

# ON THE GEOMETRY OF CLASSIFYING SPACES AND HORIZONTAL SLICES

By ZHIQIN LU

---

*Abstract.* We study the local properties of the moduli space of a polarized Calabi-Yau manifold. Let  $U$  be a neighborhood of the moduli space. Then we know the universal covering space  $V$  of  $U$  is a smooth manifold. Suppose  $D$  is the classifying space of a polarized Calabi-Yau manifold with the automorphism group  $G$ . Then we prove that the map from  $V$  to  $D$  induced by the period map is a pluriharmonic map. We also give a Kähler metric on  $V$ , which is called the Hodge metric. We prove that the Ricci curvature of the Hodge metric is negative away from zero. We also proved the nonexistence of the Kähler metric on the classifying space of a Calabi-Yau threefold which is invariant under a cocompact lattice of  $G$ .

**1. Introduction.** Let  $(X, \omega)$  be a polarized simply connected Calabi-Yau manifold. That is,  $X$  is a simply connected compact Kähler manifold of dimension  $n$  with zero first Chern class and  $\omega$  is a Kähler form of  $X$  such that  $[\omega] \in H^2(X, \mathbb{Z})$ . In this paper, we study the local properties of the moduli space  $\mathcal{M}$  of the polarized Calabi-Yau manifold  $(X, \omega)$ . By definition  $\mathcal{M}$  is the parameter space of the complex structures over  $X$  for the fixed polarization  $[\omega]$ ;  $\mathcal{M}$  is a quasi-projective variety by a theorem of Viehweg [17].

Suppose  $X \in \mathcal{M}$  is a Calabi-Yau manifold. Let

$$N = \dim\{\eta \in H^1(X, T_X) \mid \eta \lrcorner \omega = 0\}$$

where  $T_X$  is the holomorphic tangent bundle of  $X$ . By a theorem of Tian [16], we know that the polarized universal deformation space of the Calabi-Yau manifold is smooth and has dimension  $N$ . Since for each  $X' \in \mathcal{M}$  there are no nonzero holomorphic tangent vectors on  $X'$ , we conclude that in each neighborhood  $\tilde{U}$  of  $X'$ , there is an open neighborhood  $U$  in  $\mathbb{C}^N$  such that  $U$  is a finite covering of  $\tilde{U}$ . Thus the moduli space is a complex orbifold. A good reference of the theorem of Tian can be found in [5].

By the above observation, we need to study only the local properties of  $U$ ; since  $U$  is only a finite covering of  $\tilde{U}$ , it will be easy to pass the properties of  $U$  to  $\tilde{U}$ . Let  $D$  be the classifying space corresponding to the Calabi-Yau manifold  $X$ . The period map  $U \rightarrow D$  is a holomorphic immersion. Furthermore, the image of  $U$  under the period map is a horizontal slice of  $D$ . That is,  $U$  is a complex integral

submanifold of the horizontal distribution of  $D$ . For the precise definitions of the classifying space and the horizontal slices, see Sections 2–3.

Theorem 1.1 and 1.3 are given below for the horizontal slices. Thus our results are more general than merely on moduli spaces. However, few examples of horizontal slices can be found which are not moduli spaces because of the nonintegrability of the horizontal distributions.

Let  $D = G/V$  where  $G$  is a semi-simple Lie group of noncompact type without compact factors, and  $V$  be a connected compact subgroup. Let  $K$  be the maximum connected compact subgroup containing  $V$ . Then  $D_1 = G/K$  is a symmetric space.

We prove the following:

**THEOREM 1.1.** *Suppose  $U \subset D$  is a horizontal slice. Then*

$$p: U \rightarrow D_1$$

*defined by the natural projection  $D \rightarrow D_1$  is again an immersion. Furthermore, it is a pluriharmonic map, i.e., it restricts to a harmonic map from any holomorphic curve of  $U$ . Or in other words, it satisfies*

$$\nabla_{p_*X}p_*X + \nabla_{p_*JX}p_*JX + p_*J[X, JX] = 0$$

*for any vector field  $X \in T(U)$ , where  $J$  is the complex structure on  $U$  and  $\nabla$  is the Riemannian connection on  $G/K = D_1$ .*

Theorem 1.1 is of interest on its own. But the most important application of it is the following:

**THEOREM 1.2.** *The restriction of the invariant Riemannian metric on  $D_1$  to  $U$  is Kähler. Moreover, the holomorphic bisectional curvature of such a metric is nonpositive. Furthermore, the Ricci curvature and the holomorphic sectional curvature are negative away from zero.*

In the last section of this paper, we prove the nonexistence of invariant Kähler metrics on some classifying spaces.

Suppose  $\Gamma$  is a lattice of the group  $G$ . We say that  $\Gamma$  is cocompact, if  $\Gamma \backslash G$  is a compact topological space.

**THEOREM 1.3.** *Let  $D$  be the classifying space of some Calabi-Yau threefold. Then there are no Kähler metrics on  $D$  which are  $\Gamma$  invariant if  $\Gamma$  is cocompact.*

The paper is arranged as follows: In Sections 2–3 we set up the definitions and notations and prove some basic properties of horizontal slices, classifying space and the period map. Theorem 1.1 is proved in Section 4. In Section 5, we prove that the Hodge metric is Kähler and the Ricci curvature is negative away from zero, which is Theorem 1.2. In the last section, we prove Theorem 1.3 of the nonexistence of the Kähler metric with some kind of invariance.

To prove Theorem 1.1 and Theorem 1.3 we consider the Kähler form  $\omega$  on the classifying space  $D$ . It is known that if the projection  $D \rightarrow D_1$  is holomorphic, then  $D$  is Kählerian and  $\omega$  is the Kähler form. However,  $D \rightarrow D_1$  is in general not holomorphic. Nevertheless, there is a relation between  $\omega$  and the pull back of the invariant metric on  $D_1$ . From this we conclude that although  $d\omega \neq 0$  as a differential form, in some directions,  $d\omega$  is indeed zero. In particular, we use the fact that if  $X, Y, Z$  are horizontal, then  $d\omega(X, Y, Z) = 0$  in Theorem 1.1, and we use the fact that if  $X, Y$  are vertical and  $Z$  is horizontal, then  $d\omega(X, Y, Z) = 0$  in Theorem 1.3.

A result similar to Theorem 1.1 of the pluriharmonicity of the projection was studied in Bryant [2], Burstall and Salamon [3], and Black [1]. Those papers only considered the compact cases and cannot apply to the cases we examine in this paper.

*Acknowledgments.* This paper is a refinement of a part of the author's Ph.D. thesis. The author would like to thank his advisor, Professor G. Tian, for his advice and constant encouragement during four years of study. He also thanks Professor S. T. Yau for his constant encouragement and many important ideas stimulating further study of this problem, and Professor J. Jost for the paper [10] and for pointing out some mistakes in the references.

