Representations of Quantum Tori and G-bundles on Elliptic Curves

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Dedicated to A. A. Kirillov on the occasion of his 65th birthday

ABSTRACT We study a BGG-type category of infinite-dimensional representations of $\mathcal{H}[W]$, a semidirect product of the quantum torus with parameter $\mathbf{q}$, built on the root lattice of a semisimple group G, and the Weyl group of G. Irreducible objects of our category turn out to be parametrized by semistable G-bundles on the elliptic curve $\mathbb{C}^*/\mathbf{q}^{\mathbb{Z}}$.

1 Introduction

We introduce a noncommutative deformation of the algebra of regular functions on a torus. This deformation $\mathcal{H}$, called *quantum torus algebra*, depends on a complex parameter $\mathbf{q} \in \mathbb{C}^*$. We further introduce a certain category $\mathcal{M}(\mathcal{H}, \mathcal{A})$ of representations of $\mathcal{H}$ which are locally-finite with respect to a commutative subalgebra $\mathcal{A} \subset \mathcal{H}$ whose "size" is one-half of that of $\mathcal{H}$ (our definition is modeled on the definition of the category $\mathcal{O}$ of Bernstein–Gelfand–Gelfand). We classify all simple objects of $\mathcal{M}(\mathcal{H}, \mathcal{A})$ and show that any object of $\mathcal{M}(\mathcal{H}, \mathcal{A})$ has finite length.

In §3 we consider quantum tori arising from a pair of lattices coming from a finite reduced root system. Let W be the Weyl group of this root system. We classify all simple modules over the twisted group ring $\mathcal{H}[W]$ which belong to $\mathcal{M}(\mathcal{H}, \mathcal{A})$ as $\mathcal{H}$-modules. In §4 we show that the twisted group ring $\mathcal{H}[W]$ is Morita equivalent to $\mathcal{H}^W$, the ring of W-invariants.

In §5 we establish a bijection between the set of simple modules over the algebra $\mathcal{H}[W]$ associated with a semisimple simply-connected group G, and the set of pairs (P, α), where P is a semistable principal G-bundle on the elliptic curve $\mathcal{E} = \mathbb{C}^*/\mathbf{q}^{\mathbb{Z}}$, and α is a certain "admissible representation" (cf. Definition 5.4) of the finite group $\mathrm{Aut}(P)/(\mathrm{Aut}\ P)^\circ$. Our bijection is constructed by combining the results of §3 with a bijection between $\mathbf{q}$-conjugacy classes in a loop group and G-bundles the elliptic curve $\mathcal{E}$, established earlier by some of us in [BG].

2 Holonomic modules over quantum tori

Choose to a finite rank abelian group $\mathbf{V}$, referred to as a *lattice*, and a skew symmetric $\mathbb{Z}$-valued bilinear form $\omega : \mathbf{V} \times \mathbf{V} \to \mathbb{Z}$. Associated to these data is the Heisenberg central extension

$$0 \to \mathbb{Z} \to \tilde{\mathbf{V}} \to \mathbf{V} \to 0.$$

Here $\tilde{\mathbf{V}} = \mathbf{V} \oplus \mathbb{Z}$ as a set, and the group law on $\tilde{\mathbf{V}}$ is given by

$$(v_1, z_1) \circ (v_2, z_2) = (v_1 + v_2, z_1 + z_2 + \omega(v_1, v_2)) \quad , \quad v_i \in \mathbf{V}, z_i \in \mathbb{Z}.$$

Let $\mathbb{C}\tilde{\mathbf{V}}$ denote the group algebra of $\tilde{\mathbf{V}}$ formed by all $\mathbb{C}$-linear combinations $\sum_{g \in \tilde{\mathbf{V}}} c_g[g]$. Given a complex number $\mathbf{q} \in \mathbb{C}^*$, we define a *quantum torus*, $\mathcal{H}_{\mathbf{q}}(\mathbf{V}, \omega)$, as the quotient of $\mathbb{C}\tilde{\mathbf{V}}$ modulo the two-sided ideal generated by the (central) element $[(0, 1)] - \mathbf{q} \cdot [(0, 0)]$. We write e^v for the image of $[(v, 0)] \in \mathbb{C}\tilde{\mathbf{V}}$ in $\mathcal{H}_{\mathbf{q}}(\mathbf{V}, \omega)$. The elements $\{e^v, v \in \mathbf{V}\}$ form a $\mathbb{C}$-basis of $\mathcal{H}_{\mathbf{q}}(\mathbf{V}, \omega)$, and we have

$$e^{v_1} \cdot e^{v_2} = \mathbf{q}^{\omega(v_1, v_2)} \cdot e^{v_1 + v_2} \quad , \quad \forall v_1, v_2 \in \mathbf{V}.$$

