ON THE GROUND STATE OF QUANTUM LAYERS

ZHIQIN LU

1. Introduction

The problem is from mesoscopic physics: let $p : \Sigma \to \mathbb{R}^3$ be an embedded surface in $\mathbb{R}^3$, we assume that

1. Σ is orientable, complete, but non-compact;
2. Σ is not totally geodesic;
3. Σ is asymptotically flat in the sense that the second fundamental form goes to zero at infinity.

On can build a quantum layer Ω over such a surface Σ as follows: as a differentiable manifold, $\Omega = \Sigma \times [-a, a]$ for some positive number a. Let $\vec{N}$ be the unit normal vector of Σ in $\mathbb{R}^3$. Define

$$\tilde{p} : \Omega \to \mathbb{R}^3$$

by

$$\tilde{p}(x, t) = p(x) + t\vec{N}_x.$$

Obviously, if a is small, then $\tilde{p}$ is an embedding. The Riemannian metric ds^2_Ω is defined as the pull-back of the Euclidean metric via $\tilde{p}$. The Riemannian manifold (Ω, ds^2_Ω) is called the quantum layer.

Let $\Delta = \Delta_\Omega$ be the Dirichlet Laplacian. Then we make the following

Conjecture 1. *Using the above notations, and further assume that*

$$\int_\Sigma |K| d\Sigma < +\infty. \tag{1.1}$$

Then the ground state of Δ *exists.*

We make the following explanation of the notations and terminology:

1. Ω is a smooth manifold with boundary. The Dirichlet Laplacian is the self-adjoint extenstion of the Laplacian acting on $C_0^\infty(\Omega)$;
2. By a theorem of Huber [4], if (1.1) is valid, then Σ is differmorphic to a compact Riemann surface with finitely many points removed. Moreover, White [10] proved that if

$$\int_\Sigma K^- d\Sigma < +\infty,$$

Date: July 21, 2007.

The author is partially supported by NSF Career award DMS-0347033 and the Alfred P. Sloan Research Fellowship.

then

$$\int_{\Sigma} |K| d\Sigma < +\infty.$$

Thus (1.1) can be weakened.

(3) Since Δ is a self-adjoint operator, the spectrum of Δ is the disjoint union of two parts: pure point spectrum (eigenvalues of finite multiplicity) and the essential spectrum. The ground state is the smallest eigenvalue with finite multiplicity.

(4) The conjecture was proved under the condition

$$\int_{\Sigma} K d\Sigma \leq 0$$

in [2, 1] by Duclos, Exner and Krejčiřík and later by Carron, Exner, and Krejčiřík. Thus the remaining case is when

$$\int_{\Sigma} K d\Sigma > 0.$$

By a theorem of Hartman [3], we know that

$$\int_{\Sigma} K = 2\pi\chi(\Sigma) - \sum \lambda_i$$

where λ_i are the isoperimetric constants at each end of Σ. Thus we have

$$\chi(\Sigma) > 0$$

and $g = 0$. The surface must be differmorphic to $\mathbb{R}^2$. However, even through the topology of the surface is completely known, this is the most difficult case for the conjecture.

2. Variational Principle

It is well known that

$$\sigma_0 = \inf_{f \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla f|^2 d\Omega}{\int_\Omega f^2 d\Omega}$$

is the infimum of the Laplacian, and

$$\sigma_{ess} = \sup_K \inf_{f \in C_0^\infty(\Omega \backslash K)} \frac{\int_\Omega |\nabla f|^2 d\Omega}{\int_\Omega f^2 d\Omega} \tag{2.1}$$

is the infimum of the essential spectrum, where K is running over all the compact subset of Ω. Since $\Omega = \Sigma \times [-a, a]$, it is not hard to see that

$$\sigma_{ess} = \sup_{K \subset \Sigma} \inf_{f \in C_0^\infty(\Omega \backslash K \times [-a,a])} \frac{\int_\Omega |\nabla f|^2 d\Omega}{\int_\Omega f^2 d\Omega}, \tag{2.2}$$

where K is running over all the compact set of Σ.