**2. The Hodge structure and the classifying space.** Let  $X$  be a compact Kähler manifold of dimension  $n$ . A  $C^\infty$  form on  $X$  decomposes into  $(p, q)$ -components according to the number of  $dz^i$ 's and  $d\bar{z}^j$ 's. Denoting the  $C^\infty$   $n$ -forms and the  $C^\infty(p, q)$  forms on  $X$  by  $A^n(X)$  and  $A^{p,q}(X)$  respectively, we have the decomposition

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X).$$

The cohomology group is defined as

$$\begin{aligned} H^{p,q}(X) &= \{\text{closed}(p, q) - \text{forms}\} / \{\text{exact}(p, q) - \text{forms}\} \\ &= \{\phi \in A^{p,q}(X) \mid d\phi = 0\} / dA^{p-1,q}(X) \cap A^{p,q}(X). \end{aligned}$$

We have:

**THEOREM 2.1.** (Hodge decomposition theorem) *Let  $X$  be a compact Kähler manifold of dimension  $n$ . Then the  $n$ -th complex de Rham cohomology group of  $X$  can be written as the direct sum*

$$H^n(X, \mathbb{C}) \otimes \mathbb{C} = H_{DR}^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X).$$

Suppose  $\omega$  is the Kähler form of  $X$ . Define

$$L: H^k(X, \mathbb{C}) \rightarrow H^{k+2}(X, \mathbb{C}), \quad k = 0, \dots, 2n - 2$$

to be the multiplication by  $\omega$ .

Suppose  $X$  is a smooth algebraic variety. Then  $\omega$  is called a polarization of  $X$  if  $[\omega]$  is the first Chern class of an ample line bundle over  $X$ . In that case,  $(X, \omega)$  is called the polarized algebraic variety.

The following two famous Lefschetz theorems are extremely important in defining the classifying space and the period map.

**THEOREM 2.2.** (Hard Lefschetz theorem) *On a polarized algebraic variety  $(X, \omega)$  of dimension  $n$ ,*

$$L^k: H^{n-k}(X, \mathbb{C}) \rightarrow H^{n+k}(X, \mathbb{C})$$

*is an isomorphism for every positive integer  $k \leq n$ .*

The primitive cohomology  $P^k(X, \mathbb{C})$  is then defined to be the kernel of  $L^{n-k+1}$  on  $H^k(X, \mathbb{C})$ .

**THEOREM 2.3.** (Lefschetz Decomposition Theorem) *On a polarized algebraic variety  $(X, \omega)$ , we have the following decomposition:*

$$H^n(X, \mathbb{C}) = \bigoplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} L^k P^{n-2k}(X, \mathbb{C}).$$

Define  $H_Z = P^n(X, \mathbb{C}) \cap H^n(X, \mathbb{Z})$  and  $H^{p,q} = P^n(X, \mathbb{C}) \cap H^{p,q}(X)$ . Then we have

$$H_Z \otimes \mathbb{C} = \sum H^{p,q}, \quad H^{p,q} = \overline{H^{q,p}}$$

for  $p + q = n$ . Set  $H = H_Z \otimes \mathbb{C}$ . We call  $\{H^{p,q}\}$  the Hodge decomposition of  $H$ .

*Remark 2.1.* We define a filtration of  $H_Z \otimes \mathbb{C} = H$  by

$$0 \subset F^n \subset F^{n-1} \subset \dots \subset F^1 = H$$

such that

$$H^{p,q} = F^p \cap \overline{F^q}, \quad F^p \oplus \overline{F^{n-p+1}} = H.$$

The sets  $\{H^{p,q}\}$  and  $\{F^p\}$  are equivalent to describe the Hodge decomposition of  $H$ . In the remainder of this paper, we will use both notations interchangeably.

Now suppose that  $Q$  is the quadratic form on  $H_Z$  induced by the cup product of the cohomology group.  $Q$  can be represented by

$$Q(\phi, \psi) = (-1)^{n(n-1)/2} \int \phi \wedge \psi$$

for  $\phi, \psi \in H$ .  $Q$  is a nondegenerated form, and is skewsymmetric if  $n$  is odd and is symmetric if  $n$  is even. It satisfies the two Hodge-Riemannian relations:

- (1)  $Q(H^{p,q}, H^{p',q'}) = 0$  unless  $p' = n - p, q' = n - q$ ;
- (2)  $(\sqrt{-1})^{p-q} Q(\phi, \bar{\phi}) > 0$  for any nonzero element  $\phi \in H^{p,q}$ .

*Definition 2.1.* A polarized Hodge structure of weight  $n$ , denoted by  $\{H_Z, F^p, Q\}$ , is given by a filtration of  $H = H_Z \otimes \mathbb{C}$

$$0 \subset F^n \subset F^{n-1} \subset \dots \subset F^0 \subset H$$

such that

$$H = F^p \oplus \bar{F}^{n-p+1}$$

together with a bilinear form

$$Q: H_Z \otimes H_Z \rightarrow \mathbb{Z},$$

which is skewsymmetric if  $n$  is odd and symmetric if  $n$  is even such that it satisfies the two Hodge-Riemannian relations:

- (3)  $Q(F^p, F^{n-p+1}) = 0$  for  $p = 1, \dots, n$ ;
- (4)  $(\sqrt{-1})^{p-q} Q(\phi, \bar{\phi}) > 0$  if  $\phi \in H^{p,q}$  and  $\phi \neq 0$ ,

where  $H^{p,q}$  is defined by

$$H^{p,q} = F^p \cap \bar{F}^q$$

for  $p + q = n$ .

*Definition 2.2.* The classifying space  $D$  for the polarized Hodge structure is the set of all the filtrations

$$0 \subset F^n \subset \dots \subset F^1 \subset H, \quad F^p \oplus \overline{F^{n-p+1}} = H$$

or the set of all the decompositions

$$\sum H^{p,q} = H, \quad H^{p,q} = \overline{H^{q,p}}$$

on which  $Q$  satisfies the two Hodge-Riemannian relations (1), (2), or (3), (4) above.

Over the classifying space  $D$  we have the holomorphic vector bundles  $\underline{F}^n, \dots, \underline{F}^1, \underline{H}$  whose fibers at each point are the vector spaces  $F^n, \dots, F^1, H$ , respectively. These bundles are called Hodge bundles.

It is well known that the holomorphic tangent bundle  $T(D)$  can be realized as a subbundle of  $\text{Hom}(\underline{H}, \underline{H})$ :

$$T(D) \subset \bigoplus \text{Hom}(\underline{F}^p, \underline{H}/\underline{F}^p) = \bigoplus_{r>0} \text{Hom}(\underline{H}^{p,q}, \underline{H}^{p-r,q+r})$$

such that the following compatible condition holds

$$\begin{array}{ccc} F^p & \longrightarrow & F^{p-1} \\ \downarrow & & \downarrow \\ H/F^p & \longleftarrow & H/F^{p-1}. \end{array}$$

*Definition 2.3.* A subbundle  $T_h(D)$  is called the horizontal bundle of  $D$ , if

$$T_h(D) = \{\xi \in T(D) \mid \xi F^p \subset F^{p-1}, p = 1, \dots, n\}.$$

For any point  $p \in D$  such that  $p$  is defined as subspaces  $\{H^{p,q}\}$  of  $H$ , define the two vector spaces

$$\begin{aligned} H^+ &= H^{n,0} + H^{n-2,2} + \dots \\ H^- &= H^{n-1,1} + H^{n-3,3} + \dots \end{aligned}$$

Now we fix a point  $p_0 \in D$ . Suppose the corresponding vector spaces are  $\{H_0^{p,q}\}$  and  $\{H_0^+, H_0^-\}$ . We define  $V$  to be the connected compact subgroup of  $G$  leaving  $\{H_0^{p,q}\}$  unchanged for any  $p, q$  with  $p+q = n$ , and  $K$  to be the connected compact subgroup of  $G$  leaving  $H_0^+$  invariant. We give the basic properties of the classifying spaces in the following three lemmas. The proofs are easy and are omitted.

