{2} - \frac{\pi^2}{8} - (\frac{\pi^2}{32} - \frac{1}{6})\nu]\frac{1}{a_0} - \frac{(\frac{8}{3\pi} - \frac{\pi}{4})^2}{[-1 + (12 - \pi^2)\frac{1}{\nu}]}\right)c > 0.
\end{aligned}$$

Let τ be the constant in (11). Then

$$\begin{aligned}
\sigma c^2 &= \frac{\tau c}{\left([\frac{3}{2} - \frac{\pi^2}{8} - (\frac{\pi^2}{32} - \frac{1}{6})\nu]\frac{1}{a_0} - \frac{(\frac{8}{3\pi} - \frac{\pi}{4})^2}{[-1 + (12 - \pi^2)\frac{1}{\nu}]}\right)}, \\
&\le \frac{\tau\nu\delta}{\left([\frac{3}{2} - \frac{\pi^2}{8} - (\frac{\pi^2}{32} - \frac{1}{6})\nu]\frac{1}{a_0} - \frac{(\frac{8}{3\pi} - \frac{\pi}{4})^2}{[-1 + (12 - \pi^2)\frac{1}{\nu}]}\right)}
\end{aligned}$$

and

$$\delta - \sigma c^2 > 0.68\delta.$$

□

4. Appendix

We now give a lemma to verify the expression for C', which is used in the proof of Theorem 2.

Lemma 4 (Manual Computation)**.**

$$C' = -\frac{4\sin t}{\cos^5 t}\left(2\cos^3 t - 3\pi\cos^2 t - 24\cos t - 6\frac{t}{\sin t} - 18t\sin t + 12\pi\right)$$

Proof of Lemma 4.

$$C = p' \sin t + 2p \cos t - 3.$$
$$C' = (p'' - 2p) \sin t + p'(3 \cos t)$$
$$p(t) = \frac{2(\frac{4}{\pi} \cos t + \frac{4}{\pi} t \sin t - \sin^2 t - 1)}{\cos^3 t}.$$
$$U = \frac{4}{\pi} \cos t + \frac{4}{\pi} t \sin t - \sin^2 t - 1$$
$$V = \frac{1}{\cos^3 t}$$
$$p/2 = UV = (\frac{4}{\pi} \cos t + \frac{4}{\pi} t \sin t - \sin^2 t - 1)\frac{1}{\cos^3 t}$$
$$p'/2 = U'V + UV'$$
$$p''/2 = U''V + 2U'V' + UV''$$

$$U' = \frac{4}{\pi} t \cos t - 2 \sin t \cos t = (\frac{2}{\pi} t - \sin t) 2 \cos t$$
$$U'' = \frac{4}{\pi} \cos t - \frac{4}{\pi} t \sin t - 2 \cos^2 t + 2 \sin^2 t$$
$$V' = 3 \sin t (\frac{1}{\cos^4 t})$$
$$V'' = 3(\frac{1}{\cos^3 t}) + 12 \sin^2 t (\frac{1}{\cos^5 t})$$

$$C'/2 = (p''/2 - p) \sin t + (p'/2)(3 \cos t)$$
$$= (U''V + 2U'V' + UV'' - 2UV) \sin t + (U'V + UV')(3 \cos t)$$

$$U''V + 2U'V' + UV'' - 2UV$$

$$= (\frac{4}{\pi} \cos t - \frac{4}{\pi} t \sin t - 2 \cos^2 t + 2 \sin^2 t)\frac{1}{\cos^3 t}$$
$$+2(\frac{2}{\pi} t - \sin t) 2 \cos t 3 \sin t (\frac{1}{\cos^4 t})$$
$$U[3(\frac{1}{\cos^3 t}) + 12 \sin^2 t (\frac{1}{\cos^5 t})]$$
$$-2U \frac{1}{\cos^3 t}$$
$$= (\frac{4}{\pi} \cos t - \frac{4}{\pi} t \sin t - 2 \cos^2 t + 2 \sin^2 t + U)\frac{1}{\cos^3 t}$$
$$+\frac{24}{\pi} t \sin t (\frac{1}{\cos^3 t}) - 12 \sin^2 t (\frac{1}{\cos^3 t})$$
$$+12U \sin^2 t (\frac{1}{\cos^5 t})$$

$$= (\frac{4}{\pi} \cos t - \frac{4}{\pi} t \sin t - 2 \cos^2 t + 2 \sin^2 t + U + \frac{24}{\pi} t \sin t - 12 \sin^2 t)\frac{1}{\cos^3 t}$$
$$+12U \sin^2 t (\frac{1}{\cos^5 t})$$

