Solutions to a Class of Multidimensional SPDEs

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Abstract

In this paper, we consider multi-dimensional SPDEs of parabolic type with space-time white noise. We discretize the space-time white noise to independently identically distributed time white noise located on configuration space and then consider the convergence of the discretized solution $u(t,x,n)$. We first prove that in general the laws of $u(t,x,n)dtdx$ form a tight sequence and the limit is the law of some measure-valued random variable which gives a weak integral solution in the linear case. For the stochastic multi-dimensional KPP equation, we prove that the solution is real-valued. This is the first example of SPDEs discovered so far in multi-dimensions with real-valued solutions.

1 Introduction

In this paper, we study the multi-dimensional SPDE of the following form:

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) + f[u(t,x)] \cdot \dot{W}(t,x), \tag{1.1}$$

where $\dot{W}(t,x)$ is the space-time white-noise on $R^d \times R_+$ (see [24], [10], [20]). There have been many good results in the case where $d=1$ (see [24], [17], [20], [22], [10] for references). However, in the higher dimensional ($d>1$) case, there are only results for the linear case where $f(u)=u$ (see [16], [18], [1], [23]). The main difficulty is that the solutions in multi-dimensional case was known as only taking values in L. Schwartz's distribution space ([10]). Certainly it is difficult to understand the non-linear function of such a generalized random field (see [20]).

We are going to use a random homogenization method ([8], [19], [13]) to discretize Equations (1.1) when $f(u)=u$ and $f(u)=u(1-u)$. While we denote by $\{u(t,x,n)\}$ the discretized solutions, we are going to show that the laws of $\{u(t,x,n)dxdt\}_n$ form a tight sequence and any of their limits is the law of some measure-valued random variable. So it suggests to consider any of the limits as an integral solution to (1.1). We hope this method will enable us to understand multi-dimensional SPDE in an alternative way.

When the above integral solution is differentiable with derivative $u(s,x)$, it is natural to consider $u(s,x)$ as a solution to (1.1) (see Theorem 3.3.)

In sections 4 and 5, we study the following stochastic reaction diffusion equation with space-time white noise in R^d:

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) + (1-u(t,x))u(t,x) \cdot \dot{W}(t,x), \tag{1.2}$$

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with an initial condition $u(0,x)=u_0(x)$. Here we assume that $u_0\in C^2$ and $0\le u_0\le 1$. This equation (when replace $\dot W(t,x)$ by 1) is known as KPP equation arised in mathematical biology, chemical engineering and population dynamics. With space-time white noise, the case for $d=1$ was considered in [22] and the solution is real-valued. But this equation used to be considered meaningless for $d>1$ as solution for high dimensional SPDE used to be considered as a distribution so that $(1-u)u$ did not make sense if u is just a distribution-valued random variable. However in this paper we will prove that this equation has a real-valued solution for any dimension. It is interesting that the solution to the non-linear equation behaves better than linear equations. This phenomenon for the deterministic equation was noted in [5].

The method for Equation (1.2) can be generalized easily to more general equation in R^d e.g.

$$\frac{\partial}{\partial t}u(t,x)=\frac{1}{2}\Delta u(t,x)+(a-u(t,x))u(t,x)h(u(t,x))+(a-u(t,x))u(t,x)g(u(t,x))\cdot\dot W(t,x), \tag{1.3}$$

for any C^2 functions h and g and assuming that if $u\ge 0$, $u(1-u)h(u)\le cu$ for a positive constant $c>0$. Here $a>0$ is a constant. For instance, $h(u)=a^m+a^{m-1}u+\cdots+au^{m-1}+u^m$, where m is an integer, satisfies the requirements. Assume that $0\le u_0(x)\le a$. We then still have the existence of a real-valued solution to Equation (1.3).

2 Discretize the space noise

Decompose R^d into the squares:

$$L^{(n)}_{k_1,\cdots,k_d}=\{(x_1,x_2,\cdots,x_d):\tfrac{1}{n}k_j\le x_j\le\tfrac{1}{n}(k_j+1),\ \text{for } j=1,2,\cdots,d\},$$
$$k_1,k_2,\cdots,k_d=0,\pm1,\pm2,\cdots.$$

Denote $\mathcal{K}=(k_1,k_2,\cdots,k_d)$ and define

$$W_n(t,x,\omega)=n^{\frac{d}{2}}\sum_{\mathcal{K}}a(x,\mathcal{K})\dot w(t,\mathcal{K}) \tag{2.1}$$

where $w(t,\mathcal{K})$, $\mathcal{K}\in Z^d$, are independent Wiener processes of 1-parameter t on a probability space $(\Omega,\mathcal{F},P)$ and $\dot w(t,\mathcal{K})$ denotes the Ito derivative of the Wiener process. Assume $a(x,\mathcal{K})$ is a C^∞ function and satisfies the following conditions: for any $\mathcal{K}:L^n_{\mathcal{K}}\subset\{x\in R^d:|x_i|\le n, i=1,2,\cdots,d\}$, $\int_{L^{(n)}_{\mathcal{K}}}a(x,\mathcal{K})dx=\frac{1}{n^d}$ and $a(x,\mathcal{K})=0$ for $x\in R^d-L^{(n)}_{\mathcal{K}}$ and $a(x,\mathcal{K})=0$ for all other $\mathcal{K}$. For simplicity we assume that all $a(x,\mathcal{K})$ are identical except for a shift.

Then it is easy to see from the central limit theorem that for any block $B=\{(t,x_1,x_2,\cdots,x_d): t_1\le t\le t_2,\ a_j\le x_j\le b_j,\ \ j=1,2,\cdots,d\}$,

$$\begin{aligned}\int_B W_n(t,x)dxdt &= \int_{[a_1,b_1]\times\cdots\times[a_d,b_d]}\int_{t_1}^{t_2}n^{\frac{d}{2}}\sum_{\mathcal{K}}a(x,\mathcal{K})w(dt,\mathcal{K})dx\\ &= \int_{t_1}^{t_2}\sum_{\mathcal{K}}\int_{L_{\mathcal{K}}\cap([a_1,b_1]\times\cdots\times[a_d,b_d])}n^{\frac{d}{2}}a(x,\mathcal{K})dxw(dt,\mathcal{K})\\ &= n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_{L_{\mathcal{K}}\cap([a_1,b_1]\times\cdots\times[a_d,b_d])}a(x,\mathcal{K})dx(w(t_2,\mathcal{K})-w(t_1,\mathcal{K})).\end{aligned}$$

Therefore

$$\begin{aligned}
& E(\int_B W_n(t,x)dxdt)^2 \\
= \quad & n^d \sum_{\mathcal{K}} \{ \int_{L_{\mathcal{K}} \cap ([a_1,b_1]\times\cdots\times[a_d,b_d])} a(x,\mathcal{K})dx\}^2 E(w(t_2,\mathcal{K}) - w(t_1,\mathcal{K}))^2 \\
= \quad & n^d \sum_{\mathcal{K}} \{ \int_{L_{\mathcal{K}} \cap ([a_1,b_1]\times\cdots\times[a_d,b_d])} a(x,\mathcal{K})dx\}^2 (t_2 - t_1) \\
\to \quad & (b_1 - a_1) \times (b_2 - a_2) \times \cdots \times (b_d - a_d) \times (t_2 - t_1), \text{ as } n \to \infty.
\end{aligned}$$

So $(\int_B W_n dxdt)$ as a sequence of sum of independent random variables, as $n \to \infty$, converges in law to a normally distributed random variable with mean 0 and variance $\int_B dxdt$. Moreover, if B_1 and B_2 are disjoint, then $(\int_{B_1} W_n dxdt)$ is independent of $(\int_{B_2} W_n dxdt)$. Thus the limiting random process is a multi-parameter Brownian sheet. Thus, we can say that W_n converges in law to a "white-noise" which is regarded as the weak derivative of the Brownian sheet. By using the celebrated Skorohod's lemma, we can select a new probability space and assume that W_n converges almost everywhere to the white-noise $\dot{W}$. We will fix this new probability space and still denote it by Ω.

Consider the following stochastic parabolic equation with initial condition in R^d:

$$\begin{aligned}
& du(t,x,n) = \frac{1}{2}\Delta u(t,x,n)dt + n^{\frac{d}{2}} \sum_{\mathcal{K}} a(x,\mathcal{K})u(t,x,n)dw(t,\mathcal{K}), \\
& u(0,x,n) = u_0(x).
\end{aligned} \tag{2.2}$$

Here u_0 is a nonnegative and bounded C^2 function.

The solution of (2.2) is given by the Feynman-Kac integration. The proof of the following Lemma is standard (see, for example, Kunita [11], and Lemma 7.1 in [6] for the proof of (2.4)).