**LEMMA 2.1.**  *$K$  is the maximal compact subgroup of  $G$  containing  $V$ . Thus  $V$  itself is a compact subgroup.*

We define the Weil operator

$$C: H^{p,q} \rightarrow H^{p,q}, \quad C|_{H^{p,q}} = (\sqrt{-1})^{p-q}.$$

Then we have

$$C|_{H^+} = (\sqrt{-1})^n, \quad C|_{H^-} = -(\sqrt{-1})^n.$$

Let

$$Q_1(x, y) = Q(Cx, \bar{y}).$$

Then we have:

LEMMA 2.2.  $Q_1$  is an Hermitian inner product.

LEMMA 2.3. Let

$$D_1 = \{H^{n,0} + H^{n-2,2} + \dots \mid \{H^{p,q}\} \in D\}.$$

Then the group  $G$  acts on  $D_1$  transitively with the stable subgroup  $K$  at  $H_0^+$ , and  $D_1$  is a symmetric space.

Definition 2.4. We call map  $p$

$$p: G/V \rightarrow G/K, \quad \{H^{p,q}\} \mapsto H^{n,0} + H^{n-2,2} + \dots$$

the natural projection of the classifying space. It is the same as the map  $aV \mapsto aK$  between the coset spaces of  $V$  and  $K$ .

With the above discussions, we can prove:

PROPOSITION 2.1. Suppose  $T_v(D)$  is the distribution of the tangent vectors of the fibers of the canonical map

$$p: D \rightarrow G/K;$$

then

$$T_v(D) \cap T_h(D) = \{0\}.$$

*Proof.* Let  $\mathfrak{g}$  be the Lie algebra of the Lie group  $G$ . Let  $\mathfrak{g} = \mathfrak{f} + \mathfrak{p}$  be the Cartan decomposition such that  $\mathfrak{f}$  is the Lie algebra of  $K$ . Then

$$T_v(D) = G \times_V \mathfrak{v}_1$$

where  $\mathfrak{f} = \mathfrak{v} + \mathfrak{v}_1$  and  $\mathfrak{v}_1$  is the orthonormal complement of the Lie algebra  $\mathfrak{v}$  of  $V$ . On the other hand,  $T_h(D) \subset G \times_V \mathfrak{p}$ . So we have  $T_v(D) \cap T_h(D) = \{0\}$ .  $\square$

Definition 2.5. A horizontal slice  $\mathcal{M}$  of  $D$  is a complex integral submanifold of the distribution  $T_h(D)$ .

Definition 2.6. Let  $U$  be an open neighborhood of the universal deformation space at a Calabi-Yau manifold  $X$ . Then for each  $X'$  near  $X$ , we have an isomorphism  $H^n(X', C) = H^n(X, C)$ . Under this isomorphism,  $\{H^{p,q}(X') \cap P^n(X', C)\}_{p+q=n}$

can be considered as a point of  $D$ . The map

$$U \rightarrow D \quad X' \mapsto \{H^{p,q}(X') \cap P^n(X', C)\}_{p+q=n}$$

is called the period map.

The most important property of the period map is the following [6]:

**THEOREM 2.4.** (Griffiths) *The period map  $p: U \rightarrow D$  is an immersion and  $p(U)$  is a horizontal slice of the classifying space.*

From the above theorem and Proposition 2.1 in this section, we can prove:

**COROLLARY 2.1.** *With the notations as above, the map*

$$p: U \subset D \rightarrow G/K$$

*is an immersion.*

**3. The invariant complex structure.** It was proved by Griffiths that the classifying space  $D$  is actually a homogeneous complex manifold. We are going to study this fact in some detail via the Lie group point of view.

Define  $H_R = H_Z \otimes R$  and let

$$\begin{aligned} G_R &= \text{Aut}(H_R, Q) = \{g: H_R \rightarrow H_R \mid Q(g\phi, g\psi) = g(\phi, \psi), \phi, \psi \in H_R\} \\ G_C &= \text{Aut}(H_C, Q) = \{g: H_R \rightarrow H_R \mid Q(g\phi, g\psi) = g(\phi, \psi), \phi, \psi \in H\} \end{aligned}$$

Let  $G = G_R$ . Then  $G$  acts on  $D$  transitively and thus  $D$  is a homogeneous space.

Let  $V$  be the isotropy group fixing one point of  $D$ , then  $V$  is a compact subgroup (see Lemma 2.1).

Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . Then we have the standard Cartan decomposition

$$\mathfrak{g} = \mathfrak{f} + \mathfrak{p}$$

into the compact part  $\mathfrak{f}$  and noncompact part  $\mathfrak{p}$ . We assume that the Lie algebra  $\mathfrak{v}$  of  $V$  is contained in  $\mathfrak{f}$ . Since the Killing form on  $\mathfrak{f}$  is negative definite, there is a subset  $\mathfrak{v}_1$  of  $\mathfrak{f}$  such that

$$\mathfrak{f} = \mathfrak{v} + \mathfrak{v}_1$$

is an orthonormal decomposition of  $\mathfrak{f}$ . Thus we have

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{v}_1 \oplus \mathfrak{p}.$$

There is a natural representation of  $\mathfrak{v}$  to  $\mathfrak{v}_1 \oplus \mathfrak{p}$  such that the tangent bundle of  $D$  is the associated bundle of the principle bundle  $G \rightarrow G/V$  with respect to this representation.

Suppose that  $\mathfrak{a}^c$  is the complexification of a Lie algebra  $\mathfrak{a}$ . Then we have

$$\mathfrak{g}^c = \mathfrak{v}^c \oplus \mathfrak{v}_1^c \oplus \mathfrak{p}^c.$$

Let  $\tau$  be the complex conjugate of  $\mathfrak{g}^c$  with respect to the compact real form of  $\mathfrak{g}^c$ . Then let

$$\mathfrak{v}_1^c \oplus \mathfrak{p}^c = \mathfrak{n}_- \oplus \tau(\mathfrak{n}_-)$$

be the splitting into  $(\pm \sqrt{-1})$   $\tau$ -spaces. Then the complex structure of  $D$  is determined by  $\tau(\mathfrak{n}_-)$ .

Suppose  $J$  is the invariant complex structure of  $D$ . It is well known that to give an invariant complex structure on  $G/V$  is equivalent to give a linear transformation  $J$  on  $\mathfrak{v}_1 + \mathfrak{p}$  such that

$$\begin{aligned} J^2 &= -id_{(\mathfrak{v}_1 + \mathfrak{p})} \\ \rho(h)J &= J\rho(h), \quad \forall h \in V \end{aligned}$$

where

$$\rho: V \rightarrow \mathfrak{v}_1 + \mathfrak{p}$$

is the standard adjoint representation.

