# Finiteness of volume of moduli spaces

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Abstract: We give a "physics proof" of a conjecture made by the first author at Strings 2005, that the moduli spaces of certain conformal field theories are finite volume in the Zamolodchikov metric, using an RG flow argument.

## Contents



## 1. A conjecture on moduli spaces

A rather fundamental question in string/M theory is whether the number of candidate physical vacua is finite or infinite, and if finite to estimate its number [3, 4]. While addressing this question in all generality is beyond present abilities, it leads to many relatively well posed subquestions, as the first author discussed at Strings 2005 [7].

One of these questions is whether the volume of any moduli space of two-dimensional conformal field theories, defined by integrating the volume form derived from the Zamolodchikov metric, is finite. This type of question was first asked in the physics literature (as far as we know) by Horne and Moore [9]. They were motivated by cosmology, and pointed out that if the total volume were infinite, while the volume for "physically viable" models were finite, there could be a danger that early cosmological dynamics would never produce the viable models. More recently, the question arises because the volume of moduli space is the leading estimate for the number of flux vacua in certain string compactifications [5, 6].

In the primary examples of sigma models with Ricci flat target spaces, part or all of the moduli space metric receives no quantum corrections, and this part is the restriction of the standard metric on the space of metrics to the subspace of Ricci flat metrics. Thus it is a standard metric studied mathematically, such as the Weil-Petersson (WP) metric on complex structure moduli space, or a symmetric space metric. In many cases, these moduli spaces have already been shown to be finite volume – the case of symmetric spaces modulo arithmetic lattices is well known, and another is the WP metric on the moduli space of complex structures of a Calabi-Yau threefold [16, 13, 14]. For more on the Weil-Petersson geometry, we refer to [15, 13].

We only know one counterexample to the claim, namely the $c = 1$ bosonic and the $(1, 1)$ $\hat{c} = 1$ SCFT's, *i.e.* sigma models with target space $S^1$. Of course, if we consider volumes of submanifolds of a moduli space as well, the existence of infinite length paths in moduli space is well known. The basic example of how this is compatible with finite volume is $T^2$ target space, as discussed in [9].

It is not obvious to us whether the case of target space $S^1$ is an isolated exception, or whether it signals a general class of examples. Even if there were other examples, if they were all associated with the large volume limit, they would not be a problem for the finiteness question posed in [4, 7], as there we require finiteness of the number of vacua with a lower bound on the KK scale.

A related and somewhat simpler conjecture is finiteness of the volume of the part of CFT moduli space in which there is a lower bound on non-zero operator dimensions, which we will call the "gap."[1] Our original motivation for considering this version is that a similar condition, namely an upper bound on the diameter, is often made when discussing related questions in Riemannian geometry. In the Strings talk, we explained this in the context of Cheeger's theorem, and we discuss this further in [1]. Another motivation for this form of the conjecture is Kontsevich's work on CFT [10]. He has conjectured that the space of CFT's with fixed $c$ and a lower bound on the gap is precompact, meaning that there is a distance function on the space such that any infinite sequence has a Cauchy subsequence. While this is a weaker, topological claim, it is far more general in the sense that one is not limited to talking about moduli spaces; one can talk about any sequence of CFT's.

In the following, we outline a physical argument for finiteness given a gap, inspired by work of the second author, and which we are pursuing in joint work to appear, along with discussions of the related mathematics and an attempt to compute the actual volume of the moduli space of quintic hypersurfaces.

## 2. An RG approach

At present there is no general definition of CFT which is concrete enough to address the conjecture in general, but a very useful definition for certain cases, such as Calabi-Yau target space, is the gauged linear sigma model (GLSM) [17]. This produces a CFT as the IR limit of an RG flow from a non-trivial but weakly coupled UV theory, and thus the UV limit of the Zamolodchikov metric is computable in perturbation theory. Thus, we adopt a two-step approach, of first showing that the UV metric is finite volume, and then arguing that the RG flow will preserve this finiteness.