Lemma 2.1 *For any fixed n, equation (2.2) has a unique solution $u(t,x,n)$ which is C^2 in x and and continuous in t and*

$$\begin{aligned}
u(t,x,n) = \quad & \hat{E}u_0(X_{t,0}(x)) \exp\{-\frac{1}{2}n^d \sum_{\mathcal{K}} \int_0^t a^2(X_{t,s}(x),\mathcal{K})ds \\
& + n^{\frac{d}{2}} \sum_{\mathcal{K}} \int_0^t a(X_{t,s}(x),\mathcal{K})dw(s,\mathcal{K})\}.
\end{aligned} \tag{2.3}$$

Here $X_{t,s}$ is the inverse of the Brownian flow in R^d on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ and $\hat{E}$ denotes the expectation over that probability space. The stochastic integral $\int_0^t a(X_{t,s}(x),\mathcal{K})dw(s,\mathcal{K})$ is defined to be Stratonovich integral on the product probability space $(\Omega \times \hat{\Omega}, \mathcal{F} \times \hat{\mathcal{F}}, P \times \hat{P})$. It coincides with the Ito integral as $X_{t,s}(x)$ is independent of $w(s,\mathcal{K})$ for any $\mathcal{K}$. Moreover, we have the following estimate, for $x \in L_{\tilde{\mathcal{K}}}$, for certain $\tilde{\mathcal{K}} \in Z^d$,

$$\begin{aligned}
& \hat{E}\exp\{n^{\frac{d}{2}} \sum_{\mathcal{K}} \int_0^t a(X_{t,s}(x),\mathcal{K})dw(s,\mathcal{K})\} \\
\leq \quad & \exp\{n^{\frac{d}{2}} a(x,\tilde{\mathcal{K}})w(t,\tilde{\mathcal{K}})\} \left\{ \hat{E}\exp\{2n^d \sum_{\mathcal{K}} \int_0^t (Da(X_{t,s}(x),\mathcal{K})w(s,\mathcal{K}))^2 ds\} \right\}^{\frac{1}{2}}.
\end{aligned} \tag{2.4}$$

Therefore (2.3) is bounded P a.s..

In general, we consider

$$du(t,x,n) = \frac{1}{2}\Delta u(t,x,n)dt + n^{\frac{d}{2}}\sum_{\mathcal{K}} a(x,\mathcal{K})f(u(t,x,n))dw(t,\mathcal{K}),$$
$$u(0,x,n) = u_0(x). \tag{2.5}$$

Lemma 2.2 *Suppose for any n, the solution $u(t,x,n)$ to (2.5) exists and is nonnegative and $u_0 \in C_0^2(R^d)$ with a compact support $G \subset R^d$, then there exist constants $C_2 \geq C_1 > 0$ such that for any $n > 0$,*

$$C_1 \exp\{-\frac{(\mathrm{dist}(x,G))^2}{2t}\} \leq Eu(t,x,n) \leq C_2 \exp\{-\frac{(\mathrm{dist}(x,G))^2}{2t}\}, \tag{2.6}$$

and therefore

$$E\int_{R^d} u(t,x,n)dx < \infty. \tag{2.7}$$

Proof. Taking the expectation to both sides of (2.5), we have

$$\frac{\partial Eu(t,x,n)}{\partial t} = \frac{1}{2}\Delta Eu(t,x,n). \tag{2.8}$$

So $v(t,x) = Eu(t,x,n)$ is the solution of the deterministic heat equation with initial condition $v(0,x) = u_0(x)$, and is independent of n. As $u_0(x)$ has the compact support $G \subset R^d$, it is well known that there exist $C_2 \geq C_1 > 0$ (independent of n of course) such that

$$C_1 \exp\{-\frac{(\mathrm{dist}(x,G))^2}{2t}\} \leq v(t,x) \leq C_2 \exp\{-\frac{(\mathrm{dist}(x,G))^2}{2t}\}, \tag{2.9}$$

for a constant $C > 0$. So we have (2.6). It turns out that

$$\int_{R^d} v(t,x)dx < \infty,$$

for any $t \geq 0$. This is followed by (2.7) easily. ‡‡

In the next section, we will always assume $u_0 \in C_0^2(R^d)$ and nonnegative.

3 Tightness results

Let $\overline{R^d} = R^d \bigcup \{\infty\}$. Denote by $\overline{\mathcal{P}}_b$ the set of all measure on $[0,1] \times \overline{R^d}$ bounded by positive b. So $\overline{\mathcal{P}}_b$ is a compact polish space equipped with the topology of measure convergence, which is the least fine topology to make all the mapping $\mu : \int_{\overline{R^d}} f(x)\mu(dx)$ continuous for all bounded continous function $f(x)$ defined on $\overline{R^d}$ (see, for example, [3] III, 60). Denote $\tilde{\mathcal{P}} = \bigcup_b \overline{\mathcal{P}}_b$. When we equip $\tilde{\mathcal{P}}$ again with the topology of measure convergence, it is easy to see that for each $0 < b < \infty$,

$$\{\mu \in \tilde{\mathcal{P}},\ \mu[\overline{R^d}] < b\}$$

is an open set and its closure is $\overline{\mathcal{P}}_b$. Thus $\tilde{\mathcal{P}}$ is locally compact polish space. Denote by $\overline{\mathcal{P}}$ its one-point compactification. Finally, denote by $\mathcal{P}$ the set of all bounded measure on $[0,1]\times R^d$. Consider $V_n(dt,dx)=u(t,x,n)dtdx$ as a sequence of random variable taking values in $\mathcal{P}$. Here we assume $u(t,x,n)$, the solution to (2.5), exists and is nonnegative.

Theorem 3.1 *Suppose $u_0\in C_0^2(R^d)$ and nonnegative. There is a subsequence V_{n_k} which converges to a $\mathcal{P}-$valued random variable V in law.*

Proof. 1) Denote by P_n the laws of V_n on $\overline{\mathcal{P}}$. Since $\overline{\mathcal{P}}$ is compact and seperable, there is a subsequence P_{n_k} which converges to some P_∞. We are going to show that P_∞ is carried by $\tilde{\mathcal{P}}$. We have

$$P_n[V_n([0,1]\times\overline{R^d})>c]\le c^{-1}E[V_n([0,1]\times\overline{R^d})]=c^{-1}\int_0^1\int_{R^d}Eu(t,x,n)dxdt.$$

Since $\{\nu\in\overline{\mathcal{P}};\ \nu([0,1]\times\overline{R^d})>c\}$ is open in $\overline{\mathcal{P}}$,

$$P_\infty[\nu\in\overline{\mathcal{P}};\ \nu([0,1]\times\overline{R^d})>c]\le c^{-1}\liminf_k\int_0^1\int_{R^d}Eu(t,x,n_k)dxdt,$$

where we used the fact that weak convergent measures reduce their probabilities in the limit on open sets (see, for example, [7] p.108). When $c\to\infty$, the right-hand side of the above inequality tends to 0. That is, P_∞ is carried by $\tilde{\mathcal{P}}$.

2) Now let us show that P_∞ is carried by $\mathcal{P}$. Indeed, for fixed positive pairs κ and c,

$$\{\nu\in\mathcal{P},\ \nu([0,1]\times\{x\in R^d;\ |x|>c\})>\kappa\}$$

is open in $\tilde{\mathcal{P}}$. We have easily

$$P_n[V_n([0,1]\times\{x\in R^d;\ |x|>c\})>\kappa]\le\kappa^{-1}E[\int_0^1\int_{|x|>c}u(t,x,n)dxdt],$$

where the right-hand tends to 0 uniformly when $c\to\infty$. Thus, we deduce that P_∞ carries on $\mathcal{P}$. ‡‡

Since $\mathcal{P}$ is a Polish space, by the celebrated Skorohod's Lemma, we can assume that V_n and V are all defined on the same probability space and V_n converges to V almost surely.

Consider equation (2.2) and its solution $u(t,x,n)$. Given any C_0^2-function $\phi(t,x)$ on $[0,1)\times R^d$, by (2.2),

$$\begin{aligned}&\int_{R^d}\int_0^1\phi(t,x)\frac{\partial}{\partial t}u(t,x,n)dtdx-\int_{R^d}\frac{1}{2}\int_0^1\phi(t,x)\Delta u(t,x,n)dtdx\\ =\ &\int_{R^d}\int_0^1\phi(t,x)W_n(t,x,\omega)u(t,x,n)dtdx.\end{aligned}$$

Using integration by parts formula,

$$\begin{aligned}&\int_{R^d}\int_0^1\phi(t,x)W_n(t,x,\omega)u(t,x,n)dtdx\\ =\ &-\int_{R^d}\phi(0,x)u_0(x)dx-\int_{R^d}\int_0^1u(t,x,n)\frac{\partial}{\partial t}\phi(t,x)dtdx-\int_{R^d}\frac{1}{2}\int_0^1u(t,x,n)\Delta\phi(t,x)dtdx\\ \to\ &-\int_{R^d}\phi(0,x)u_0(x)dx-\int_{R^d}\int_0^1\frac{\partial}{\partial t}\phi(t,x)V(dt,dx)-\int_{R^d}\frac{1}{2}\int_0^1\Delta\phi(t,x)V(dt,dx),\end{aligned}$$

as $n \to \infty$, where $V(dt, dx)$ is given by the previous Theorem. Thus we may consider the last side as the definition of

$$\int_{R^d} \int_0^1 \phi(t,x)\dot{W}(t,x).V(dt,dx)$$

where $\dot{W}(.,.)$ is the space-time whit noise. Thus V gives a weak solution to (1.1) when $f(u) = u$ with initial condition $u(0,x) = u_0(x)$.