By the structure theory of the complex semi-simple Lie algebra, the complexification of  $\mathfrak{v}_1 + \mathfrak{p}$  can be written as the sum of the root spaces

$$(\mathfrak{v}_1 + \mathfrak{p})^c = \sum_{\alpha \in I} \mathfrak{g}_\alpha$$

for some index set  $I$ . Suppose  $\mathfrak{h}$  is the Cartan subalgebra of  $\mathfrak{g}^c$ . Let  $\mathfrak{h}_R$  be its real part. Then

$$\mathfrak{h}_R \subset \mathfrak{v}.$$

Thus  $J$  is  $V$ -invariant, which implies that  $J$  is  $\mathfrak{h}$  invariant. In particular

$$[h, JX] = J[h, X]$$

for  $h \in \mathfrak{h}, X \in (\mathfrak{v}_1 + \mathfrak{p})^c$ . Now let  $X \in \mathfrak{g}_\alpha$ . Then

$$[h, JX] = J[h, X] = J\alpha(h)X = \alpha(h)JX.$$

Thus

$$JX \in \mathfrak{g}_\alpha.$$

Since  $J^2 = -1$ , we know  $JX = \pm\sqrt{-1}X$ . In particular, if  $J_1, J_2$  are the two invariant complex structures, we have

$$J_1J_2 = J_2J_1.$$

Suppose  $p$  is the projection of  $\mathfrak{v}_1 + \mathfrak{p}$  to the second factor.

*Definition 3.1.* We call  $\omega$  the fundamental form on  $D$  if  $\omega$  is  $G$  invariant and if on  $\mathfrak{v}_1 + \mathfrak{p}$ , the tangent space of  $D$  at  $T_e(D)$ ,  $\omega$  is defined as

$$\omega(X, Y) = -B(pX, pJY)$$

where  $B$  is the Killing form of the Lie algebra.

From the following proposition, we know  $\omega$  is well defined.

PROPOSITION 3.1. *We have:*

- (1)  $\omega(X, Y) = -\omega(Y, X)$ , for  $X, Y \in \mathfrak{v}_1 + \mathfrak{p}$ ;
- (2)  $\omega$  is  $V$  invariant.

*Proof.* We recall the root decomposition

$$(\mathfrak{v}_1 + \mathfrak{p}_0)^c = \sum_{\alpha \in I} \mathfrak{g}_\alpha$$

and

$$J\mathfrak{g}_\alpha \subset \mathfrak{g}_\alpha \quad \text{for } \alpha \in I.$$

In particular, if  $X \in \mathfrak{v}_1, JX \in \mathfrak{v}_1$ ; and  $X \in \mathfrak{p}, JX \in \mathfrak{p}$ .

So if  $X \in \mathfrak{v}_1$ , then  $pX = 0, pJX = 0$ . So

$$\omega(X, Y) = 0 = -\omega(Y, X).$$

To verify the first claim, we only need to check the proposition for  $X \in \mathfrak{g}_\alpha$  and  $Y \in \mathfrak{g}_\beta$  for noncompact roots  $\alpha$  and  $\beta$ . Suppose  $JX = \sigma_X X, JY = \sigma_Y Y$ . Then  $\omega(X, Y) = \sigma_Y B(X, Y), \omega(Y, X) = \sigma_X B(Y, X)$ . If  $\alpha + \beta \neq 0$ , then  $B(X, Y) = 0$ . So we need only assume that  $\alpha + \beta = 0$ . In this case, suppose that  $JX = \sigma X$ . Then  $J\bar{X} = -\sigma\bar{X}$ . But  $\bar{X} \in \mathfrak{g}_{-\alpha}$ . So  $JY = -\sigma Y$ . Thus we have  $\sigma_X = -\sigma_Y$  and then  $\omega(X, Y) = -\omega(Y, X)$ .

Next,  $\forall h \in V$ , we have

$$\omega(Ad(h)X, Ad(h)Y) = \omega(X, Y)$$

because  $Ad(h)$  commutes  $p$  and  $J$ .

From the above theorem we know that  $\omega$  is well defined. □

**4. The pluriharmonic.** In this section, we prove Theorem 1.1. The notations are as in the previous sections.

Let  $D = G/V$  be a classifying space and  $\mathfrak{g}$  be the Lie algebra of  $G$ . We begin with the following key observation:

**THEOREM 4.1.** *Let  $\mathfrak{g} = \mathfrak{v} + \mathfrak{v}_1 + \mathfrak{p}$  be the decomposition in the previous sections. Suppose  $U$  is an open neighborhood of a horizontal slice. Then if  $X, Y, Z \in \Gamma(U, G \times_V \mathfrak{p})$  or  $X, Y \in \Gamma(U, G \times_V \mathfrak{v}_1)$  and  $Z \in \Gamma(U, G \times_V \mathfrak{p})$ , then*

$$d\omega(X, Y, Z) = 0,$$

where  $\omega$  is the differential form defined in Definition 3.1.

*Proof.* We have

$$T(G/V) = T_{\mathfrak{v}}(G/V) + G \times_V \mathfrak{p}$$

where

$$T_{\mathfrak{v}}(G/V) = G \times_V \mathfrak{v}_1.$$

Suppose  $\sigma_0: G \rightarrow G$  is the involution. That is,  $\sigma_0$  is an isomorphism of  $G$  such that  $\sigma_0^2 = 1$ .  $\sigma_0$  induces the Lie algebra isomorphism  $\sigma_0: \mathfrak{g} \rightarrow \mathfrak{g}$ . It is easy to see that  $\sigma_0(X) = X$  for  $X \in \mathfrak{f}$ ,  $\sigma_0(X) = -X$  for  $X \in \mathfrak{p}$ .

$\sigma_0$  also induces a  $C^\infty$  map

$$G/V \rightarrow G/V, \quad aV \mapsto \sigma(a)V.$$

Let  $\sigma_g$  be the map

$$(4.1) \quad \sigma_g = (L_g)\sigma_0(L_g^{-1})$$

on  $G/V$  where  $L_g$  is the left transformation, then  $\sigma_g(gV) = gV$ .

**LEMMA 4.1.**  $\sigma_g(\omega) = \omega$  where  $\omega$  is the differential form defined in Definition 3.1.

*Proof.* By the definition of  $\sigma_g$ , we need only prove that  $\sigma_0(\omega) = \omega$ . Next we observe that  $\sigma_0(\omega)$  is also an invariant form. So we only check that  $\sigma_0(\omega) = \omega$  at the original point.

We see that

$$\sigma_0 J = J\sigma_0, p\sigma_0 = \sigma_0 p$$

where  $p: D \rightarrow D_1$  is the projection.