While the argument is general, for concreteness, let us discuss the case of the quintic hypersurface in $\mathbb{CP}^4$. The GLSM is now a $(2,2)$ sigma model with a $U(1)$ gauge symmetry and six chiral superfields, five of $U(1)$ charge $Q=+1$ denoted $Z^i$ for $1 \le i \le 5$, and a sixth of $U(1)$ charge $Q=-5$ denoted $P$. Finally, there is a generic superpotential of degree 5 in $Z$ and linear in $P$.

The bare couplings in the model include 126 coefficients in the superpotential, the gauge coupling (a relevant operator), and a Fayet-Iliopoulos term for the $U(1)$ which we denote $\zeta$, complexified by a "theta angle" $\theta$ controlling the operator $\int F$. In addition, one can vary the kinetic term; the only operators here which are not irrelevant are the matrix entering in the quadratic term $g_{i\bar{j}} Z^i \bar{Z}^{\bar{j}}$ and the $P\bar{P}$ term. These are redundant with the overall scale of the superpotential coefficients but we will keep them for now.

The basic structure of the RG flow is simply a flow down to the nonlinear sigma model whose target space is the moduli space of "vacua" (constant solutions of the equations of motion). This can be found in two steps. First, solving the D-term conditions and

[1] We understand that Cumrun Vafa has also considered this variation, in a paper which appeared as this one was finished.

quotienting by $U(1)$, one finds two results depending on the sign of $\zeta$. For $\zeta > 0$, the quotient is the noncompact Calabi-Yau fivefold $\mathcal{O}_{\mathbb{CP}^4}(-5)$, while for $\zeta < 0$ it is the orbifold $\mathbb{C}^5/\mathbb{Z}^5$. The first is just the blowup of the second at zero and is the only crepant resolution of this orbifold singularity, so the two possibilities are closely related in algebraic geometric terms. Second, one considers the superpotential constraint $W' = 0$ in the two cases. In the case of $\mathcal{O}_{\mathbb{CP}^4}(-5)$, the F-term constraint $\partial W/\partial P = 0$ defines a quintic hypersurface, while the others force $P = 0$ so the hypersurface sits in $\mathbb{CP}^4$. In the case of $\mathbb{C}^5/\mathbb{Z}^5$, one has $P \neq 0$, so the conditions $\partial W/\partial Z^i = 0$ are nontrivial and effectively produce a Landau-Ginzburg (LG) model, whose $\mathbb{Z}^5$ quotient is the Gepner model construction. One can check that the two regimes of $\zeta$ are continuously connected if $\theta \neq 0$, and this was Witten's original proof of the CY-LG equivalence.

A slightly more detailed picture of the RG flow can be obtained by noting that the order in which these two steps (D and F term) are implemented depends on the ratio of the energy scales set by the gauge coupling (which controls the D-term potential) and the overall coefficient superpotential. Both are relevant operators and thus one expects their couplings (both of mass dimension 2) to set these scales directly.

Far above both scales, perturbation theory is good, and we can compute a good approximation to the Zamolodchikov metric in free field theory. We will concentrate on the metric on complex structure moduli space; the rest can be computed as well and is easily seen to decouple from the complex structure metric in leading order.

A basis for the variations of complex structure are the 126 operators

$$O_I \equiv O_{i_1 i_2 i_3 i_4 i_5} = P Z^{i_1} Z^{i_2} Z^{i_3} Z^{i_4} Z^{i_5}.$$

whose couplings $t^I$ parameterize $V \equiv \mathbb{C}^{126}$. In free field theory,

$$\langle Z^i(1) \bar{Z}^{\bar{j}}(0) \rangle = g^{i\bar{j}} G(1,0); \qquad \langle P(1) \bar{P}(0) \rangle = G(1,0),$$

where $g^{i\bar{j}}$ is the inverse of the metric appearing in the bare $Z\bar{Z}$ kinetic term, and $G(z_1, z_2)$ is the two-point function for a free boson. Multiplying these to get the two-point function and Zamolodchikov metric

$$G_{I\bar{J}} = \langle O_I(1) \bar{O}_{\bar{J}}(0) \rangle,$$

one finds that this metric is diagonal and non-degenerate, with coefficients given by simple combinatoric factors which will not concern us below.