Let $F_{n,t}$ be the natural filtration of $\{W_n(t,x)\}_x$. Let us consider a sequence of new probability space $(\Omega \times R^d, F_{n,1} \times \mathcal{B}, P \times \mu)$ with filtration $(F_{n,t} \times \mathcal{B})_{t \le 1}$ where $\mathcal{B}$ is the Borel $\sigma-$algebra of subsets in R^d and $\mu(dx)$ gives the standard normal law on R^d. Now we consider $u(t,x,n)$ as a process (depending only on t) on this new probability space $R^d \times \Omega$,

$$\sup_n \{\int_{\Omega \times R^d} [\int_0^1 u(t,x,n)dtd(P \times \mu)\} \le \sup_n \int_0^1 \int_{R^d} Eu(t,x,n)dxdt < \infty. \tag{3.1}$$

Then we can obtain from Theorem 4 in [15],

Theorem 3.2 *$\{\{\int_0^s u(t,.,n,.)dt\}_{s \le 1}\}_n$ forms a tight sequence and any limiting process is of the form $\{\xi(s,x)\}_{s \le 1}$ where $\xi(s,.)$ is an increasing process of s for almost all (x,ω).*

If we can have more uniform control on $u(t,x,n)$, then we have the following

Theorem 3.3 *Suppose there is $p > 1$ such that the following condition is satisfied:*

$$\sup_n \int_{\Omega \times R^d} [\int_0^1 |u(t,x,n)|^p dt]d(P \times \mu) < \infty. \tag{3.2}$$

Then $\xi(s,x)$ in Theorem 3.2 is differentiable with respect to s almost surely under $dP \times d\mu$. Denote $u(s,x) = \frac{\partial \xi(s,x)}{\partial s}$. Moreover, $u(s,x) = \frac{\partial^{d+1} V}{\partial s \partial x_1 \cdots \partial x_d}$ where V is given by Theorem 3.1. So u gives a weak solution to (1.1).

Proof. According to Theorem 3 of [25], (3.2) implies

$$\int_{\Omega \times R^d} [\int_0^1 |\frac{\partial \xi(s,x)}{\partial s}|^p ds]d(P \times \mu) < \infty.$$

So we get the first statement.

Let us consider the second statement. By using the Skorohod's lemma, we may assume that for almost every (ω, x), and all bounded continuous $\phi(s)$,

$$\int_0^1 \phi(s)u(s,x,n,\omega)ds \to \int_0^1 \phi(s)u(s,x,\omega)ds, \text{ as } n \to \infty.$$

Let $\psi(s,x)$ be any bounded continuous function on $[0,1] \times \overline{R^d}$. For any $M < \infty$, by the dominated convergence theorem

$$\int_{|x|<M} \int_0^1 \psi(s,x)u(s,x,n,\omega)dsdx \to \int_{|x|<M} \int_0^1 \psi(s,x)u(s,x,n,\omega)dsdx, \text{ as } n \to \infty.$$

On the other hand, by (2.6), $\int_{|x|\geq M}\int_0^1 \psi(s,x)u(s,x,n,\omega)dsdx$ tends to 0 in probability (uniformly in n) when $M\to\infty$. Hence for any given $\epsilon>0$,

$$P[|\int_{R^d}\int_0^1 \psi(s,x)u(s,x,n,\omega)dsdx - \int_{R^d}\int_0^1 \psi(s,x)u(s,x,\omega)dsdx| > \epsilon] \to 0, \qquad \text{as } n\to\infty.$$

Take a sequence of bounded continuous functions $\{\psi_k(s,x)\}$ which are dense in uniform convergence norm in the space of all bounded contionuous functions on $[0,1]\times\overline{R^d}$. By diagonal line method, we may find a subsequence $u(s,x,n_m,\omega)$ such that

$$P[\int_{R^d}\int_0^1 \psi_k(s,x)u(s,x,n_m,\omega)dsdx \to \int_{R^d}\int_0^1 \psi_k(s,x)u(s,x,\omega)dsdx \text{ as } m\to\infty] = 1.$$

On the other hand, $u(s,x,n_m,\omega)dsdx$ tends to V. So we get the second conclusion. ‡‡

4 Nonlinear stochastic reaction diffusion equations with time white noise

In this section, we study the following semilinear SPDE with global Lipschitz nonlinear term

$$\begin{aligned} dv(t,x,n) = & \quad \frac{1}{2}\Delta v(t,x,n)dt + n^{\frac{d}{2}}\sum_{\mathcal{K}} a(x,\mathcal{K})F(v(t,x,n))dw(t,\mathcal{K}), \\ v(0,x,n) = & \quad u_0(x). \end{aligned} \tag{4.1}$$

Assume that $u_0\in C^2(R^d)$ is bounded and $F(v)$ satisfies the following global Lipschitz condition: there exists an $L>0$ such that

$$|F(v_1)-F(v_2)| \leq L|v_1-v_2|. \tag{4.2}$$

The mild solution of the equation (4.1) is defined as the solution of the following integral equation if exists ([2])

$$v(t,x,n) = \int_{R^d} p_t(y,x)u_0(y)dy + n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d} p_{t-s}(y,x)a(y,\mathcal{K})F(v(s,y,n))dydw(s,\mathcal{K}). \tag{4.3}$$

Here $p_t(y,x)$ is the heat kerkel of the Laplacian operator $\frac{1}{2}\Delta$ on R^d:

$$p_t(y,x) = p_t(y-x) = \frac{1}{(2\pi t)^{\frac{d}{2}}}\exp\{-\frac{|y-x|^2}{2t}\}.$$

It has been proved in Walsh [24] that the mild solution is equivalent to the weak solution which is defined using test functions.

The following is a special case of the well-known Burkholder inequality. We list it here ready to use in the proof of lemma 4.2.

Lemma 4.1 *(Burkholder inequality) Assume $f(s)$ is $\mathcal{F}_s$ measurable and $X(t) = \int_0^t f(s)dw(s)$, then for any $p \geq 1$,*

$$EX^{2p}(t) \leq (2p-1)^p t^{p-1} \int_0^t Ef^{2p}(s)ds. \tag{4.4}$$

The following theorem first gives the existence and uniqueness of mild solution associated with (4.3). Then under some extra conditions on F and u_0, we prove that the mild solution turns out to be a strong solution to equation (4.1). Some of the ideas in our proof are inspired by stochastic flow theory for stochastic ordinary differential equations (see e.g. [4], [11]). We don't strike here the minimal conditions on F and u_0 needed for the regularity results as our purpose in this paper is to apply the results to stochastic KPP equation (5.1) with time noise. Actually following our proof, one can find easily that if F, F', F'' satisfy global Lipschitz conditions and linear growth conditions, the results of the following theorem still holds. Actually, one only needs one more estimate that for any $p \geq 1$,

$$E\{v(t,x,n)\}^{2p} \leq C(n,p,t), \tag{4.5}$$

for a constant $C(n,p,t) > 0$. This follows from our proof and the linear growth condition of F easily.

Theorem 4.2 *Assume F satisfies the global Lipschitz condition (4.2) and u_0 is bounded. Then there exists a unique solution $u(t,x,n) \in L^2(\Omega)$ to equation (4.3) for any $t \geq 0$. Assume further that $u_0 \in C^2(R^d)$ and $F \in C^2(R^1)$ and F, F' and F'' are bounded, then the mild solution is a strong solution to the equation (4.1) that is C^2 with respect to x.*

Proof. Define a Banach space:

$$\begin{aligned} S = \quad & \{v : \text{ for each } t \in [0,1], v(t,x,\omega) \text{ is continuous in } t, \text{ and } x \text{ and is } \mathcal{F}_t \text{ measurable} \\ & \text{and } v \in L^2(\Omega)\}, \end{aligned}$$

with the norm:

$$|||v||| = \sup_{0 \leq t \leq 1} \sup_x \sqrt{Ev^2(t,x)}.$$

Define a map: $\theta : S \to S$ by

$$\begin{aligned} \theta(v)(t,x,n) = \quad & \int_{R^d} p_t(y,x)u_0(y)dy \\ & + n^{\frac{d}{2}} \sum_{\mathcal{K}} \int_0^t \int_{R^d} p_{t-s}(y,x)a(y,\mathcal{K})F(v(s,y,n))dydw(s,\mathcal{K}). \end{aligned} \tag{4.6}$$

Then for $v_1, v_2 \in S$,

$$\begin{aligned} & E(\theta(v_1)(t,x,n) - \theta(v_2)(t,x,n))^2 \\ = \quad & n^d E\{\sum_{\mathcal{K}} \int_0^t \int_{R^d} p_{t-s}(y,x)a(y,\mathcal{K})(F(v_1(s,y,n)) - F(v_2(s,y,n)))dydw(s,\mathcal{K})\}^2 \end{aligned}$$

$$
\begin{aligned}
&= n^d \sum_{\mathcal{K}} E \int_0^t \{ \int_{R^d} p_{t-s}(y,x) a(y,\mathcal{K}) (F(v_1(s,y,n)) - F(v_2(s,y,n))) dy \}^2 ds \\
&\le n^d \sum_{\mathcal{K}} E \int_0^t \int_{R^d} p_{t-s}(y,x) a^2(y,\mathcal{K}) dy \\
&\qquad \times \int_{R^d} p_{t-s}(y,x) (F(v_1(s,y,n)) - F(v_2(s,y,n)))^2 dy ds \\
&\le L ||a||_\infty^2 n^d \sum_{\mathcal{K}} \int_0^t \int_{R^d} p_{t-s}(y,x) E(v_1(s,y,n) - v_2(s,y,n))^2 dy ds \\
&\le L ||a||_\infty^2 n^d \sum_{\mathcal{K}} \sup_{0 \le s \le t} \sup_y E(v_1(s,y,n) - v_2(s,y,n))^2 \int_0^t \int_{R^d} p_{t-s}(y,x) dy ds \\
&\le L ||a||_\infty^2 n^d |\mathcal{K}| t \sup_{0 \le t \le 1} \sup_x E(v_1(t,x,n) - v_2(t,x,n))^2. \qquad (4.7)
\end{aligned}
$$