In particular, if $\{u_1, \ldots, u_n\}$ is a $\mathbb{Z}$-basis of V, then the algebra $\mathcal{H}_{\mathbf{q}}(\mathbf{V}, \omega)$ is generated over $\mathbb{C}$ by the elements $X_i = e^{u_i}$ subject to the relations $X_i X_j = q_{ij}^2 X_j X_i$, where $q_{ij} = q^{\omega(u_i, u_j)}$. Therefore, $\mathcal{H}_{\mathbf{q}}(\mathbf{V}, \omega)$ is an iterated skew polynomial extension (cf. [MR]) and hence Noetherian ([MR, 1.2.9(iv)]).

Lemma 2.1. *If the form ω is nondegenerate, and $\mathbf{q}$ is not a root of unity, then the algebra $\mathcal{H}_q(\mathbf{V}, \omega)$ is simple.*

Proof. Suppose $h = \sum_{i=1}^s c_i e^{v_i}$ is an element of a two-sided ideal $J \subset \mathcal{H}_{\mathbf{q}}(\mathbf{V}, \omega)$, where all $v_i \in \mathbf{V}$ are distinct, and all the $c_i \in \mathbb{C}$ are nonzero. We claim that $e^{v_i} \in J$ for every i, whence $J = \mathcal{H}_{\mathbf{q}}(\mathbf{V}, \omega)$ since the elements e^{v_i} are invertible.

To prove the claim, we use the nondegeneracy of ω and the assumption that all the vectors v_i are distinct to find an element $v \in \mathbf{V}$ such that $\omega(v, v_i) \neq \omega(v, v_j)$, for any $i \neq j$. Hence, since q is not a root of unity, we conclude that

$$\mathbf{q}^{k \cdot \omega(v, v_i)} \neq \mathbf{q}^{k \cdot \omega(v, v_j)} \quad , \quad \forall k = 1, 2, \ldots, \quad \text{whenever} \quad i \neq j. \tag{2.1}$$

Now, for any $k = 0, 1, \ldots,$ set $u_k := e^{k \cdot v} h e^{-k \cdot v} \in J$. We have

$$u_k = e^{k \cdot v} h e^{-k \cdot v} = \sum c_i \cdot e^{k \cdot v} e^{v_i} e^{-k \cdot v} = \sum_{i=1}^{s} c_i \cdot \mathbf{q}^{k \cdot \omega(v, v_i)} \cdot e^{v_i}.$$

Observe that the determinant of the matrix $a_{ik} := \mathbf{q}^{k \cdot \omega(v, v_i)}$ is the Vandermonde determinant $\prod_{i > j} (\mathbf{q}^{k \cdot \omega(v, v_i)} - \mathbf{q}^{k \cdot \omega(v, v_j)})$. By (2.1) this determinant is nonzero, so that the matrix is invertible. Hence, each of the elements $e^{v_1}, \ldots, e^{v_s}$ can be expressed as a linear combination of the $u_0, \ldots, u_{s-1} \in J$, and the claim follows. $\square$

Remark. If $\mathbf{q}^m = 1$, then the elements $(1 - e^{mv}),\, v \in \mathbf{V}$ are in the center of $\mathcal{H}_{\mathbf{q}}(\mathbf{V}, \omega)$, hence any such element generates a nontrivial two-sided ideal.