By definition, we have $\sigma \leq \sigma_{ess}$. Furthermore, we have

Proposition 2.1. *If $\sigma_0 < \sigma_{ess}$, then the ground state exists and is equal to σ_0.*

□

Let (x_1, x_2) be a local coordinate system of Σ. Then (x_1, x_2, t) defines a local coordinate system of Ω. Such a local coordinate system is called a *Fermi coordinate system*. Let $x_3 = t$ and let $ds_\Omega^2 = G_{ij}dx_idx_j$. Then we have

$$G_{ij} = \begin{cases} (p + t\vec{N})_{x_i}(p + t\vec{N})_{x_j} & 1 \le i, j \le 2; \\ 0 & i = 3, \text{or } j = 3, \text{ but } i \ne j; \\ 1 & i = j = 3. \end{cases} \tag{2.3}$$

We make the following defintion: let f be a smooth function of Ω. Then we define

$$Q(f, f) = \int_\Omega |\nabla f|^2 d\Omega - \kappa^2 \int_\Omega f^2 d\Omega; \tag{2.4}$$

$$Q_1(f, f) = \int_\Omega |\nabla' f|^2 d\Omega; \tag{2.5}$$

$$Q_2(f, f) = \int_\Omega \left(\frac{\partial f}{\partial t}\right)^2 d\Omega - \kappa^2 \int_\Omega f^2 d\Omega, \tag{2.6}$$

where $|\nabla' f|^2 = \sum_{i,j=1}^2 G^{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}$.

Obviously, we have

$$Q(f, f) = Q_1(f, f) + Q_2(f, f).$$

It follows that

$$\int_\Omega |\nabla f|^2 d\Omega = \int_\Omega |\nabla' f|^2 d\Omega + \int_\Omega \left(\frac{\partial f}{\partial t}\right)^2 d\Omega$$

for a smooth function $f \in C^\infty(\Omega)$, where

$$|\nabla' f|^2 = \sum_{1 \le i,j \le 2} G^{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}$$

is the norm of the horizontal differential. Apprently, we have

$$\int_\Omega |\nabla f|^2 d\Omega \ge \int_\Omega \left(\frac{\partial f}{\partial t}\right)^2 d\Omega.$$

Let $ds_\Sigma^2 = g_{ij}dx_idx_j$ be the Riemannian metric of Σ under the coordinates (x_1, x_2). Then we are above to compare the matrices $(G_{ij})_{1 \le i,j \le 2}$ and (g_{ij}), at least outside a big compact set of Σ. By (2.3), we have

$$G_{ij} = g_{ij} + tp_{x_i}\vec{N}_{x_j} + tp_{x_j}\vec{N}_{x_i} + t^2\vec{N}_{x_i}\vec{N}_{x_j}.$$

We assume that at the point x, $g_{ij} = \delta_{ij}$. Then we have

$$|G_{ij} - \delta_{ij}| \le 2a|B| + a^2|B|^2,$$

where B is the second fundamental form of the surface Σ. Thus we have the following conclusion:

Proposition 2.2. *For any $\varepsilon > 0$, there is a compact set K of Σ such that on $\Sigma\backslash K$ we have*

$$(1-\varepsilon)\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \leq \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \leq (1+\varepsilon)\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

In particular, we have

$$(1-\varepsilon)^2 d\Sigma dt \leq d\Omega \leq (1+\varepsilon)^2 d\Sigma dt.$$

□

Let $\kappa = \frac{\pi}{2a}$. Then we proved the followng:

Lemma 2.1. *Using the above notations, we have*

$$\sigma_{ess} \geq \frac{\pi^2}{4a^2}.$$

Proof. Let K be any compact set of Σ. If $f \in C_0^\infty(\Omega\backslash K)$, then by Proposition 2.2, we have

$$\int_\Omega \left(\frac{\partial f}{\partial t}\right)^2 d\Omega \geq (1-\varepsilon)^2 \int_\Sigma \int_{-a}^{a} \left(\frac{\partial f}{\partial t}\right)^2 dt d\Sigma \geq (1-\varepsilon)^2\kappa^2 \int_\Sigma \int_{-a}^{a} f^2 dt d\Sigma,$$

where the last inequality is from the 1-dimensional Poincaré inequality. Thus by using Proposition 2.2 again, we have

$$\int_\Omega |\nabla f|^2 d\Omega \geq \frac{(1-\varepsilon)^2}{(1+\varepsilon)^2}\kappa^2 \int_\Omega f^2 dt d\Sigma.$$

for any ε. Thus we have

$$\sigma_{ess} \geq \frac{(1-\varepsilon)^2}{(1+\varepsilon)^2}\kappa^2$$

and the lemma is proved.