So for any tangent vector  $X, Y$ ,

$$\begin{aligned} (\sigma_0 \omega)(X, Y) &= \omega(\sigma_0(X), \sigma_0(Y)) = -B(p\sigma_0 X, pJ\sigma_0 Y) \\ &= -B(\sigma_0 pX, \sigma_0 pJY) = -B(pX, pJY) = \omega(X, Y) \end{aligned}$$

where  $B$  is the Killing form. Thus the lemma is proved.  $\square$

*Continuation of the Proof of Theorem 4.1.* Now if  $X, Y, Z \in \Gamma(U, G \times_V \mathfrak{p})$  or  $X, Y \in \Gamma(U, G \times_V \mathfrak{v}_1)$  and  $Z \in \Gamma(U, G \times_V \mathfrak{p})$ , then

$$\begin{aligned} d\omega(X, Y, Z) &= -d\omega(\sigma_0 X, \sigma_0 Y, \sigma_0 Z) \\ &= -(\sigma_0 d\omega)(X, Y, Z) = -d(\sigma_0 \omega)(X, Y, Z) \\ &= -d\omega(X, Y, Z). \end{aligned}$$

So

$$d\omega(X, Y, Z) = 0. \quad \square$$

The invariant Riemannian metric on  $G/K$  is defined by (up to a multiplication of a constant),

$$(X, Y)_{gV} = B((L_{g^{-1}})_* X, (L_{g^{-1}})_* Y)$$

where  $L_{g^{-1}}$  is the left translation of the group  $G$  by  $g^{-1}$ .

LEMMA 4.2. *Let  $X, Y$  be vector fields on  $D$ . Let  $p: D \rightarrow D_1$  be the projection. Then*

$$\omega(X, Y) = -(p_* X, p_* JY).$$

*Proof.* At the origin point, the lemma is trivially true by definition. At a general point  $gV$  of  $D$ , note that the left translation  $L_{g^{-1}}$  commutes with the projection  $p$ . We have

$$\begin{aligned} \omega_{gV}(X, Y) &= \omega_0((L_{g^{-1}})_* X, (L_{g^{-1}})_* Y) = -B(p(L_{g^{-1}})_* X, pJ(L_{g^{-1}})_* Y) \\ &= -B((L_{g^{-1}})_* p_* X, (L_{g^{-1}})_* p_* JY) = -(p_* X, p_* JY). \quad \square \end{aligned}$$

*Definition 4.1.* An immersion

$$p: M \rightarrow N$$

from a complex manifold  $M$  to a Riemannian manifold  $N$  is called pluriharmonic, if for any vector field  $X$  on  $M$

$$\nabla_{p_*X}p_*X + \nabla_{p_*JX}p_*JX + p_*J[X, JX] = 0$$

for the Riemannian connection  $\nabla$  of  $N$ .

Now we begin to prove Theorem 1.1:

**THEOREM 4.2.** *Suppose  $U$  is a horizontal slice of  $D$ . Then*

$$p: U \subset D \rightarrow G/K$$

*is a pluriharmonic immersion for the invariant Riemannian connection  $\nabla$  on  $G/K$ .*

*Proof.* We have already proved that the map is an immersion in Corollary 2.1. So it remains to prove the pluriharmonicity of the map. We say a complex submanifold  $S$  is integrable at a point  $q$ , if

$$T_q(S) \subset (G \times_V \mathfrak{p})_q$$

and if at a neighborhood of  $q$ , the map  $p$  is an immersion at  $q$  from  $S$  to  $G/K$ . Let  $U$  be an open neighborhood of  $S$ . Let  $X, Y, Z \in \Gamma(U, TU)$  and  $X_q, Y_q, Z_q \in (G \times_V \mathfrak{p})_q$ . Then by Theorem 4.1, at  $q$ ,

$$d\omega(X, Y, Z) = 0;$$

$p_*X, p_*Y, p_*Z$  are well defined and  $C^\infty$  in a neighborhood of  $U$ . Then by Lemma 4.2

$$\begin{aligned} 0 &= -d\omega(X, Y, Z) = -X\omega(Y, Z) + Y\omega(X, Z) - Z\omega(X, Y) \\ &\quad - \omega(X, [Y, Z]) + \omega(Y, [X, Z]) - \omega(Z, [X, Y]) \\ &= (p_*X)(p_*Y, p_*JZ) - (p_*Y)(p_*X, p_*JZ) + (p_*Z)(p_*X, p_*JY) \\ &\quad - (p_*JX, [p_*Y, p_*Z]) + (p_*JY, [p_*X, p_*Z]) - (p_*JZ, [p_*X, p_*Y]) \\ &= (\nabla_{p_*X}p_*Y, p_*JZ) + (p_*Y, \nabla_{p_*X}p_*JZ) - (\nabla_{p_*Y}p_*X, p_*JZ) \\ &\quad - (p_*X, \nabla_{p_*Y}p_*JZ) + (\nabla_{p_*Z}p_*X, p_*JY) + (p_*X, \nabla_{p_*Z}p_*JY) \\ &\quad - (p_*JX, \nabla_{p_*Y}p_*Z) + (p_*JX, \nabla_{p_*Z}p_*Y) + (p_*JY, \nabla_{p_*X}p_*Z) \\ &\quad - (p_*JY, \nabla_{p_*Z}p_*X) - (p_*JZ, \nabla_{p_*X}p_*Y) + (p_*JZ, \nabla_{p_*Y}p_*X) \\ &= (p_*Y, \nabla_{p_*X}p_*JZ) + (p_*JY, \nabla_{p_*X}p_*Z) - (p_*X, \nabla_{p_*Y}p_*JZ) \\ &\quad + (p_*X, \nabla_{p_*Z}p_*JY) - (p_*JX, \nabla_{p_*Y}p_*Z) + (p_*JX, \nabla_{p_*Z}p_*Y). \end{aligned}$$

If we substitute  $X$  by  $JX$ ,  $Y$  by  $JY$  and  $Z$  by  $JZ$ , we have

$$0 = -(p_*JY, \nabla_{p_*JX}p_*Z) - (p_*Y, \nabla_{p_*JX}p_*JZ) + (p_*JX, \nabla_{p_*JY}p_*Z) \\ - (p_*JX, \nabla_{p_*JZ}p_*Y) + (p_*X, \nabla_{p_*JY}p_*JZ) - (p_*X, \nabla_{p_*JZ}p_*JY).$$

Substituting  $X$  by  $JX$ , we have

$$0 = (p_*Y, \nabla_{p_*JX}p_*JZ) + (p_*JY, \nabla_{p_*JX}p_*Z) - (p_*JX, \nabla_{p_*JY}p_*JZ) \\ + (p_*JX, \nabla_{p_*JZ}p_*JY) + (p_*X, \nabla_{p_*JY}p_*Z) - (p_*X, \nabla_{p_*JZ}p_*Y).$$

Comparing the above two equations, we get

$$0 = -2(p_*JY, \nabla_{p_*JX}p_*Z) - 2(p_*Y, \nabla_{p_*JX}p_*JZ) + (p_*JX, [p_*JY, p_*Z]) \\ - (p_*JX, [p_*JZ, p_*Y]) + (p_*X, [p_*JY, p_*JZ]) - (p_*X, [p_*Y, p_*Z])$$

where the last four terms are equal to

$$-(p_*X, p_*J[JY, Z]) + (p_*X, p_*J[JZ, Y]) \\ + (p_*X, p_*[JY, JZ]) - (p_*X, p_*[Y, Z]) = 0$$

because of the integrability condition of  $J$ .