Here $||a||_\infty = \sup_x |a(x,\mathcal{K})|$, $|\mathcal{K}|$ is the number of the indices in $\mathcal{K}$. We have used the Lipschitz condition of F, the identity $\int_{R^d} p_{t-s}(y,x) dy = 1$ and the Hölder inequality. Therefore we have proved that

$$
|||\theta(v_1) - \theta(v_2)||| \le (\sqrt{L ||a||_\infty^2 n^d |\mathcal{K}| t}) |||v_1 - v_2|||. \qquad (4.8)
$$

It then follows from the contraction principle that there is a unique fixed point $v \in S$ of θ for all $t \le t_0$ for t_0 satisfying $L||a||_\infty^2 n^d |\mathcal{K}| t_0 < 1$. It is evident that v is the solution of the stochastic integral equation (4.3). This solution can be extended to $t \in [0,T]$ for any $T > 0$ as we can use the same argument on $[t_0, 2t_0]$, etc.

For the regularity, we first need to prove the Lipschitz continuity of $v(t,x,n)$ in x. For this we consider for $x_1, x_2 \in R^d$,

$$
\begin{aligned}
& v(t,x_1,n) - v(t,x_2,n) \\
&= \int_{R^d} p_t(y,x_1) u_0(y) dy - \int_{R^d} p_t(y,x_2) u_0(y) dy \\
&\quad + n^{\frac{d}{2}} \sum_{\mathcal{K}} \int_0^t \int_{R^d} p_{t-s}(y,x_1) a(y,\mathcal{K}) F(v(s,y,n)) dy dw(s,\mathcal{K}) \\
&\quad - n^{\frac{d}{2}} \sum_{\mathcal{K}} \int_0^t \int_{R^d} p_{t-s}(y,x_2) a(y,\mathcal{K}) F(v(s,y,n)) dy dw(s,\mathcal{K}) \\
&= \int_{R^d} p_t(y,x_1) u_0(y) dy - \int_{R^d} p_t(y,x_2) u_0(y) dy \\
&\quad + n^{\frac{d}{2}} \sum_{\mathcal{K}} \int_0^t \int_{R^d} p_{t-s}(y) a(y+x_1,\mathcal{K}) F(v(s,y+x_1,n)) dy dw(s,\mathcal{K}) \\
&\quad - n^{\frac{d}{2}} \sum_{\mathcal{K}} \int_0^t \int_{R^d} p_{t-s}(y) a(y+x_2,\mathcal{K}) F(v(s,y+x_2,n)) dy dw(s,\mathcal{K}) \\
&= \int_{R^d} p_t(y,x_1) u_0(y) dy - \int_{R^d} p_t(y,x_2) u_0(y) dy \\
&\quad + n^{\frac{d}{2}} \sum_{\mathcal{K}} \int_0^t \int_{R^d} p_{t-s}(y) (a(y+x_1,\mathcal{K}) - a(y+x_2,\mathcal{K})) F(v(s,y+x_1,n)) dy dw(s,\mathcal{K})
\end{aligned}
$$

$$
\begin{aligned}
&+n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d}p_{t-s}(y)a(y+x_2,\mathcal{K})\\
&\qquad\qquad\{F(v(s,y+x_1,n))-F(v(s,y+x_2,n))\}dydw(s,\mathcal{K}).
\end{aligned}\tag{4.9}
$$

We need moment estimates. For this we use the inequality $(a+b+c)^{2p}\le 2^{2(2p-1)}(a^{2p}+b^{2p}+c^{2p})$ (see e.g. [4]), Burkholder inequality (4.4) and Hölder inequality,

$$
\begin{aligned}
& E(v(t,x_1,n)-v(t,x_2,n))^{2p}\\
\le\ & 2^{2(2p-1)}(\int_{R^d}p_t(y,x_1)u_0(y)dy-\int_{R^d}p_t(y,x_2)u_0(y)dy)^{2p}\\
& +2^{2(2p-1)}(2p-1)^p n^{pd}t^{p-1}\sum_{\mathcal{K}}\int_0^t E\{\int_{R^d}p_{t-s}(y)(a(y+x_1,\mathcal{K})-a(y+x_2,\mathcal{K}))\\
&\qquad\qquad F(v(s,y+x_1,n))dy\}^{2p}ds\\
& +2^{2(2p-1)}(2p-1)^p n^{pd}t^{p-1}\sum_{\mathcal{K}}\int_0^t E\{\int_{R^d}p_{t-s}(y)a(y+x_2,\mathcal{K})\\
&\qquad\qquad \{F(v(s,y+x_1,n))-F(v(s,y+x_2,n))\}dy\}^{2p}ds\\
\le\ & M|x_1-x_2|^{2p}+2^{2(2p-1)}(2p-1)^p n^{pd}t^{p-1}||a||_\infty^{2p}|\mathcal{K}|\int_0^t E\int_{R^d}p_{t-s}(y)\\
&\qquad\qquad \{F(v(s,y+x_1,n))-F(v(s,y+x_2,n))\}^{2p}dyds\\
\le\ & M|x_1-x_2|^{2p}+2^{2(2p-1)}(2p-1)^p n^{pd}t^{p-1}||a||_\infty^{2p}|\mathcal{K}|L^{2p}\int_0^t\int_{R^d}p_{t-s}(y)\\
&\qquad\qquad E\{v(s,y+x_1,n)-v(s,y+x_2,n)\}^{2p}dyds\\
\le\ & M|x_1-x_2|^{2p}+2^{2(2p-1)}(2p-1)^p n^{pd}t^{p-1}||a||_\infty^{2p}|\mathcal{K}|L^{2p}\\
&\qquad \int_0^t\sup_{x_1,x_2:0<|x_1-x_2|<h_0}E\{v(s,x_1,n)-v(s,x_2,n)\}^{2p}ds.
\end{aligned}
$$

Here we have used the Lipschitz condition on F and the differentiability of $\int_{R^d}p_t(y,x)u_0(y)dy$ with respect to x (see [9]). The constant M depends on $n, a(x,\mathcal{K})$, upper bound of $|F|$ and u_0. Therefore by Gronwall inequality, we have for any $h_0>0$

$$
\begin{aligned}
& \sup_{x_1,x_2:0<|x_1-x_2|<h_0}E(v(t,x_1,n)-v(t,x_2,n))^{2p}\\
\le\ & M|x_1-x_2|^{2p}\exp\{2^{2(2p-1)}(2p-1)^p n^{pd}||a||_\infty^{2p}|\mathcal{K}|L^{2p}t^p\}.
\end{aligned}\tag{4.10}
$$

The following estimate holds for any $p\ge 1$.

In the following, for a scalar $h>0$ and a vector $e\in R^d$, denote by ∇_e the directional derivative along $e\in R^d$ (see [4]). From (4.9), and Lipschitz estimate (4.10), we have

$$
\begin{aligned}
& v(t,x+he,n)-v(t,x,n)\\
=\ & \nabla_e(\int_{R^d}p_t(y,x)u_0(y)dy)h+h\alpha_1(h)\\
& +n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d}p_{t-s}(y)((\nabla_e a(y+x,\mathcal{K}))h+h\alpha_2(h))\\
&\qquad (F(v(s,y+x,n))+\alpha_3(h,y+x,s))dydw(s,\mathcal{K})
\end{aligned}
$$

$$+n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d}p_{t-s}(y)a(y+x,\mathcal{K})\{F'(v(s,y+x,n))(v(s,y+x+he,n)-v(s,y+x,n))$$
$$+\alpha_4(h,y+x,s)\}dydw(s,\mathcal{K}).$$

Here $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ are infinitely small in the following sense

$$\alpha_1(h) = O(h) \text{ uniformly in } x,t, \tag{4.11}$$
$$\alpha_2(h) = O(h) \text{ uniformly in } x,t, \tag{4.12}$$
$$\sup_x E\alpha_3^{2p}(h,x,s) = O(h^{2p}) \text{ uniformly in } s, \tag{4.13}$$
$$\sup_x E\alpha_4^{2p}(h,x,s) = O(h^{4p}) \text{ uniformly in } s. \tag{4.14}$$

The last two estimates for α_3 and α_4 follow from (4.10). Define

$$J(t,x,n,h) = \frac{v(t,x+he,n)-v(t,x,n)}{h}. \tag{4.15}$$

It is evident that $J(t,x,n,h)$ satisfies the following integral equation:

$$\begin{aligned}J(t,x,n,h) = \quad & \nabla_e\Big(\int_{R^d}p_t(y,x)u_0(y)dy\Big)+\alpha_1(h)\\ & +n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d}p_{t-s}(y)(\nabla_e a(y+x,\mathcal{K})+\alpha_2(h))\\ & \qquad (F(v(s,y+x,n))+\alpha_3(h,y+x,s))dydw(s,\mathcal{K})\\ & +n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d}p_{t-s}(y)a(y+x,\mathcal{K})\{F'(v(s,y+x,n))J(t,x,n,h)\\ & \qquad +\frac{1}{h}\alpha_4(h,y+x,s)\}dydw(s,\mathcal{K}).\end{aligned}$$