Fix a lattice $\mathbf{X}$, let $\mathbf{Y} = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{X}, \mathbb{Z})$, be the dual lattice and write $\langle\, ,\rangle : \mathbf{X} \times \mathbf{Y} \to \mathbb{Z}$ for the canonical pairing. From now on, we take $\mathbf{V} = \mathbf{X} \oplus \mathbf{Y}$, where the form ω on $\mathbf{X} \oplus \mathbf{Y}$ is given by

$$\omega(x \oplus y, x' \oplus y') := \langle x, y'\rangle - \langle x', y\rangle \quad , \quad x, x' \in \mathbf{X},\ y, y' \in \mathbf{Y}\,.$$

Let $\mathcal{H} = \mathcal{H}_{\mathbf{q}}(\mathbf{X} \oplus \mathbf{Y}, \omega)$ denote the corresponding algebra. The elements $\{e^x, x \in \mathbf{X}\}$, resp. $\{e^y, y \in \mathbf{Y}\}$, span the commutative subalgebra $\mathbb{C}\mathbf{X} \subset \mathcal{H}$, resp. $\mathbb{C}\mathbf{Y} \subset \mathcal{H}$, and there is a natural vector space (but not *algebra*) isomorphism $\mathcal{H} \simeq \mathbb{C}\mathbf{X} \otimes_{\mathbb{C}} \mathbb{C}\mathbf{Y}$. The algebra structure is determined by the commutation relations

$$e^y e^x = \mathbf{q}^{<x,y>} e^x e^y \quad , \quad \forall x \in \mathbf{X},\ y \in \mathbf{Y}. \tag{2.2}$$

We introduce the complex torus $T := \mathrm{Hom}(\mathbf{X}, \mathbb{C}^*)$ so $\mathbf{X} \simeq \mathrm{Hom}_{\text{alg group}}(T, \mathbb{C}^*)$. Any element $x \in \mathbf{X}$ may be viewed as a $\mathbb{C}^*$-valued regular function $t \mapsto x(t)$ on T. Any $y \in \mathbf{Y}$ gives a well-defined element $\phi_y \in \mathrm{Hom}_{\text{alg group}}(\mathbb{C}^*, T) = Hom(\mathbf{X}, \mathbb{Z})$. We set $\mathbf{q}^y := \phi_y(\mathbf{q}) \in T$. The assignment $y \mapsto \mathbf{q}^y$ identifies the lattice $\mathbf{Y}$ with a finitely generated discrete subgroup $\mathbf{q}^{\mathbf{Y}} \subset T$.

Let A be a commutative $\mathbb{C}$-algebra and $\alpha : A \to \mathbb{C}$ an algebra homomorphism, referred to as a *weight*. For an A-module M, let $M(\alpha) := \{m \in M \mid am = \alpha(a) \cdot m\ ,\ \forall a \in A\}$ denote the corresponding weight subspace.

Definition. Given a $\mathbb{C}$-algebra H with a *commutative* subalgebra $A \subset H$, define

- $\mathcal{M}(H, A)$ to be the category of finitely generated H-modules M such that the H-action on M restricted to A is *locally finite*, that is, for any $m \in M$ we have $\dim_{\mathbb{C}} A \cdot m < \infty$.
- $\mathcal{M}^{ss}(H, A)$ to be the full subcategory of $\mathcal{M}(H, A)$ consisting of A-*diagonalizable* H-modules, i.e., H-modules M of the form

$$M = \bigoplus_{\alpha \in \text{ Weights of } A} M(\alpha) \quad \text{and} \quad \dim_{\mathbb{C}} M(\alpha) < \infty\,,\ \forall \alpha.$$

Note that if $A = \mathbb{C}$, then $\mathcal{M}(H, A) = \mathcal{M}^{ss}(H, A)$ is just the category of finitely generated H-modules.

In this section we will be concerned with the special case $H = \mathcal{H}$, $A = \mathcal{A} := \mathbb{C}\mathbf{X} \subset \mathcal{H}$, (we also fix $\mathbf{q} \in \mathbb{C}^*$, not a root of unity). Since $\mathcal{H}$ is Noetherian (as explained before Lemma 2.1) any submodule N of any object $M \in \mathcal{M}(\mathcal{H}, \mathcal{A})$ is finitely generated over $\mathcal{H}$, hence $\mathcal{M}(\mathcal{H}, \mathcal{A})$ is an *abelian* category. Note the canonical algebra isomorphism $\mathbb{C}\mathbf{X} \simeq \mathbb{C}[T]$, where $\mathbb{C}[T]$ stands for the algebra of regular polynomial functions on T. Thus, the set of weights of the algebra $\mathcal{A} = \mathbb{C}\mathbf{X}$ is canonically identified with T.