The true complex structure moduli space is the 101 complex dimensional quotient of $V \equiv \mathbb{C}^{126}$ by a $GL(5)$ action induced by the linear action on the vector $Z^i$. This $GL(5)$ symmetry is broken to $U(5)$ by any explicit choice of the bare metric $g_{i\bar{j}}$ and thus its restoration in the IR is a result of the RG flow, which if non-singular will lose the information of the choice of bare coupling $g_{i\bar{j}}$. Thus, to relate the volume of the moduli space between UV and IR, we must choose a 101-dimensional slice through the space of bare couplngs $V$ which intersects each orbit once. The metric on this slice will then be obtained by restricting the original Zamolodchikov metric on $V$. Dependence on the choice of slice should of course drop out in the IR.

Such a slice can be naturally obtained by symplectic reduction, *i.e.* imposing constraint equations

$$1 = G_{I\bar{J}} t^I \bar{t}^{\bar{J}}$$

and quotienting by the $U(5)$ which preserves $G_{I\bar{J}}$. This can be done in two steps, first quotienting by the $GL(1)$ acting as an overall rescaling of the fields, and then by $SL(5)$. The result of the first quotient is the compact moduli space $\mathbb{CP}^{125}$ with a manifestly nonsingular metric, similar to the Fubini-Study metric, and thus its volume is manifestly finite. The second quotient can introduce singularities in general but nevertheless the volume remains finite.

Before moving on, we recall that not all points in $\mathbb{CP}^{125}//SL(5)$ correspond to non-singular Calabi-Yau manifolds. For this to be the case, the defining equation $f = 0$ must be non-singular, meaning that $\partial f/\partial Z^i \neq 0$ at every point such that $f = 0$. This condition defines an open subset of the quotient space whose complement is a codimension one subvariety called the discriminant locus. Now the actual Weil-Petersson metric on complex structure moduli space is typically singular on the discriminant locus, and thus the claim that the moduli space volume in this metric is finite is rather subtle as one needs to control this divergence. However, the UV approximation we have computed so far is nonsingular at these points. This is no surprise as physically the singularities are related to properties of the Calabi-Yau metrics in these regions of moduli space (for example, the formation of long throat regions) which are not true of the UV metric. It does mean that we should not trust an argument for finiteness which does not address this subtlety in some way.

### 2.1 The RG flow

We hvae just seen that the volume of complex moduli space in the UV is finite. We now need to argue that this property is preserved under RG flow.

The basic physical intuition for this is that the metric will only flow by a finite amount in any finite amount of RG "time" (ratio of energy scales), and non-trivial RG evolution takes place only over a finite time. In principle, this would be made precise by using the RG flow equations for the couplings, to derive an RG flow equation for the Zamolodchikov metric.

We first note that standard nonrenormalization theorems imply that the complex structure moduli $t^I$ themselves do not flow. The flow equation for the Zamolodchikov metric takes the form

$$\mu \frac{\partial}{\partial \mu} G_{I\bar{J}} = \gamma_{I\bar{J}}(t, \rho) \sim \gamma_I(t, \rho) + \gamma_{\bar{J}}(t, \rho).$$

Note that this is not literally a beta function as $G_{I\bar{J}}$ is not a sigma model coupling but a two-point function. However its dependence under coordinate rescaling amounts to a flow, which in some approximation is given by the sum of the anomalous dimension coefficients $\gamma_I$ for $O_I$ and $\gamma_{\bar{J}}$ for $\bar{O}_{\bar{J}}$. In general, this flow will depend on both the point in complex structure moduli space $t$, and on the Kähler moduli $\rho$. Iit can produce a fairly arbitrary metric after a finite flow; let us denote this as

$$G_{I\bar{J}}(\mu) \equiv \langle O_I(\mu^{-1}) \bar{O}_{\bar{J}}(0) \rangle.$$

Indeed, the simple metric we derived in the UV need have little resemblance to the actual Weil-Petersson metric we expect to recover in the IR.