$$= (\frac{4}{\pi}\cos t + \frac{20}{\pi}t\sin t + 8\cos^2 t - 10 + U)\frac{1}{\cos^3 t}$$
$$+12U\sin^2 t(\frac{1}{\cos^5 t})$$

$$U'V + UV'$$
$$= (\frac{2}{\pi}t - \sin t)2\cos t\frac{1}{\cos^3 t} + U3\sin t(\frac{1}{\cos^4 t})$$
$$= (\frac{4}{\pi}t - 2\sin t)\frac{1}{\cos^2 t} + U3\sin t(\frac{1}{\cos^4 t})$$

$$C'/2 = (p''/2 - p)\sin t + (p'/2)(3\cos t)$$
$$= (U''V + 2U'V' + UV'' - 2UV)\sin t + (U'V + UV')(3\cos t)$$

$$= (\frac{4}{\pi}\cos t + \frac{20}{\pi}t\sin t + 8\cos^2 t - 10 + U)\frac{\sin t}{\cos^3 t}$$
$$+12U\sin^3 t(\frac{1}{\cos^5 t}) + (\frac{12}{\pi}t - 6\sin t)\frac{1}{\cos t} + 9U(\frac{\sin t}{\cos^3 t})$$
$$= (\frac{4}{\pi}\cos t + \frac{20}{\pi}t\sin t + 8\cos^2 t - 10 + 10U)\frac{\sin t}{\cos^3 t}$$
$$+(\frac{12}{\pi}t - 6\sin t)\frac{1}{\cos t} + 12U\sin^3 t(\frac{1}{\cos^5 t})$$
$$= (\frac{4}{\pi}\cos t + \frac{20}{\pi}t\sin t + 8\cos^2 t - 10 + 10U)\frac{\sin t}{\cos^3 t}$$
$$+(\frac{12}{\pi}t - 6\sin t)\frac{1}{\cos t} + 12U\frac{\sin t}{\cos^5 t} - 12U\frac{\sin t}{\cos^3 t}$$
$$= (\frac{4}{\pi}\cos t + \frac{20}{\pi}t\sin t + 8\cos^2 t - 10 - 2U)\frac{\sin t}{\cos^3 t}$$
$$+(\frac{12}{\pi}t - 6\sin t)\frac{1}{\cos t} + 12U\frac{\sin t}{\cos^5 t}$$

$$= (\frac{4}{\pi}\cos t + \frac{20}{\pi}t\sin t + 8\cos^2 t - 10 - \frac{8}{\pi}\cos t - \frac{8}{\pi}t\sin t + 2\sin^2 t + 2)\frac{\sin t}{\cos^3 t}$$
$$+(\frac{12}{\pi}t - 6\sin t)\frac{1}{\cos t} + 12(\frac{4}{\pi}\cos t + \frac{4}{\pi}t\sin t - \sin^2 t - 1)\frac{\sin t}{\cos^5 t}$$
$$= (-\frac{4}{\pi}\cos t + \frac{12}{\pi}t\sin t + 6\cos^2 t - 6)\frac{\sin t}{\cos^3 t}$$
$$+(\frac{12}{\pi}t - 6\sin t)\frac{1}{\cos t} + 12(\frac{4}{\pi}\cos t + \frac{4}{\pi}t\sin t - \sin^2 t - 1)\frac{\sin t}{\cos^5 t}$$

$$= (-\frac{4}{\pi}\cos t - 6)\frac{\sin t}{\cos^3 t} + \frac{12}{\pi}\frac{t}{\cos^3 t} + (\frac{4}{\pi}\cos t + \frac{4}{\pi}t\sin t - \sin^2 t - 1)\frac{12\sin t}{\cos^5 t}$$

$$= \left(-\frac{2}{\pi}\cos^3 t - 3\cos^2 t + \frac{6t\cos^2 t}{\pi\sin t} + \frac{24}{\pi}\cos t + \frac{24}{\pi}t\sin t - 6\sin^2 t - 6\right)\frac{2\sin t}{\cos^5 t}$$

$$= \left(-2\cos^3 t + 3\pi\cos^2 t + \frac{6t\cos^2 t}{\sin t} + 24\cos t + 24t\sin t - 12\pi\right)\frac{2\sin t}{\cos^5 t}$$

$$= \frac{-2\sin t}{\cos^5 t}\left(2\cos^3 t - 3\pi\cos^2 t - \frac{6t\cos^2 t}{\sin t} - 24\cos t - 24t\sin t + 12\pi\right)$$

$$= \frac{-2\sin t}{\cos^5 t}\left(2\cos^3 t - 3\pi\cos^2 t - 24\cos t - 6\frac{t}{\sin t} - 18t\sin t + 12\pi\right)$$

□

The following are some Theorems and Lemmas in [18]. We enclose them here for convenience.