□

Remark 2.1. Although not needed in this paper, we can actually prove that $\sigma_{ess} = \kappa^2$. To see this, we first observe that since the second fundamental form of Σ is bounded, there is a lower bound for the injectivity radius. As a result, the volume of the surface Σ is infinite. By the assumption, the Gauss curvature is integrable. Thus Σ is *parabolic* (cf. [5]). From the above, we conclude that for any $\varepsilon, C > 0$ and any compact sets $K \subset\subset K'$ of Σ, there is a smooth function $\varphi \in C_0^\infty(\Sigma\backslash K')$ such that

$$\varphi \equiv 1 \text{ on } K, \qquad \int_\Sigma \varphi^2 d\Sigma > C, \qquad \text{and } \int_\Sigma |\nabla\varphi|^2 d\Sigma < \varepsilon.$$

Let $\tilde{\varphi} = \varphi\chi$, where $\chi = \cos\kappa t$. Then $\tilde{\varphi}$ is a function on Ω with compact support. Since the second fundamental form goes to zero at infinity, by Proposition 2.2, for K' large enough, we have

$$\int_\Omega |\nabla\tilde{\varphi}|^2 d\Omega < 4a\varepsilon.$$

Thus from (2.4) and Proposition 2.2 again, we have

$$Q(\tilde{\varphi},\tilde{\varphi}) < 4a\varepsilon + (1+\varepsilon)^2 \int_\Sigma \varphi^2 d\Sigma \int_{-a}^{a} \left(\frac{\partial\chi}{\partial t}\right)^2 dt \\ - (1-\varepsilon)^2\kappa^2 \int_\Sigma \varphi^2 d\Sigma \int_{-a}^{a} \chi^2 dt.$$

A straightforward computation gives

$$\int_{-a}^{a} \left(\frac{\partial\chi}{\partial t}\right)^2 dt = \kappa^2 \int_{-a}^{a} \chi^2 dt.$$

Thus

$$Q(\tilde{\varphi},\tilde{\varphi}) \leq 4a\varepsilon + 4a\varepsilon \int_\Sigma \varphi^2 d\Sigma.$$

By the definition of σ_{ess}, we have

$$\sigma_{ess} - \kappa^2 \leq \frac{Q(\tilde{\varphi},\tilde{\varphi})}{\int_\Omega \tilde{\varphi}^2 d\Omega} \leq \frac{4\varepsilon(1+\int_\Sigma \varphi^2 d\Sigma)}{(1-\varepsilon)^2 \int_\Sigma \varphi^2 d\Sigma}.$$

We let $\varepsilon \to 0$ and $C \to \infty$, then we have $\sigma_{ess} \leq \kappa^2$, as needed.

3. The upper bound of σ_0

It is usually more difficult to estimate σ_0 from above. In [7, Theorem 1.1], we proved the following

Theorem 3.1. *Let Σ be a convex surface in $\mathbb{R}^3$ which can be represented by the graph of a convex function $z = f(x,y)$. Suppose 0 is the minimum point of the function and suppose that at 0, f is strictly convex. Furthermore suppose that the second fundamental form goes to zero at infinity. Let C be the supremum of the second fundamental form of Σ. Let $Ca < 1$. Then the ground state of the quantum layer Ω exists.*

In this section, we generalize the above result into the following:

Theorem 3.2. *Let Σ be a complete surface in $\mathbb{R}^3$ with nonnegative Gauss curvature but not totally geodesic. Furthermore suppose that the second fundamental form of Σ goes to zero at infinity. Let C be the supremum of the second fundamental form of Σ. Let $Ca < 1$. Then the ground state of the quantum layer Ω built over Σ with width a exists.*

Remark 3.1. Since for all convex function f in Theorem 3.1, the Gauss curvature is nonnegative, the above theorem is indeed a generalization of Theorem 3.1. On the other hand, by a theorem of Sacksteder [9], any complete surface of nonnegative curvature is either a developable surface or the graph of some convex function. At a first glance, it seems that there is not much difference between the surfaces in both theorems. However, we have to use a complete different method to prove this slight generalization.