So we have

$$(4.2) \quad (p_*Y, \nabla_{p_*JX}p_*JZ) + (p_*JY, \nabla_{p_*JX}p_*Z) = 0.$$

Let  $X = Z$ . We have

$$(p_*Y, \nabla_{p_*JX}p_*JX) + (p_*JY, \nabla_{p_*JX}p_*X) = 0.$$

Substituting  $X$  by  $JX$ , we have

$$(p_*Y, \nabla_{p_*X}p_*X) - (p_*JY, \nabla_{p_*X}p_*JX) = 0.$$

So

$$(p_*Y, \nabla_{p_*X}p_*X) + (p_*Y, \nabla_{p_*JX}p_*JX) + (p_*JY, p_*[JX, X]) = 0.$$

Then the theorem follows from the fact that for any  $\tilde{X}, \tilde{Y}, \tilde{Z} \in T_{p(q)}(G/K)$ , there is an integral submanifold  $S$  at  $q$  and  $X, Y, Z \in T_q(D)$  such that  $p_*X = \tilde{X}$ ,  $p_*Y = \tilde{Y}$ ,  $p_*Z = \tilde{Z}$ .

**COROLLARY 4.1.** *Use the same notation as above. We also have*

$$(p_*Z, \nabla_{p_*JX}p_*JY) + (p_*JZ, \nabla_{p_*JX}p_*Y) = 0.$$

*Proof.* This is the same as Equation 4.2.

**5. The Hodge metric.** We use the notations in the previous sections. Now suppose  $U$  is a horizontal slice. We also denote  $\omega$  the restriction to  $U$  of the fundamental form in Definition 3.1. Then we have:

LEMMA 5.1. *On the horizontal slice  $U$*

- (1)  $d\omega = 0$ ;
- (2)  $\omega$  is a 1-1 form;
- (3)  $\omega(X, JX) > 0, X \neq 0$ .

*Proof.* The first assertion is easy from Theorem 4.1. To prove the second assertion, let's assume  $X_\alpha, X_\beta \in T^{1,0}(G/V)$ , where  $\alpha, \beta$  are roots of  $\mathfrak{g}^c$ . Then  $\omega(X_\alpha, X_\beta) \neq 0$  if and only if  $\beta = -\alpha$ . In that case, we can assume

$$X_\beta = -\bar{X}_\alpha,$$

contradicting the fact that both of them are in  $T^{1,0}(G/V)$ . So  $\omega$  is a 1-1 form on  $G/V$ , thus a 1-1 form on  $U$ . If  $X \neq 0$ ,

$$\omega(X, JX) = (p_*X, p_*X) > 0,$$

if  $X \neq 0$  by Lemma 4.2.

Thus  $\omega$  defines a Kähler metric whose underlying Riemannian metric is the restriction of the invariant Riemannian metric on  $U$ . □

*Definition 5.1.* The Kähler metric  $\omega$  is called the Hodge metric of the horizontal slice  $U$ .

The main theorem of this section is:

THEOREM 5.1. *With respect to the Hodge metric, the holomorphic bisectional curvature and the Ricci curvature are nonpositive. Furthermore, the holomorphic sectional curvature and the Ricci curvature are negative away from zero by a constant number.*

*Proof.* By Lemma 4.2, the Hodge metric is the restriction of the invariant Riemannian metric on  $G/K$ . Suppose  $R, \tilde{R}$  are the curvature tensors of  $G/K$  and  $U$ , respectively. Then we have the Gauss formula:

$$\begin{aligned} &\tilde{R}(X, Y, X, Y) + \tilde{R}(X, JY, X, JY) \\ &= R(X, Y, X, Y) + R(X, JY, X, JY) \\ &\quad + (h(X, X), h(Y, Y) + h(JY, JY)) - |h(X, Y)|^2 - |h(X, JY)|^2 \end{aligned}$$

where  $h(\cdot, \cdot)$  refers to the second fundamental form and  $X, Y \in TU$ . Now the pluriharmonicity of the map (Theorem 1.1)  $p: U \rightarrow G/K$  implies that

$$h(Y, Y) + h(JY, JY) = 0. \quad \square$$

Thus the negativity of the bisectional curvature follows from the following lemma:

LEMMA 5.2.  *$G/K$  has nonpositive sectional curvature. Furthermore, at the original point*

$$R(X, Y, X, Y) = -\|[X, Y]\|^2.$$

*Proof.* Since  $G/K$  is a symmetric space of noncompact type, the curvature formula follows from [8]. In particular, the curvature tensor is nonpositive.  $\square$

*Continuation of the proof of the theorem.* From the above lemma, we see that the holomorphic bisectional curvature is nonpositive. Now we consider the holomorphic sectional curvature. By the above computation, we have

$$\tilde{R}(X, JX, X, JX) \leq R(X, JX, X, JX).$$

That  $R(X, JX, X, JX)$  is negative away from zero is hinted by Griffiths and Schmidt [7]: if we can prove that for any  $X \in T_e U$ ,

$$\|[X, JX]\| \geq c\|X\|^2$$

for  $c > 0$ , then we have

$$R(X, JX, X, JX) \leq -c^2\|X\|^4.$$

Thus the holomorphic sectional curvature is negative away from zero at the original point. Because of the homogeneity of the metric, we know that it is negative everywhere.

Now we let

$$X = \sum_i a_i g_i + \bar{a}_i g_{-i}$$

where  $Jg_i = \sqrt{-1}g_i$  and  $g_{-j} = \bar{g}_j$ . Then

$$[X, JX] = -2\sqrt{-1} \sum_i |a_i|^2 [g_i, g_{-i}] + \cdots \cdot$$

Here  $\cdot$  refers to the terms which are not in the maximum torus. Since

$\sum_i |a_i|^2 [g_i, g_{-i}]$  belongs to the cone of positive roots, it will never be zero unless  $a_i \equiv 0$ . Thus we have proved that  $\|[X, JX]\| \geq c\|X\|^2$ .

The Ricci curvature is the sum of the holomorphic bisectional curvature of different directions. So its negativity follows from the negativity of the holomorphic sectional curvature.

One application of the above theorem is that we might be able to use it to give a proof of the compactification theorem of the moduli space. It is easy to see that the Hodge metric can be defined on the moduli space as a Kähler orbifold metric. With this, we can prove:

**COROLLARY 5.1.** *If  $\mathcal{M}$  is complete and smooth with bounded sectional curvature with respect to the Hodge metric, then  $\mathcal{M}$  is quasi-projective.*

*Proof.* It follows from a theorem of Yeung [18].

*Remark 5.1.* The above corollary is valid in a much weaker assumption. We will write out the proof in a subsequent paper.

**6. On  $\Gamma$ -invariant Kähler metrics.** Let  $D = G/V$  and let  $\Gamma$  be a discrete subgroup of  $G$ . We say that  $\Gamma$  is cocompact if  $\Gamma \backslash G$  is a compact topological space. In this section, we restrict ourselves to the case that  $D$  is the classifying space of some Calabi-Yau threefold. We consider the problem of the existence of Kähler metrics on  $D$  which are  $\Gamma$ -invariant.