Denote by $J(t,x,n)$ the solution of the following stochastic integral equation:

$$\begin{aligned}J(t,x,n) = \quad & \nabla_e\Big(\int_{R^d}p_t(y,x)u_0(y)dy\Big)\\ & +n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d}p_{t-s}(y-x)(\nabla_e a(y,\mathcal{K}))F(v(s,y,n))dydw(s,\mathcal{K})\\ & +n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d}p_{t-s}(y-x)a(y,\mathcal{K})F'(v(s,y,n))J(t,y,n)dydw(s,\mathcal{K}).\end{aligned} \tag{4.16}$$

It turns out that

$$\begin{aligned}J(t,x,n) = \quad & \nabla_e\Big(\int_{R^d}p_t(y,x)u_0(y)dy\Big)\\ & +n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d}p_{t-s}(y)(\nabla_e a(y+x,\mathcal{K}))F(v(s,y+x,n))dydw(s,\mathcal{K})\\ & +n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d}p_{t-s}(y)a(y+x,\mathcal{K})F'(v(s,y+x,n))J(t,y+x,n)dydw(s,\mathcal{K}).\end{aligned}$$

This is then followed by

$$
\begin{aligned}
& J(t,x,n,h) - J(t,x,n) \\
= \quad & \alpha_1(h) + n^{\frac{d}{2}} \sum_{\mathcal{K}} \int_0^t \int_{R^d} p_{t-s}(y)\alpha_2(h)(F(v(s,y+x,n)) + \alpha_3(h,y+x,s))dydw(s,\mathcal{K}) \\
& + n^{\frac{d}{2}} \sum_{\mathcal{K}} \int_0^t \int_{R^d} p_{t-s}(y)\nabla_e a(y+x,\mathcal{K})\alpha_3(h,y+x,s)dydw(s,\mathcal{K}) \\
& + n^{\frac{d}{2}} \sum_{\mathcal{K}} \int_0^t \int_{R^d} p_{t-s}(y)a(y+x,\mathcal{K})\{F'(v(s,y+x,n))(J(t,y+x,n,h) - J(t,y+x,n)) \\
& \qquad + \frac{1}{h}\alpha_4(h,y+x,s)\}dydw(s,\mathcal{K}).
\end{aligned}
$$

So again using Hölder inequality, Burkholder inequality and $(a+b+c+d)^{2p} \leq 2^{2(2p-1)}(a^{2p}+b^{2p}+c^{2p}+d^{2p})$,

$$
\begin{aligned}
& E(J(t,x,n,h) - J(t,x,n))^{2p} \\
= \quad & 2^{2(2p-1)}(\alpha_1(h))^{2p} + 2^{2(2p-1)} n^{pd} t^{p-1} \sum_{\mathcal{K}} \int_0^t \\
& \qquad E(\int_{R^d} p_{t-s}(y)\alpha_2(h)(F(v(s,y+x,n)) + \alpha_3(h,y+x,s))dy)^{2p} ds \\
& + 2^{2(2p-1)} n^{pd} t^{p-1} \sum_{\mathcal{K}} \int_0^t E\{\int_{R^d} p_{t-s}(y)\nabla_e a(y+x,\mathcal{K})\alpha_3(h,y+x,s)dy\}^{2p} ds \\
& + 2^{2(2p-1)} n^{pd} t^{p-1} \sum_{\mathcal{K}} \int_0^t E(\int_{R^d} p_{t-s}(y)a(y+x,\mathcal{K}) \\
& \qquad \{F'(v(s,y+x,n))(J(t,y+x,n,h) - J(t,y+x,n)) + \frac{1}{h}\alpha_4(h,y+x,n)\}dy)^{2p} ds \\
= \quad & \alpha(h) + 2^{2(2p-1)} ||a||_\infty^{2p} n^{pd} t^{p-1} \sum_{\mathcal{K}} \int_0^t \\
& \qquad E\int_{R^d} p_{t-s}(y)\{F'(v(s,y+x,n))(J(s,y+x,n,h) - J(s,y+x,n))\}^{2p} dyds \\
\leq \quad & \alpha(h) + 2^{2(2p-1)} ||a||_\infty^{2p} n^{pd} t^{p-1} ||F'||_\infty^{2p} |\mathcal{K}| \int_0^t \sup_x E(J(s,x,n,h) - J(s,x,n))^{2p} ds.
\end{aligned}
$$

Here

$$
\begin{aligned}
\alpha(h) = \quad & 2^{2(2p-1)}(\alpha_1(h))^{2p} + 2^{3(2p-1)} n^{pd} |\mathcal{K}| \alpha_{2p}(h)^2 t^{p-1} \{t||F||_\infty^{2p} + \int_0^t \sup_x E\alpha_3^{2p}(h,x,s))ds\} \\
& + 2^{2(2p-1)} n^{pd} |\mathcal{K}| \cdot ||\nabla_e a||_\infty^{2p} \int_0^t \sup_x E\alpha_3^{2p}(h,x,s)ds \\
& + 2^{2(2p-1)} n^{pd} |\mathcal{K}| \cdot ||a|_\infty^{2p} ||F'||_\infty^{2p} \int_0^t \sup_x E(\frac{1}{h}\alpha_4(h,x,s))^{2p} ds \\
= \quad & O(h^{2p}).
\end{aligned}
$$

The estimate $O(h^{2p})$ follows from (4.11)–(4.14). Again using the Gronwall inequality

$$
\sup_x \sup_{0<h<h_0} E(J(t,x,n,h) - J(t,x,n))^{2p}
$$

$$\leq \quad \alpha(h_0)\exp\{2^{2(2p-1)}||a||_\infty^{2p}n^{pd}||F'||_\infty^{2p}|\mathcal{K}|t^p\}. \tag{4.17}$$

That is to say that $J(t,x,n,h) \to J(t,x,n)$ in $L^{2p}(\Omega)$ as $h \to 0$ uniformly in t and x. Therefore by Chebyshev's inequality, $J(t,x,n,h) \to J(t,x,n)$ as $h \to 0$ in probability, so that in law. Therefore there exists a sequence $h_k \to 0$ as $k \to \infty$ such that $J(t,x,n,h_k) \to J(t,x,n)$ P almost surely. Furthermore, by Kolmogorov's continuity criterion, $J(t,x,n,h)$ can be extended continuously to $J(t,x,n)$ for almost all $\omega \in \Omega$. That is to say $J(t,x,n,h) \to J(t,x,n)$ as $h \to 0$ for a.e. $\omega \in \Omega$. Similar to the proof of (4.10), we can deduce that

$$E(J(t,x_1,n) - J(t,x_2,n))^{2p} \leq C_1(n,p,t)|x_1 - x_2|^{2p}, \tag{4.18}$$

for any $p \geq 1$. Here $C_1(n,p,t)$ is a constant depending on p, n and t. Actually the proof is the same as the proof of (4.28) given below. Again by Kolmogorov's continuity criterion, $J(t,x,n)$ is a continuous process for a.e. ω. Therefore $v(t,x,n)$ is differentiable with respect to x for a.e. ω.

For C^2 property of $v(t,x,n)$, denote $J_i(t,x,n) = \frac{\partial}{x_i}v(t,x,n)$ and $\nabla_i = \frac{\partial}{\partial x_i}$ for $i = 1,2,\cdots,d$. Then by (4.16), $J_i(t,x,n)$ satisfies the following linear integral equation

$$\begin{aligned} J_i(t,x,n) = \quad & \nabla_i(\int_{R^d} p_t(y,x)u_0(y)dy) \\ & + n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d} p_{t-s}(y-x)(\nabla_i a(y,\mathcal{K}))F(v(s,y,n))dydw(s,\mathcal{K}) \\ & + n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d} p_{t-s}(y-x)a(y,\mathcal{K})F'(v(s,y,n))J_i(s,y,n)dydw(s,\mathcal{K}). \end{aligned} \tag{4.19}$$

Note that for any $p \geq 1$, using Burkholder inequality (4.4) and Hölder inequality,

$$E\{\nabla_i(\int_{R^d} p_t(y,x)u_0(y)dy)\}^{2p} \leq \sup_x |\nabla_i u_0(x)|^{2p}, \tag{4.20}$$

$$\begin{aligned} & E\{\int_0^t\int_{R^d} p_{t-s}(y-x)(\nabla_i a(y,\mathcal{K}))F(v(s,y,n))dydw(s,\mathcal{K})\}^{2p} \\ \leq \quad & (2p-1)^p t^{p-1}||\nabla_i a||_\infty^{2p}||F||_\infty^{2p}, \end{aligned} \tag{4.21}$$

$$\begin{aligned} & E\{\int_0^t\int_{R^d} p_{t-s}(y-x)a(y,\mathcal{K})F'(v(s,y,n))J_i(s,y,n)dydw(s,\mathcal{K})\}^{2p} \\ \leq \quad & (2p-1)^p t^{p-1}||a||_\infty^{2p}||F'||_\infty^{2p}\int_0^t \sup_x E\{J_i(s,x,n)\}^{2p}ds. \end{aligned} \tag{4.22}$$