Proof of Theorem 3.2. If the Gauss curvature is identically zero, then By [8, Theorem 2], the ground state exists.

If the Guass curvature is positive at one point, then by using the theorem of Sacksteder [9], Σ can be represented by the graph of some convex function. If we fix an orientation, we can assume that H, the mean curvature, is always nonnegative.

By a result of White [10], we know that there is an $\varepsilon_0 > 0$ such that for $R >> 0$,

$$\int_{\partial B(R)} ||B|| > \varepsilon_0,$$

where B is the second fundamental form of Σ. Since Σ is convex, we have

$$H \geq \frac{1}{2}||B||.$$

Thus we have

$$\int_{B(R_2)\backslash B(R_1)} H\, d\Sigma \geq \frac{1}{2}\varepsilon_0(R_2 - R_1) \tag{3.1}$$

provided that $R_2 > R_1$ are large enough.

We will create suitable test functions using the techniques similar to [2, 1, 7, 6, 8]. Let $\varphi \in C_0^\infty(\Sigma\backslash B(\frac{R}{2}))$ be a smooth function such that

$$\varphi \equiv 1 \quad \text{on } B(2R)\backslash B(R), \qquad \int_\Sigma |\nabla\varphi|^2 d\Sigma < \varepsilon_1,$$

where $\varepsilon_1 \to 0$ as $R \to \infty$. The existence of such a function φ is guaranteed by the parabolicity of Σ. Then we have, as in Remark 2.1, that

$$Q(\varphi\chi, \varphi\chi) < 4a\varepsilon_1 + 2a\pi^2 \int_{\Sigma\backslash B(R/2)} K\varphi^2 d\Sigma.$$

Since K is integrable, for any $\varepsilon_2 > 0$, there is an $R_0 > 0$ such that if $R > R_0$, we have

$$Q(\varphi\chi, \varphi\chi) < \varepsilon_2.$$

Now let's consider a function $j \in C_0^\infty(B(\frac{5}{3}R)\backslash B(\frac{4}{3}R))$. Consider the function $j\chi(t)t$, where j is a smooth function on Σ such that $j \equiv 1$ on $B(\frac{19}{12}R)\backslash B(\frac{17}{12}R)$; $0 \leq j \leq 1$; and $|\nabla j| < 2$. Then there is an absolute constant C_1, such that

$$Q(j\chi(t)t, j\chi(t)t) \leq C_1 R^2.$$

Finally, let's consider $Q(\varphi\chi(t), j\chi(t)t)$. Since $\operatorname{supp} j \subset \{\varphi \equiv 1\}$, by (2.5), $Q_1(\varphi\chi(t), j\chi(t)t) = 0$. Let

$$\sigma = -\int_{-a}^{a} \chi'(t)\chi(t)t dt > 0.$$

Then

$$Q(\varphi\chi(t), j\chi(t)t) = -\sigma \int_\Sigma j d\Sigma.$$

Let $\varepsilon > 0$. Then we have

$$Q(\varphi\chi(t) + \varepsilon j\chi(t)t, \varphi\chi(t) + \varepsilon j\chi(t)t) < \varepsilon_2 - 2\varepsilon\sigma \int_\Sigma j d\Sigma + \varepsilon^2 C_1 R^2.$$

By (3.1), we have

$$Q(\varphi\chi(t) + \varepsilon j\chi(t)t, \varphi\chi(t) + \varepsilon j\chi(t)t) < \varepsilon_2 - \frac{1}{3}\varepsilon\sigma R + \varepsilon^2 C_1 R^2.$$

If

$$\varepsilon_2 < \frac{\sigma^2}{36C_1},$$

then there is a suitable $\varepsilon > 0$ such that

$$Q(\varphi\chi(t) + \varepsilon j\chi(t)t, \varphi\chi(t) + \varepsilon j\chi(t)t) < 0.$$

Thus $\sigma_0 < \kappa^2$.