Theorem A.[Theorem 8 in [18]] *If $a > 0$ and $\mu\delta \leq \frac{4}{\pi^2}a$ for a constant $\mu \in (0, 1]$, then*

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \mu\alpha.$$

Theorem B.[Theorem 9 in [18]] *If the "midrange" $a = 0$, then*

$$\lambda_1 \geq \frac{\pi^2}{d^2} + \alpha. \tag{21}$$

Lemma 5 (Lemma 13 in [18])**.** *The function ξ satisfies the following*

$$\frac{1}{2}\xi'' \cos^2 t - \xi' \cos t \sin t - \xi = 2\cos^2 t \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}), \tag{22}$$

$$\xi' \cos t - 2\xi \sin t = 4t\cos t \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}), \tag{23}$$

$$\int_0^{\frac{\pi}{2}} \xi(t)\, dt = -\frac{\pi}{2}, \tag{24}$$

$$1 - \frac{\pi^2}{4} = \xi(0) \leq \xi(t) \leq \xi(\pm\frac{\pi}{2}) = 0 \quad \text{on } [-\frac{\pi}{2}, \frac{\pi}{2}],$$

$$\xi' \text{ is increasing on } [-\frac{\pi}{2}, \frac{\pi}{2}] \text{ and } \xi'(\pm\frac{\pi}{2}) = \pm\frac{2\pi}{3},$$

$$\xi'(t) < 0 \text{ on } (-\frac{\pi}{2}, 0) \text{ and } \xi'(t) > 0 \text{ on } (0, \frac{\pi}{2}),$$

$$\xi''(\pm\frac{\pi}{2}) = 2,\ \xi''(0) = 2(3 - \frac{\pi^2}{4}) \text{ and } \xi''(t) > 0 \text{ on } [-\frac{\pi}{2}, \frac{\pi}{2}],$$

$$(\frac{\xi'(t)}{t})' > 0 \text{ on } (0, \pi/2) \text{ and } 2(3 - \frac{\pi^2}{4}) \leq \frac{\xi'(t)}{t} \leq \frac{4}{3} \text{ on } [-\frac{\pi}{2}, \frac{\pi}{2}],$$

$$\xi'''(\frac{\pi}{2}) = \frac{8\pi}{15}, \xi'''(t) < 0 \text{ on } (-\frac{\pi}{2}, 0) \text{ and } \xi'''(t) > 0 \text{ on } (0, \frac{\pi}{2}).$$

Lemma 6 (Lemma 14 in [18])**.** *The function η satisfies the following*

$$\frac{1}{2}\eta'' \cos^2 t - \eta' \cos t \sin t - \eta = -\sin t \qquad \text{in} \ \ (-\frac{\pi}{2}, \frac{\pi}{2}), \tag{25}$$

$$\eta' \cos t - 2\eta \sin t = \frac{8}{\pi} \cos t - 2 \qquad \text{in} \ \ (-\frac{\pi}{2}, \frac{\pi}{2}), \tag{26}$$

$$-1 = \eta(-\frac{\pi}{2}) \le \eta(t) \le \eta(\frac{\pi}{2}) = 1 \qquad \text{on} \ \ [-\frac{\pi}{2}, \frac{\pi}{2}],$$

$$0 < 2(\frac{4}{\pi} - 1) = \eta'(0) \le \eta'(t) \le \eta'(\pm\frac{\pi}{2}) = \frac{8}{3\pi} \qquad \text{on} \ \ [-\frac{\pi}{2}, \frac{\pi}{2}],$$

$$-1/2 = \eta''(-\frac{\pi}{2}) \le \eta''(t) \le \eta''(\frac{\pi}{2}) = 1/2 \qquad \text{on} \ \ [-\frac{\pi}{2}, \frac{\pi}{2}],$$

$$\eta'''(t) > 0 \quad \text{on} \ \ [-\frac{\pi}{2}, \frac{\pi}{2}] \quad \text{and} \quad \eta'''(\pm\frac{\pi}{2}) = \frac{32}{15\pi}.$$

Lemma 7. *The function $r(t) = \xi'(t)/\eta'(t)$ is an increasing function on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, i.e., $r'(t) > 0$, and $|r(t)| \le \frac{\pi^2}{4}$ holds on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.*

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