Similar to the proof of (4.10), using Gronwall inequality, there exists a constant $C_2(n,p,t) > 0$ such that

$$\sup_x E\{J_i(t,x,n)\}^{2p} \leq C_2(n,p,t). \tag{4.23}$$

Furthermore, for any $x_1 \in R^d, x_2 \in R^d$,

$$\begin{aligned} & J_i(t,x_1,n) - J_i(t,x_2,n) \\ = \quad & \nabla_i(\int_{R^d} p_t(y,x_1)u_0(y)dy) - \nabla_i(\int_{R^d} p_t(y,x_2)u_0(y)dy) \end{aligned}$$

$$
\begin{aligned}
&+n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d}p_{t-s}(y)\{(\nabla_i a(y+x_1,\mathcal{K}))F(v(s,y+x_1,n)) \\
&\qquad -(\nabla_i a(y+x_2,\mathcal{K}))F(v(s,y+x_2,n))\}dydw(s,\mathcal{K}) \\
&+n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d}p_{t-s}(y)[a(y+x_1,\mathcal{K})F'(v(s,y+x_1,n))J_i(s,y+x_1,n) \\
&\qquad -a(y+x_2,\mathcal{K})F'(v(s,y+x_2,n))J_i(s,y+x_2,n)]dydw(s,\mathcal{K}).
\end{aligned}
\tag{4.24}
$$

First we note that the smooth property of the heat semigroup implies

$$
\{\nabla_i(\int_{R^d}p_t(y,x_1)u_0(y)dy)-\nabla_i(\int_{R^d}p_t(y,x_2)u_0(y)dy)\}^{2p}\leq C_3|x_1-x_2|^{2p}. \tag{4.25}
$$

Secondly, by Burkholder inequality (4.4) and Hölder inequality, we have

$$
\begin{aligned}
& E\{\int_0^t\int_{R^d}p_{t-s}(y)\{(\nabla_i a(y+x_1,\mathcal{K}))F(v(s,y+x_1,n)) \\
&\qquad -(\nabla_i a(y+x_2,\mathcal{K}))F(v(s,y+x_2,n))\}dydw(s,\mathcal{K})\}^{2p} \\
= \quad & E\{\int_0^t\int_{R^d}p_{t-s}(y)\{[(\nabla_i a(y+x_1,\mathcal{K}))-(\nabla_i a(y+x_2,\mathcal{K}))]F(v(s,y+x_1,n)) \\
&\qquad +(\nabla_i a(y+x_2,\mathcal{K}))[F(v(s,y+x_1,n))-F(v(s,y+x_2,n))]\}dydw(s,\mathcal{K})\}^{2p} \\
\leq \quad & 2^{2p-1}(2p-1)^p t^{p-1}E\int_0^t\{\int_{R^d}p_{t-s}(y)[(\nabla_i a(y+x_1,\mathcal{K}))-(\nabla_i a(y+x_2,\mathcal{K}))] \\
&\qquad F(v(s,y+x_1,n))dy\}^{2p}ds \\
& +2^{2p-1}(2p-1)^p t^{p-1}E\int_0^t\{\int_{R^d}p_{t-s}(y)(\nabla_i a(y+x_2,\mathcal{K})) \\
&\qquad [F(v(s,y+x_1,n))-F(v(s,y+x_2,n))]dy\}^{2p}ds \\
\leq \quad & C_4(p,t)|x_1-x_2|^{2p} \\
& +2^{2p-1}(2p-1)^p t^{p-1}||\nabla_i a||_\infty^{2p}L^{2p}\int_0^t\int_{R^d}p_{t-s}(y) \\
&\qquad E[(v(s,y+x_1,n))-F(v(s,y+x_2,n))]\}^{2p}dyds \\
\leq \quad & C_4(p,t)|x_1-x_2|^{2p}+2^{2p-1}(2p-1)^p t^{p-1}||\nabla_i a||_\infty^{2p}L^{2p} \\
&\qquad \int_0^t\sup_y E[v(s,y+x_1,n)-v(s,y+x_2,n)]\}^{2p}ds \\
\leq \quad & C_4(p,t)|x_1-x_2|^{2p}+C_5(n,p,t)|x_1-x_2|^{2p}.
\end{aligned}
\tag{4.26}
$$

The last inequality follows form (4.10). Similarly for the last term in (4.24),

$$
\begin{aligned}
& E\{\int_0^t\int_{R^d}p_{t-s}(y)[a(y+x_1,\mathcal{K})F'(v(s,y+x_1,n))J_i(s,y+x_1,n) \\
&\qquad -a(y+x_2,\mathcal{K})F'(v(s,y+x_2,n))J_i(s,y+x_2,n)]dydw(s,\mathcal{K})\}^{2p} \\
= \quad & E\{\int_0^t\int_{R^d}p_{t-s}(y)\{[a(y+x_1,\mathcal{K})F'(v(s,y+x_1,n)) \\
&\qquad -a(y+x_2,\mathcal{K})F'(v(s,y+x_2,n))]J_i(s,y+x_1,n) \\
&\qquad +a(y+x_2,\mathcal{K})F'(v(s,y+x_2,n))[J_i(s,y+x_1,n)-J_i(s,y+x_2,n)]\}dydw(s,\mathcal{K})\}^{2p}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & E\{\int_0^t \int_{R^d} p_{t-s}(y)\{[a(y+x_1,\mathcal{K})-a(y+x_2,\mathcal{K})]F'(v(s,y+x_1,n)) \\
& \qquad +a(y+x_2,\mathcal{K})[F'(v(s,y+x_1,n))-F'(v(s,y+x_2,n))]J_i(s,y+x_1,n) \\
& \qquad +a(y+x_2,\mathcal{K})F'(v(s,y+x_2,n))[J_i(s,y+x_1,n)-J_i(s,y+x_2,n)]\}dydw(s,\mathcal{K})\}^{2p} \\
\leq \quad & 2^{2(2p-1)}(2p-1)^p t^{p-1}\int_0^t E\{\int_{R^d} p_{t-s}(y)[a(y+x_1,\mathcal{K})-a(y+x_2,\mathcal{K})]F'(v(s,y+x_1,n))dy\}^{2p}ds \\
& +2^{2(2p-1)}(2p-1)^p t^{p-1}\int_0^t E\{\int_{R^d} p_{t-s}(y)a(y+x_2,\mathcal{K}) \\
& \qquad [F'(v(s,y+x_1,n))-F'(v(s,y+x_2,n))]J_i(s,y+x_1,n)dy\}^{2p}ds \\
& +2^{2(2p-1)}(2p-1)^p t^{p-1}\int_0^t E\{\int_{R^d} p_{t-s}(y)a(y+x_2,\mathcal{K})F'(v(s,y+x_2,n)) \\
& \qquad [J_i(s,y+x_1,n)-J_i(s,y+x_2,n)]\}dy\}^{2p}ds \\
\leq \quad & C_6(n,p,t)|x_1-x_2|^{2p} \\
& +2^{2(2p-1)}(2p-1)^p t^{p-1}||a||_\infty^{2p}\int_0^t \int_{R^d} p_{t-s}(y) \\
& \qquad E[F'(v(s,y+x_1,n))-F'(v(s,y+x_2,n))]^{2p}(J_i(s,y+x_1,n))^{2p}dyds \\
& +2^{2(2p-1)}(2p-1)^p t^{p-1}||a||_\infty^{2p}||F'||_\infty^{2p}\int_0^t \int_{R^d} p_{t-s}(y)E[J_i(s,y+x_1,n)-J_i(s,y+x_2,n)]^{2p}\}dyds \\
\leq \quad & C_6(n,p,t)|x_1-x_2|^{2p} \\
& +2^{2(2p-1)}(2p-1)^p t^{p-1}||a||_\infty^{2p}\int_0^t \sup_y\{E[F'(v(s,y+x_1,n))-F'(v(s,y+x_2,n))]^{4p}\}^{\frac{1}{2}} \\
& \qquad \sup_y\{E(J_i(s,y+x_1,n))^{4p}\}^{\frac{1}{2}}ds \\
& +2^{2(2p-1)}(2p-1)^p t^{p-1}||a||_\infty^{2p}||F'||_\infty^{2p}\int_0^t \sup_y E[J_i(s,y+x_1,n)-J_i(s,y+x_2,n)]^{2p}ds \\
\leq \quad & C_6(n,p,t)|x_1-x_2|^{2p}+C_7(n,p,t)|x_1-x_2|^{2p} \\
& +2^{2(2p-1)}(2p-1)^p t^{p-1}||a||_\infty^{2p}||F'||_\infty^{2p}\int_0^t \sup_y E[J_i(s,y+x_1,n)-J_i(s,y+x_2,n)]^{2p}ds. \qquad (4.27)
\end{aligned}
$$

Here $C_3,\cdots,C_6$ are constants depending on n,p,t. So similar to the proof of (4.10), The above estimates are followed by that there exists a constant $C_7(n,p,t)>0$ such that

$$
\sup_y E[J_i(t,y+x_1,n)-J_i(t,y+x_2,n)]^{2p}\leq C_7(n,p,t)|x_1-x_2|^{2p}. \qquad (4.28)
$$