□

4. Further Discussions.

We proved the following more general

Theorem 4.1. *We assume that Σ satisfies*

(1) *The isopermetric inequality holds. That is, there is a constant $\delta_1 > 0$ such that if D is a domain in Σ, we have*

$$(length(\partial D))^2 \geq \delta_1 Area(D).$$

(2) *There is another positive constant $\delta_2 > 0$ such that for any compact set K of Σ, there is a curve C outside the set K such that if $\vec{\gamma}$ is one of its normal vector in Σ, then there is a vector $\vec{a}$ such that*

$$\langle \vec{\gamma}, \vec{a} \rangle \geq \delta_2 > 0$$

for some fixed vector $\vec{a} \in R^3$.

Then the ground state exists.

Proof. We let φ be a smooth function such that $\operatorname{supp}\varphi \subset B(R)\backslash B(r)$ for $R >> R/4 >> 4r >> r > 0$ large. We also assume that on $B(R/2)\backslash B(2r)$, $\varphi \equiv 1$. Let $\varepsilon_0 > 0$ be a positive number to be determined later such that

$$\int_\Sigma |\nabla\varphi|^2 \leq \varepsilon_0, \qquad \int_\Sigma |K|\varphi^2 \leq \varepsilon_0.$$

Note that ε_0 is independent of R.

We let $\chi = \cos\frac{\pi}{2a}t$. Then there is a constant C such that

$$Q(\varphi\chi, \varphi\chi) < C\varepsilon_0.$$

Let C be a curve outside the compact set $B(4r)$ satisfying the condition in the theorem. We let R big enough that $C \subset B(R/4)$.

In order to construct the test functions, we let ρ be the cut-off function such that $\rho = 1$ if $t \leq 0$ and $\rho = 0$ if $t \geq 1$ and we assume that ρ is decreasing. Near the curve C, any point p has a coordinate (t, s), where

$s \in C$ from the exponential map. To be more precise, let (x_1, x_2) be the the local coordinates near C such that locally C can be represented by $x_1 = 0$. Let the Riemannian metric under this coordinate system be

$$g_{11}(dx_1)^2 + 2g_{12}dx_1dx_2 + g_{22}(dx_2)^2.$$

The fact that $\vec{\gamma}$ is a normal vector implies that if

$$\vec{\gamma} = \gamma_1 \frac{\partial}{\partial x_1} + \gamma_2 \frac{\partial}{\partial x_2},$$

then

$$\gamma_1 g_{12} + \gamma_2 g_{22} = 0.$$

Let $\sigma_t(x_2)$ be the geodesic lines starting from $x_2 \in C$ with initial vector $\vec{\gamma}$. Then σ_t is the exponential map. The Jacobian of the map at $t = 0$ is

$$\begin{pmatrix} \gamma_1 & \gamma_2 \\ 0 & 1 \end{pmatrix}$$

In particular, $\gamma_1 \neq 0$ since the map must be nonsingular. A simple computation shows that $\nabla t = \gamma_1 g^{1j} \frac{\partial}{\partial x_i}$. Thus ∇t is proportional to $\vec{\gamma}$.

Let φ_1 be a cut-off function such that $\varphi_1 \equiv 1$ on $B(R/4)\backslash B(4r)$ and $\text{supp}|\,(\varphi_1) \subset B(R/2)\backslash B(2r)$.

We define $\tilde{\rho}(p) = \varphi_1 \rho(t/\varepsilon_1)$, where ε_1 is a positive constant to be determined. WLOG, let $\vec{a}$ be the z-direction in the Euclidean space.

Let $\vec{n}$ be the normal vector of Σ. Let n_z be the z-component of $\vec{n}$. We compute the following term $Q(\varphi\chi, \tilde{\rho} n_z \chi_1)$, where $\chi_1 = t \cos \frac{\pi}{2a} t$. First $Q_1(\varphi\chi, \tilde{\rho} n_z \chi_1) = 0$ becasue $\text{supp}(\tilde{\rho} n_z)$ is contained in the area where $\varphi \equiv 1$. On the other hand, since $\chi\chi_1$ is an odd function, we have

$$Q_2(\varphi\chi, \tilde{\rho} n_z \chi_1) = -\int_\Sigma H\varphi\tilde{\rho} n_z d\Sigma \int_{-a}^{a} (\chi'\chi_1' t - \kappa^2 \chi\chi_1 t) dt.$$

A straight computation shows that

$$C_1 = \int_{-a}^{a} (\chi'\chi_1' t - \kappa^2 \chi\chi_1 t) dt = -1/2 \neq 0.$$