The motivation of this problem is from the recent work of Rajan [14], who proved the infinitesimal rigidity of the complex structure for a large family of homogeneous Kähler manifolds. It would be very interesting to prove the same rigidity theorem for the classifying space. However, In general,  $D$  is not a homogeneous Kähler manifold. Furthermore, we prove in this section that  $D$  does not even admit a Kähler metric which is  $\Gamma$ -invariant.

More precisely, we have:

**THEOREM 6.1.** *Suppose  $D$  is the classifying space of a Calabi-Yau threefold.  $D$  is the set of filtration*

$$0 \subset F^3 \subset F^2 \subset F^1 \subset H$$

*satisfying the two Hodge-Riemannian conditions. Suppose that  $\Gamma$  is a cocompact lattice of  $D$ . Then there are no Kähler metrics on  $D$  which are  $\Gamma$ -invariant.*

Recall that  $\Gamma$  is cocompact if and only if  $\Gamma \backslash D$  is a compact space. To prove the theorem we consider the map  $p$  of the projection

$$p: D \rightarrow G/K$$

where  $K$  is the maximum compact subgroup of  $G$  containing  $V$ .

It is easy to see that  $G = \mathfrak{Sp}(2n+2, R)$ , the real symplectic group, where  $2n+2$  is the complex dimension of  $H$ . In particular,  $G/K$  is the Hermitian symmetric space, which can be realized as the Siegal Space of the third kind.

We observed that  $p$  is not holomorphic nor anti-holomorphic. The reason is that if  $p$  is anti-holomorphic, then we can reverse the complex structure on  $D$  to make it holomorphic. By a general theorem in Murakami [13], we know that in that case,  $D$  will be a homogeneous Kähler manifold. But we have:

LEMMA 6.1.  *$D$  is not a homogeneous Kähler manifold.*

*Proof.* The proof is easy after writing out the root decomposition and the conjugate  $\tau$  explicitly. For details, see [12].  $\square$

LEMMA 6.2. *If  $D$  admits a  $\Gamma$ -invariant Kähler metric, then there is a  $\Gamma$ -equivariant holomorphic or anti-holomorphic map  $f: D \rightarrow G/K$  which is surjective at some point  $x_0 \in D$ .*

*Proof.* Suppose  $\tilde{\omega}$  is the Kähler form on  $D$  which is  $\Gamma$ -invariant. According to Eells-Sampson [4], Jost-Yau [9], and Labourie [11], there is a  $\Gamma$ -equivariant harmonic map  $f: D \rightarrow G/K$  such that  $f$  is  $\Gamma$ -equivariant homotopic to  $p$ . That is,  $f$  and  $p$  can be linked by a path of continuous  $\Gamma$ -equivariant maps. By topology, we know that  $p, f$  induce the same homomorphism between the cohomology groups:

$$p^*, f^*: H^{2N}(\Gamma \backslash G/K, \mathbb{C}) \rightarrow H^{2N_1}(\Gamma \backslash D, \mathbb{C}).$$

Here  $2N$  is the real dimension of  $\Gamma \backslash G/K$ .  $2N_1$  is the real dimension of  $\Gamma \backslash D$ . In particular, if  $\eta$  is the volume form of  $\Gamma \backslash G/K$  then we have

$$\int_{\Gamma \backslash D} f^* \eta \wedge \tilde{\omega}^{N_1-N} = \int_{\Gamma \backslash D} p^* \eta \wedge \tilde{\omega}^{N_1-N}.$$

On the other hand

$$\int_{\Gamma \backslash D} p^* \eta \wedge \tilde{\omega}^{N_1-N} = \int_{\Gamma \backslash G/K} \eta \int \tilde{\omega}^{N_1-N} > 0.$$

So

$$\int_{\Gamma \backslash D} f^* \eta \wedge \tilde{\omega}^{N_1-N} \neq 0,$$

and in particular,  $f^* \eta \neq 0$  at some point  $x_0$ .

By the rigidity theorem of Siu [15],  $f$  is a holomorphic or anti-holomorphic map.  $\square$

The invariant Riemannian metric  $g$  on  $D$  is a Hermitian metric.

LEMMA 6.3. *The map  $f$  is a harmonic map with respect to  $g$ .*

*Proof.* We assume that  $f$  is holomorphic without losing generality. First note that each fiber of  $p: D \rightarrow G/K$  is a compact Kähler submanifold of  $D$ . Thus according to Liouville's theorem  $f$  is a constant along each fiber.

Now suppose that  $(w^1, \dots, w^N)$  is the holomorphic local coordinate of  $G/K$  and  $(z^1, \dots, z^{N_1})$  is the corresponding holomorphic local coordinate such that  $f = (f^1, \dots, f^N)$  with respect to these coordinates are holomorphic functions. Thus at any point  $x \in D$ ,

$$\frac{\partial f^\alpha}{\partial \bar{z}^j}(x) = 0$$

for any  $\alpha, j$ . We assume that at  $p(x)$ ,  $(w^1, \dots, w^N)$  is normal. That is, all the connection coefficients are zero at  $p(x)$ .

Now suppose  $\Delta$  is the Laplacian of the metric  $g$ . We need to prove

$$\Delta f^\alpha(x) = 0$$

for  $\alpha = 1, \dots, N$ .

Now

$$\begin{aligned} -\Delta f^\alpha &= \delta df^\alpha = \delta \partial f^\alpha + \delta \bar{\partial} f^\alpha \\ &= - * d * \frac{\partial f^\alpha}{\partial z^i} dz^i = - * d \frac{\partial f^\alpha}{\partial z^i} * dz^i \\ &= - * \frac{\partial^2 f^\alpha}{\partial z^i \partial \bar{z}^j} dz^j \wedge * dz^i - \frac{\partial f^\alpha}{\partial z^i} * d * dz^i. \end{aligned}$$

We have

$$dz^k \wedge * dz^j = 0$$

for all  $k, j$  by the type considerations.

LEMMA 6.4.

$$\sum_j \frac{\partial f^\alpha}{\partial z^j} * d * dz^j = - \sum_j \frac{\partial f^\alpha}{\partial z^j} \Delta z^j = 0.$$

*Proof.* Suppose that  $z^j = y^j + \sqrt{-1}y^{j+N_1}$ . Here  $(y^1, \dots, y^{2N_1})$  is the real local coordinate. Suppose

$$g = g_{\bar{j}j} dz^i \otimes d\bar{z}^j$$

is the invariant Hermitian metric on  $D$ . The corresponding Riemannian metric is then

$$G = G_{ab} dy^a dy^b$$

$$\begin{aligned}
&= 2(\operatorname{Re} g_{i,\bar{j}} dy^i dy^j + \operatorname{Re} g_{i,\bar{j}} dy^{i+N_1} dy^{j+N_1} \\
&\quad - \operatorname{Im} g_{i,\bar{j}} dy^{i+N_1} dy^j + \operatorname{Im} g_{i,\bar{j}} dy^i dy^{j+N_1}).
\end{aligned}$$