This estimate will be used soon later. To prove the existence of the 2nd order derivatives, note

$$
\begin{aligned}
& J_i(t,x+he,n)-J_i(t,x,n) \\
= \quad & \nabla_i(\int_{R^d} p_t(y,x+he)u_0(y)dy)-\nabla_i(\int_{R^d} p_t(y,x)u_0(y)dy) \\
& +n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d} p_{t-s}(y)\{(\nabla_i a(y+x+he,\mathcal{K}))F(v(s,y+x+he,n)) \\
& \qquad -(\nabla_i a(y+x,\mathcal{K}))F(v(s,y+x,n))\}dydw(s,\mathcal{K}) \\
& +n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d} p_{t-s}(y)[a(y+x+he,\mathcal{K})F'(v(s,y+x+he,n))J_i(t,y+x+he,n)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad -a(y+x,\mathcal{K})F'(v(s,y+x,n))J_i(t,y+x,n)]dydw(s,\mathcal{K})\\
= \quad & \nabla_i(\int_{R^d} p_t(y,x+he)u_0(y)dy) - \nabla_i(\int_{R^d} p_t(y,x)u_0(y)dy)\\
&+n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d} p_{t-s}(y)\{[(\nabla_i a(y+x+he,\mathcal{K})) - (\nabla_i a(y+x,\mathcal{K}))]F(v(s,y+x+he,n))\\
&\qquad +(\nabla_i a(y+x,\mathcal{K}))[F(v(s,y+x+he,n)) - F(v(s,y+x,n))]\}dydw(s,\mathcal{K})\\
&+n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d} p_{t-s}(y)[a(y+x+he,\mathcal{K}) - a(y+x,\mathcal{K})]\\
&\qquad F'(v(s,y+x+he,n))J_i(t,y+x+he,n)\\
&\qquad +a(y+x,\mathcal{K})[F'(v(s,y+x+he,n)) - F'(v(s,y+x,n))]J_i(t,y+x+he,n)\\
&\qquad +a(y+x,\mathcal{K})F'(v(s,y+x,n))[J_i(t,y+x+he,n) - J_i(t,y+x,n)]dydw(s,\mathcal{K}).
\end{aligned}
$$

It turns out that

$$
G_i(t,x,n,h) = \frac{J_i(t,x+he,n) - J_i(t,x,n)}{h}.
$$

satisfies

$$
\begin{aligned}
& G_i(t,x,n,h)\\
= \quad & \nabla_e\nabla_i(\int_{R^d} p_t(y,x)u_0(y)dy) + \beta_1(h,t,x)\\
&+n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d} p_{t-s}(y)\{[\nabla_e\nabla_i a(y+x,\mathcal{K})) + \beta_2(h,x+y)][F(v(s,y+x,n)) + \beta_3(h,s,y+x)]\\
&\qquad +(\nabla_i a(y+x,\mathcal{K}))[F'(v(s,y+x,n))J(s,y+x,n,h) + \beta_4(h,s,x+y)]\}dydw(s,\mathcal{K})\\
&+n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d} p_{t-s}(y)[\nabla_e a(y+x,\mathcal{K}) + \beta_5(h,y+x)]\\
&\qquad [F'(v(s,y+x,n)) + \beta_6(h,s,y+x)][J_i(s,y+x,n) + \beta_7(h,s,x)]\\
&\qquad +a(y+x,\mathcal{K})[F''(v(s,y+x,n))J(s,y+x,n) + \beta_8(h,s,x+y)]\\
&\qquad [J_i(s,y+x,n) + \beta_9(h,s,y+x)]\\
&\qquad +a(y+x,\mathcal{K})F'(v(s,y+x,n))[G_i(s,y+x,n)]dydw(s,\mathcal{K}).
\end{aligned}
\tag{4.29}
$$

Here $\beta_1(h),\cdots,\beta_9(h)$ are infinitesimals in the following sense respectively:

$$\beta_1(h,t,x) = O(h),\ \text{uniformly in } x,t, \tag{4.30}$$

$$\beta_i(h,x) = O(h),\ \text{uniformly in } x,\ \text{for } i=2,5, \tag{4.31}$$

$$E\{\beta_i(h,t,x)\}^{2p} = O(h^{2p}),\ \text{uniformly in } x,t,\ \text{for } i=3,4,6,7, \tag{4.32}$$

$$E\{\beta_i(h,t,x)\}^{4p} = O(h^{4p}),\ \text{uniformly in } x,t,\ for\ i=8,9. \tag{4.33}$$

The estimate (4.30) for β_1 follows from the smooth property of the heat semigroup and (4.31) for β_2 and β_5 follows from the differentiability of $a(x,\mathcal{K})$. The moment estimates (4.32) and (4.33) follows from (4.10) and (4.28). We need $4p$-th moment estimates for β_8 and β_9 as we need to estimate the following term in (4.29) in procedure of the proof of (4.35):

$$
\begin{aligned}
& E\{\int_0^t \int_{R^d} p_{t-s}(y)a(y+x,\mathcal{K})[F''(v(s,y+x,n))J(s,y+x,n)\beta_9(h,s,y+x) \\
& \qquad +J_i(s,y+x,n)\beta_8(h,s,x+y)+\beta_8(h,s,x+y)\beta_9(h,s,y+x)]dydw(s,\mathcal{K})\}^{2p} \\
\leq\ & 2^{2(2p-1)}(2p-1)^p t^{p-1}\int_0^t E\{\int_{R^d} p_{t-s}(y)a(y+x,\mathcal{K}) \\
& \qquad [F''(v(s,y+x,n))J(s,y+x,n)\beta_9(h,s,y+x)dy\}^{2p}ds \\
& +2^{2(2p-1)}(2p-1)^p t^{p-1}\int_0^t E\{\int_{R^d} p_{t-s}(y)a(y+x,\mathcal{K})J_i(s,y+x,n)\beta_8(h,s,x+y)dy\}^{2p}ds \\
& +2^{2(2p-1)}(2p-1)^p t^{p-1}\int_0^t E\{\int_{R^d} p_{t-s}(y)a(y+x,\mathcal{K})\beta_8(h,s,x+y)\beta_9(h,s,y+x)]dy\}^{2p}ds \\
\leq\ & 2^{2(2p-1)}(2p-1)^p t^{p-1}||a||_\infty^{2p}||F''||_\infty^{2p}\int_0^t\int_{R^d} p_{t-s}(y)E\{J(s,y+x,n)\beta_9(h,s,y+x)\}^{2p}dyds \\
& +2^{2(2p-1)}(2p-1)^p t^{p-1}||a||_\infty^{2p}\int_0^t\int_{R^d} p_{t-s}(y)E\{J_i(s,y+x,n)\beta_8(h,s,x+y)\}^{2p}dyds \\
& +2^{2(2p-1)}(2p-1)^p t^{p-1}||a||_\infty^{2p}\int_0^t\{\int_{R^d} p_{t-s}(y)E\{\beta_8(h,s,x+y)\beta_9(h,s,y+x)\}^{2p}]dyds \\
\leq\ & 2^{2(2p-1)}(2p-1)^p t^{p-1}||a||_\infty^{2p}||F''||_\infty^{2p}\int_0^t \sup_x\{E\{J(s,x,n)\}^{4p}\}^{\frac{1}{2}}\sup_x E\{\beta_9(h,s,x)\}^{4p}\}^{\frac{1}{2}}ds \\
& +2^{2(2p-1)}(2p-1)^p t^{p-1}||a||_\infty^{2p}\int_0^t \sup_x\{E\{J_i(s,x,n)\}^{4p}\}^{\frac{1}{2}}\sup_x\{E\{\beta_8(h,s,x)\}^{4p}\}^{\frac{1}{2}}ds \\
& +2^{2(2p-1)}(2p-1)^p t^{p-1}||a||_\infty^{2p}\int_0^t \sup_x\{E\{\beta_8(h,s,x)\}^{4p}\}^{\frac{1}{2}}\sup_x\{E\{\beta_9(h,s,x)\}^{4p}\}^{\frac{1}{2}}ds \\
=\ & O(h^{2p}). \qquad (4.34)
\end{aligned}
$$

Here we have used the 4p-th moment estimate (4.33) and (4.23), Burkholder inequality and Hölder inequality.