However, given that the anomalous dimensions are finite, the change in any given metric coefficient induced by a finite flow will be finite,

$$\frac{G_{I\bar{J}}(\mu_1)}{G_{I\bar{J}}(\mu_2)} \sim \left(\frac{\mu_1}{\mu_2}\right)^\gamma$$

where $\gamma = 0$ for a truly marginal operator and might be expected to be typically (though not always) small.

While the total RG time is infinite, one can see that most of the flow takes place around the energy scales $g$ set by the gauge coupling and $|W|$ set by the superpotential, and is small in the extreme UV and IR. In the UV, this follows from standard results on the behavior of perturbation theory, while in the IR, it follows from the assumption that, at a given point in moduli space $(t, \rho)$, the flow to the IR fixed point comes in along a leading irrelevant direction controlled by an operator with dimension $2 + \epsilon$ with some $\epsilon > 0$. In this case, the estimate for the total flow in any coupling or correlation function induced by a flow from a scale $\mu_0$ at which this assumption is valid, to the IR, is

$$G_{I\bar{J}}(\mathrm{IR}) \sim G_{I\bar{J}}(\mu_0) + O(\mu_0^\epsilon).$$

Thus this is again expected to be finite.

This argument can be applied pointwise in the moduli space and thus leads to finiteness, if its assumptions are satisfied. Conversely, it suggests that any possible divergence in the Zamolodchikov metric as we approach some limiting point $p$ in moduli space, and thus a divergence in the volume form, can be traced back to the fact that the RG time required for the flow diverges as we approach $p$. The other possibility, that beta function coefficients themselves diverge, seems unphysical.

Such a divergence of the RG flow time could appear for at least two reasons. First, it could be that at $p$, the RG flow actually does not reach a $\hat{c} = 3$ SCFT, but instead stops at an intermediate fixed point with $3 < \hat{c} < 6$. In this case, as we approach $p$, the RG flows will get "hung up" near the intermediate fixed point, for a time which grows as we approach $p$ as a power controlled by the dimension of the leading relevant operator coming out of $p$. We do not know of evidence that this happens in the case at hand, but it is an interesting possibility.

Second, it could be that the assumption that there is a gap $\epsilon > 0$ to the leading irrelevant operator fails, and the flow takes infinite time in the IR. Clearly this assumption can fail; it is not at all obvious even in the large volume limit (there is no shortage of candidate operators there), and presumably must fail near a conifold point, and other limits in which metric coefficients can diverge. Indeed, in the degeneration limit of the conifold, the gap goes to zero [12], as follows from a candidate $\hat{c} = 3$ SCFT description of the noncompact CY containing the conifold region [11]. Thus this mechanism can produce the known logarithmic divergence in the moduli space metric.

Thus, up to the possible existence of intermediate fixed points, we have "physically proven" a weak form of the finiteness conjecture, that the volume of moduli space in regions with a gap is finite.

Since already the conifold CFT does not have a gap, and it presumably can be part of a physically realistic compactification (one is not sending the KK scale to zero but rather a scale associated with the throat region, which could decouple from the physical sector), this is rather weaker than the conjecture in [7], but is clearly a step in this direction, as it shows that the "bulk" of moduli space has finite volume, and relates moduli space properties to the RG in an interesting way. We believe that incorporating more of the mathematics which enters the proof of the finiteness of the total moduli space volume will lead to a better physical argument, and are pursuing this.

We conclude by mentioning a much stronger claim which is suggested by the work of the second author and which we are also exploring in our ongoing work. It is that the RG flow preserves the Kähler class of the moduli space metric, in the sense that the difference of the two Kähler forms $\omega_{IR} - \omega_{UV}$ is exact and sufficiently weakly singular that the volume does not contain boundary terms. Again, there are simple arguments for the exactness, which we will give in [8]; the subtle part of this claim is the weakness of the singularity, which allows computing the total volume of the moduli space by integrating the very simple volume form $\omega_{IR}^n/n!$. This point may be within mathematical reach and would allow us to estimate this total volume, completing the estimate of the asymptotic number of attractor points begun in [5] for a high dimensional case. Another important generalization would be to discuss the curvature form of [2].

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