Here we assume the sum over  $i, j, k, \dots$  is from 1 to  $N_1$  and the sum over  $a, b, \dots$  is from 1 to  $2N_1$ . Thus

$$\begin{aligned}
\Delta_z^k &= \frac{1}{\sqrt{\det G}} \frac{\partial}{\partial x^a} \left( \sqrt{\det G} G^{ab} \frac{\partial}{\partial x^b} z^k \right) \\
&= \frac{1}{\det g} \frac{\partial}{\partial x^a} (\det g (G^{ak} + \sqrt{-1} G^{a(k+N_1)})).
\end{aligned}$$

Now

$$\begin{aligned}
(G^{ak} + \sqrt{-1} G^{a(k+N_1)}) g_{k\bar{j}} &= \frac{1}{2} (G^{ak} + \sqrt{-1} G^{a(k+n)}) (G_{kj} + \sqrt{-1} G_{k,j+n}) \\
&= \delta_{aj} + \sqrt{-1} \delta_{a(j+N_1)}.
\end{aligned}$$

Thus

$$G^{ak} + \sqrt{-1} G^{a(k+n)} = \begin{cases} g^{k\bar{a}} & a \leq N_1 \\ \sqrt{-1} g^{s, a-N_1} & a > N_1. \end{cases}$$

So finally we have

$$\begin{aligned}
\Delta_z^k &= \frac{1}{\det g} \frac{\partial}{\partial x^l} (\det g g^{k\bar{l}}) + \frac{1}{\det g} \frac{\partial}{\partial x^{l+N_1}} (\det g \sqrt{-1} g^{k\bar{l}}) \\
&= 2 \frac{1}{\det g} \frac{\partial}{\partial \bar{z}^l} (\det g g^{k\bar{l}}) = 2 g^{k\bar{l}} g^{i\bar{j}} (d\omega)_{i\bar{j}\bar{l}},
\end{aligned}$$

where  $\omega = g_{i\bar{j}} dz^i \wedge d\bar{z}^j$  is the Kähler form of the invariant Hermitian metric  $g$  on  $D$ .

So we have

$$\sum_j \frac{\partial f^\alpha}{\partial \bar{z}^j} * d * dz^j = C \frac{\partial f^\alpha}{\partial \bar{z}^j} g^{j\bar{k}} g^{r\bar{s}} (d\omega)_{r\bar{s}\bar{k}}$$

where  $C$  is a constant. Thus we have to prove

$$C \frac{\partial f^\alpha}{\partial \bar{z}^j} g^{j\bar{k}} g^{r\bar{s}} (d\omega)_{r\bar{s}\bar{k}} = 0.$$

*Claim.*

$$C \frac{\partial f^\alpha}{\partial \bar{z}^j} g^{j\bar{k}} g^{r\bar{s}} (d\omega)_{r\bar{s}\bar{k}} = C \langle df^\alpha \wedge \omega, \bar{d}\omega \rangle.$$

*Proof.* It follows from the definition of the inner product of the differential forms.  $\square$

Now let  $x \in D$ . Then we have the symmetry  $\sigma_x$  defined in Equation 4.1. Now since  $f$  is constant along each fiber, there is an  $\tilde{f}: G/K \rightarrow G/K$  such that

$$f = \tilde{f} \circ p.$$

Using  $\sigma_x p = p \sigma_x$  and  $\sigma_x d\tilde{f} = -d\tilde{f}$ , we have

$$\sigma_x df^\alpha = \sigma_x d\tilde{f}^\alpha p = p \sigma_x d\tilde{f}^\alpha = -p d\tilde{f}^\alpha = -df^\alpha.$$

On the other hand  $\sigma_x \omega = \omega$  in Lemma 4.1. Thus

$$\langle df^\alpha \wedge \omega, \overline{d\omega} \rangle = \sigma_x \langle df^\alpha \wedge \omega, \overline{d\omega} \rangle = -\langle df^\alpha \wedge \omega, \overline{d\omega} \rangle.$$

The lemma is proved.  $\square$

*Continuation of the proof.* Now we know  $f$  is a harmonic map with respect to the metric  $g$ . In particular,  $p$  is a harmonic map with respect to  $g$ . Since  $p$  is a surjective map, by the uniqueness theorem of the harmonic map, we know  $p = f$ . But this is a contradiction, because  $f$  is a holomorphic map but  $p$  is not. So we have proved that there are no Kähler metrics on  $D$  which are  $\Gamma$ -invariant.

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NY 10027

*Electronic mail:* LU@MATH.COLUMBIA.EDU

---

#### REFERENCES

---

- [1] M. Black, *Harmonic Maps into Homogeneous Spaces*, Pitman Res. Notes Math. Ser., Longman Sci. Tech., Harlow, 1991.
- [2] R. Bryant, Lie groups and twister spaces, *Duke Math. J.* **52** (1985), 233–261.
- [3] F. Burstall and S. Salmons, Tournaments, flags, and harmonic maps, *Math. Ann.* **277** (1987), 249–265.
- [4] J. Eells and J. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* **86** (1964), 109–160.
- [5] R. Friedman, On threefolds with trivial canonical bundle, *Complex Geometry and Lie Theory (Sundance, UT, 1989)*, (S.-T. Yau, ed.), Amer. Math. Soc., Providence, RI, 1991, pp. 103–134.
- [6] P. Griffiths (ed.), *Topics in Transcendental Algebraic Geometry*, *Ann. of Math. Stud.*, vol. 106, Princeton University Press, 1984.
- [7] P. Griffiths and W. Schmid, Locally homogeneous complex manifolds, *Acta Math.* **123** (1969), 253–302.
- [8] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, Boston, 1978.
- [9] J. Jost and S.-T. Yau, Harmonic maps and group representations, *Differential Geometry, Pitman Monographs Surveys Pure Appl. Math.*, Longman Sci. Tech., Harlow, 1991, pp. 241–259.

- [10] ———, Harmonic mappings and algebraic varieties over function fields, *Amer. J. Math.* **115** (1993), 1197–1227.
- [11] F. Labourie, Existence d'applications harmoniques tordues a valeurs dans les varietes a courbure negative, *Proc. Amer. Math. Soc.* **111** (1991), 877–882.
- [12] Z. Lu, On the geometry of the moduli spaces, Ph.D. thesis, Courant Institute, New York University, New York, NY, 1997.
- [13] S. Murakami, *Introduction to Homogeneous Spaces (in Chinese)*, Science and Technology Press, Shanghai, 1983.
- [14] C. S. Rajan, Vanishing theorems for cohomologies of automorphic vector bundles, *J. Differential Geom.* **43** (1996), 376–409.
- [15] Y. T. Siu, The complex analyticity of harmonic maps and strong rigidity compact Kähler manifolds, *Ann. of Math.* **112** (1980), 73–111.
- [16] G. Tian, Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Peterson-Weil metric, *Mathematical Aspects of String Theory*, vol. 1 (S. T. Yau, ed.), World Scientific, Teaneck, NJ, 1987, pp. 629–646.
- [17] E. Viehweg, *Quasi-Projective Moduli for Polarized Manifolds*, *Ergeb. Math. Grenzgeb.*, Springer-Verlag, New York, 1991
- [18] S. K. Yeung, Compactification of Kähler manifolds with negative Ricci curvature, *Invent. Math.* **106** (1991), 13–25.