Let $G_i(t,x,n)$ be the solution of the following integral equation

$$
\begin{aligned}
& G_i(t,x,n) \\
=\ & \nabla_e\nabla_i(\int_{R^d} p_t(y,x)u_0(y)dy) \\
& +n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d} p_{t-s}(y)\{\nabla_e\nabla_i a(y+x,\mathcal{K}))F(v(s,y+x,n)) \\
& \qquad +(\nabla_i a(y+x,\mathcal{K}))F'(v(s,y+x,n))J(s,x,n,h)\}dydw(s,\mathcal{K}) \\
& +n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t\int_{R^d} p_{t-s}(y)\nabla_e a(y+x,\mathcal{K})F'(v(s,y+x,n))J_i(t,y+x,n) \\
& \qquad +a(y+x,\mathcal{K})F''(v(s,y+x,n))J(s,y+x,n)J_i(t,y+x,n) \\
& \qquad +a(y+x,\mathcal{K})F'(v(s,y+x,n))[G_i(t,y+x,n)]dydw(s,\mathcal{K}).
\end{aligned}
$$

Then similarly to the proof of (4.17), using Burkholder inequality, Hölder inequality, Gronwall inequality and (4.30)-(4.33), (4.34), now it is routine to derive that there exists a constant $C_8(n,p,t)>$

0 such that

$$E\{G_i(t,x,n,h)-G_i(t,x,n)\}^{2p} \le C_8(n,p,t)h^{2p}. \tag{4.35}$$

Then by Kolmogorov continuity criterion, we know that $G_i(t,x,n,h) \to G_i(t,x,n)$ as $h \to 0$ for a.e. $\omega \in \Omega$. Then similar to the proof (4.23) we can prove that there exists a constant $C_9(n,p,t) > 0$ such that

$$E\{G_i(t,x,n)\}^{2p} \le C_9(n,p,t). \tag{4.36}$$

Thus similar to the proof of (4.28), without any difficulty, we can prove that there exists a constant $C_{10}(n,p,t) > 0$ such that

$$E\{G_i(t,x_1,n)-G_i(t,x_2,n)\}^{2p} \le C_{10}(n,p,t)|x_1-x_2|^{2p}. \tag{4.37}$$

We leave the details to the reader. Therefore again by Kolmogorov continuity criterion, $G_i(t,x,n)$ is a continuous process. Therefore $G_i(t,x,n)$ is differentiable with respect to x for a.e. ω. That is to say $v(t,x,n) \in C^2$ with respect to x for a.e. $\omega \in \Omega$. ‡‡

5 Stochastic KPP equations

In this section, we consider stochastic KPP equation (1.2). As a mean of approximation, we consider the following stochastic KPP equation with time white noise

$$\begin{aligned} du(t,x,n) &= \frac{1}{2}\Delta u(t,x,n)dt + n^{\frac{d}{2}}\sum_{\mathcal{K}} a(x,\mathcal{K})(1-u(t,x,n))u(t,x,n)dw(t,\mathcal{K}), \\ u(t,x,n) &= u_0(x). \end{aligned} \tag{5.1}$$

We will prove that the stochastic KPP equation (5.1) with time white noise is an approximation to the stochastic KPP equation (1.2) with space-time white noise.

We will first prove in the following that equation (5.1) has a unique strong solution which is C^2 in space, and moreover, $0 \le u(t,x,n) \le 1$. Note that the linear equation does not have this boundedness property. It is this property and Meyer's pseudo-path topology argument we used in section 3 lead us to prove that $u(t,x,n) \to u(t,x)$ in measure and the $u(t,x)$ is a function of (t,x). The latter is a solution to equation (1.2).

The nonlinear term $(1-u)u$ does not satisfy global Lipschitz condition. However it is locally Lipschitz. We will prove the existence and uniqueness of (5.1) by using the results in section 4 about nonlinear equations with global Lipschitz nonlinearity.

We have the following key lemma.

Lemma 5.1 *Assume $u_0 \in C^2(R^d)$ and $0 \le u_0 \le 1$. Then the solution $u(t,x,n)$ of the stochastic KPP equation (5.1) exists and is unique and C^2 in x and satisfies:*

$$0 \le u(t,x,n) \le 1. \tag{5.2}$$

Proof. Define

$$F(u)=\begin{cases} -2 & \text{if } u\le -1\\ [u(1-u)+2]\exp\{1+\frac{1}{u^2-1}\}-2 & \text{if } -1<u<0\\ (1-u)u & \text{if } 0\le u\le 1\\ [u(1-u)+2]\exp\{1+\frac{1}{(u-1)^2-1}\}-2 & \text{if } 1<u<2\\ -2 & \text{if } u\ge 2. \end{cases}$$

Then it is evident that $F(u)$ is a global Lipschitz function and there exists a C^∞ function $C_1(u)$ such that $F(u)=C_1(u)u$. Consider the following SPDE

$$\begin{aligned} dv(t,x,n) = \quad & \frac{1}{2}\Delta v(t,x,n)dt \\ & +n^{\frac{d}{2}}\sum_{\mathcal{K}} a(x,\mathcal{K})C_1(v(t,x,n))v(t,x,n)dw(t,\mathcal{K}), \\ v(0,x,n) = \quad & u_0(x). \end{aligned} \tag{5.3}$$

This equation has a unique regular solution according to Theorem 4.2. As

$$\begin{aligned} & n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t a(x,\mathcal{K})C_1(v(s,x,n))v(s,x,n)dw(s,\mathcal{K}) \\ = \quad & n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t a(x,\mathcal{K})C_1(v(s,x,n))v(s,x,n)\circ dw(s,\mathcal{K}) \\ & -\frac{1}{2}n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t a(x,\mathcal{K})d_s\{C_1(v(s,x,n))v(s,x,n)\}dw(s,\mathcal{K}) \\ = \quad & n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t a(x,\mathcal{K})C_1(v(s,x,n))v(s,x,n)\circ dw(s,\mathcal{K}) \\ & -\frac{1}{2}n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t a(x,\mathcal{K})\{C_1'(v(s,x,n))v(s,x,n)+C_1(v(s,x,n))\}dv(s,x,n)dw(s,\mathcal{K}) \\ = \quad & n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t a(x,\mathcal{K})C_1(v(s,x,n))v(s,x,n)\circ dw(s,\mathcal{K}) \\ & -\frac{1}{2}n^{d}\sum_{\mathcal{K}}\int_0^t a^2(x,\mathcal{K})\{C_1'(v(s,x,n))v(s,x,n) \\ & \qquad +C_1(v(s,x,n))\}C_1(v(s,x,n))v(s,x,n)ds. \end{aligned} \tag{5.4}$$

So the Ito type equation (5.3) is equivalent to the following Stratonovich type equation:

$$\begin{aligned} dv(t,x,n) = \quad & \frac{1}{2}\Delta v(t,x,n)dt-\frac{1}{2}n^d\sum_{\mathcal{K}}a^2(x,\mathcal{K})\{C_1'(v(t,x,n))v(t,x,n) \\ & \qquad +C_1(v(t,x,n))\}C_1(v(t,x,n))v(t,x,n)dt \\ & +n^{\frac{d}{2}}\sum_{\mathcal{K}}a(x,\mathcal{K})C_1(v(t,x,n))v(t,x,n)\circ dw(t,\mathcal{K}). \end{aligned} \tag{5.5}$$

By the Feynman-Kac formula,

$$
\begin{aligned}
v(t,x,n) = \hat{E}u_0(X_0^t(x))\exp\{ \quad & -\frac{1}{2}n^d\sum_{\mathcal{K}}\int_0^t a^2(X_s^t(x),\mathcal{K})\{C_1'(v(s,X_s^t(x),n))v(s,X_s^t(x),n) \\
& \qquad\qquad +C_1(v(s,X_s^t(x),n))\}C_1(v(s,X_s^t(x),n))ds \\
& +n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t a(X_s^t(x),\mathcal{K})C_1(v(s,X_s^t(x),n))\circ dw(s,\mathcal{K})\} \\
= \hat{E}u_0(X_0^t(x))\exp\{ \quad & -\frac{1}{2}n^d\sum_{\mathcal{K}}\int_0^t a^2(X_s^t(x),\mathcal{K})C_1^2(v(s,X_s^t(x),n))ds \\
& +n^{\frac{d}{2}}\sum_{\mathcal{K}}\int_0^t a(X_s^t(x),\mathcal{K})C_1(v(s,X_s^t(x),n))dw(s,\mathcal{K})\}.
\end{aligned}
$$

Here the stochastic flow $X_s^t(x)$ is the same as in Lemma 2.1 and the stochastic integral with $\circ$ is a Stratonovich integral and the one without $\circ$ is an Ito integral. It is evident from the formula that

$$v(t,x,n)\geq 0. \tag{5.6}$$

Let $Z=1-v$. Then $Z(t,x,n)$ satisfies the following Ito type SPDE

$$
\begin{aligned}
dZ(t,x,n) = \quad & \frac{1}{2}\Delta Z(t,x,n)dt \\
& -n^{\frac{d}{2}}\sum_{\mathcal{K}}a(x,\mathcal{K})C_1(Z(t,x,n))Z(t,x,n)dw(t,\mathcal{K}), \\
Z(0,x,n) = \quad & 1-u_0(x).
\end{aligned} \tag{5.7}
$$

So by the Feynman-Kac formula again, we have a slution

$$Z(t,x,n)\geq 0, \tag{5.8}$$

as $1-u_0\geq 0$. Combine (5.6) and (5.8), we obtain

$$0\leq v(t,x,n)\leq 1. \tag{5.9}$$

But when $0\leq v(t,x,n)\leq 1$, $F(v(t,x,n))=(1-v(t,x,n))v(t,x,n)$. Therefore $v(t,x,n)$ satisfies actually

$$
\begin{aligned}
& dv(t,x,n) = \frac{1}{2}\Delta v(t,x,n)dt + n^{\frac{d}{2}}\sum_{\mathcal{K}}a(x,\mathcal{K})(1-v(t,x,n))v(t,x,n)\circ dw(t,\mathcal{K}), \\
& v(t,x,n) = u_0(x).
\end{aligned} \tag{5.10}
$$

That is to say that $u(t,x,n)=v(t,x,n)$ is a solution of (5.1) and satisfies (5.2). The uniqueness of the solution to (5.1) follows from the uniqueness of the solution to (5.3). ‡‡

Now, it is trivial to verify that the condition (3.2) of Theorem 3.3 is satisfied, so we get

Theorem 5.2 *Assume $u_0\in C_0^2(R^d)$ and $0\leq u_0(x)\leq 1$ for all $x\in R^d$. Then Equation (1.2) has a real-valued solution.*

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