Furthermore, we have $Hn_z = \Delta z$. As a result, we have

$$-\int_\Sigma H\varphi\tilde{\rho} n_z d\Sigma = \int_\Sigma \nabla z \nabla \tilde{\rho} = \int_{\{t \leq \varepsilon_1\}} \nabla z \nabla \tilde{\rho}$$

(Note that $\varphi \equiv 1$ on the points we are interested). We have the following Taylar expansion:

$$\nabla z \nabla \tilde{\rho}(t, x_2) = \nabla z \nabla \tilde{\rho}(0, x_2) + O(t)$$

Since $\int_{\{t \leq \varepsilon_1\}} O(t)/\varepsilon_1 = O(\varepsilon_1)\, Length(C)$, ,we have

$$\int_\Sigma \nabla z \nabla \tilde{\rho} \geq (\delta_2 - O(\varepsilon_1))\, Length(C)$$

We choose ε_1 small enough, then we have

$$\int_\Sigma \nabla z \nabla \tilde{\rho} \geq \frac{1}{2}\delta_2\, Length(C)$$

If we let $\varepsilon \to 0$, then that above becomes

$$-\int_\Sigma H\varphi\rho n_z d\Sigma = \int_\Sigma \nabla z \nabla \rho \geq \delta_2\, Length(C).$$

Finally, we have $|\rho n_z| + |\nabla(\rho n_z)| \leq 2$, thus we have

$$Q(\rho n_z \chi_1, \rho n_z \chi_1) \leq C Area(D),$$

where D is the domain C enclosed. To summary, for any $\varepsilon < 0$, we have

$$Q(\varphi\chi + \varepsilon\rho n_z\chi_1, \varphi\chi + \varepsilon\rho n_z\chi_1) \leq C\varepsilon_0 + 2\varepsilon C_1\delta_2\, Length(C) + C\varepsilon^2 Area(D).$$

Using the isopermetric inequality, we know that if $\varepsilon_0 < \delta_1\delta_2^2/C^2$ is small enough, then

$$Q(\varphi\chi + \varepsilon\rho n_z\chi_1, \varphi\chi + \varepsilon\rho n_z\chi_1) < 0$$

which proves the theorem.

□

Using the same proof, we can prove the following:

Theorem 4.2. *Using the same notations as in Conjecture 1, we assume further that*

$$||B||(x) \leq C/dist(x, x_0),$$

where $x_0 \in \Sigma$ is a reference point of Σ. Then Conjecture 1 is true.

□

References

[1] G. Carron, P. Exner, and D. Krejčiřík. Topologically nontrivial quantum layers. *J. Math. Phys.*, 45(2):774–784, 2004.

[2] P. Duclos, P. Exner, and D. Krejčiřík. Bound states in curved quantum layers. *Comm. Math. Phys.*, 223(1):13–28, 2001.

[3] P. Hartman. Geodesic parallel coordinates in the large. *Amer. J. Math.*, 86:705–727, 1964.

[4] A. Huber. On subharmonic functions and differential geometry in the large. *Comment. Math. Helv.*, 32:13–72, 1957.

[5] P. Li. Curvature and function theory on Riemannian manifolds. In *Surveys in differential geometry*, Surv. Differ. Geom., VII, pages 375–432. Int. Press, Somerville, MA, 2000.

[6] C. Lin and Z. Lu. On the discrete spectrum of generalized quantum tubes. *Communications of Partial Differential Equations*, 31:1529–1546, 2006.

[7] C. Lin and Z. Lu. Existence of bounded states for layers built over hypersurfaces in R^{n+1}. *J. Funct. Anal.*, 244(1):1–25, 2007.

[8] C. Lin and Z. Lu. Quantum layers over surfaces ruled outside a compact set. *Journal of Mathematical Physics*, 48:053522, 2007.

[9] R. Sacksteder. On hypersurfaces with no negative sectional curvatures. *Amer. J. Math.*, 82:609–630, 1960.

[10] B. White. Complete surfaces of finite total curvature. *J. Differential Geom.*, 26(2):315–326, 1987.

Department of Mathematics, University of California, Irvine, Irvine, CA 92697
E-mail address, Zhiqin Lu: zlu@math.uci.edu