For $\lambda \in T$, define an $\mathcal{H}$-module M_λ as a $\mathbb{C}$-vector space with basis $\{v_\mu\,,\ \mu \in \lambda \cdot \mathbf{q}^{\mathbf{Y}} \subset T\}$ and with $\mathcal{H}$-action given by

$$e^y(v_\mu) = v_{\mu\cdot\mathbf{q}^y} \quad , \quad e^x(v_\mu) = x(\mu)\cdot v_\mu. \tag{2.3}$$

The module M_λ has the following interpretation. Write I_μ for the maximal ideal in $\mathbb{C}[T]$ corresponding to a point $\mu \in T$, and let $\mathbb{C}_\mu := \mathbb{C}[T]/I_\mu$ be the skyscraper sheaf at μ. Let $\mathbb{C}[\lambda \cdot \mathbf{q}^{\mathbf{Y}}] := \bigoplus_{\mu\in\lambda\cdot\mathbf{q}^{\mathbf{Y}}} \mathbb{C}_\mu$ be the (not finitely generated) $\mathbb{C}[T]$-module formed by all $\mathbb{C}$-valued, finitely supported functions on the set $\lambda \cdot \mathbf{q}^{\mathbf{Y}}$. Define an $\mathcal{H}$-action on $\mathbb{C}[\lambda \cdot \mathbf{q}^{\mathbf{Y}}]$ by the formulas

$$e^x(f) : t \mapsto x(t)\cdot f(t) \quad , \quad e^y(f) : t \mapsto f(\mathbf{q}^y \cdot t). \tag{2.4}$$

Thus, $x \in \mathbf{X}$ and $y \in \mathbf{Y}$ act via multiplication by the function $x(t)$ and shift by $\mathbf{q}^y$, respectively. It is straightforward to verify that sending $v_\mu \in \mathsf{M}_\lambda\,,\ \mu \in \lambda\cdot\mathbf{q}^{\mathbf{Y}}$ to the characteristic function of the one-point set $\{\mu\}$ establishes an isomorphism of $\mathcal{H}$-modules $\mathsf{M}_\lambda \xrightarrow{\sim} \mathbb{C}[\lambda \cdot \mathbf{q}^{\mathbf{Y}}]$ intertwining the actions (2.3) and (2.4), respectively.

Clearly, $\mathsf{M}_\lambda \in \mathcal{M}^{ss}(\mathcal{H}, \mathcal{A})$. Moreover, it is obvious from the isomorphism $\mathsf{M}_\lambda \simeq \mathbb{C}[\lambda \cdot \mathbf{q}^{\mathbf{Y}}]$ that $\mathsf{M}_\lambda \simeq \mathsf{M}_\mu$ if $\mu \in \lambda \cdot \mathbf{q}^{\mathbf{Y}}$. Thus, the modules M_λ are effectively parametrized (up to isomorphism) by the points of the variety: $\Lambda := T/\mathbf{q}^{\mathbf{Y}}$. When $|\mathbf{q}| \neq 1$, Λ is an abelian variety. Observe that the modules corresponding to two different points of Λ have disjoint weights, hence are nonisomorphic.

Proposition 2.5.

(i) M_λ *is a simple* $\mathcal{H}$*-module, for any* $\lambda \in \Lambda$*. Moreover, the set* $\{\mathsf{M}_\lambda, \lambda \in \Lambda\}$ *is a complete collection of (the isomorphism classes of) simple objects of the category* $\mathcal{M}(\mathcal{H}, \mathcal{A})$*.*

(ii) *Any object of the category* $\mathcal{M}^{ss}(\mathcal{H}, \mathcal{A})$ *is isomorphic to a finite direct sum* $\bigoplus_{\lambda\in\Lambda} \mathsf{M}_\lambda$*, in particular, the category* $\mathcal{M}^{ss}(\mathcal{H}, \mathcal{A})$ *is semisimple.*

(iii) *Any object of the category* $\mathcal{M}(\mathcal{H}, \mathcal{A})$ *has finite length.*

Proof. Let $M \in \mathcal{M}(\mathcal{H}, \mathcal{A})$. An easy straightforward calculation shows that, for any nonzero element $m \in M(\lambda)$, the $\mathcal{H}$-submodule in M generated by m is isomorphic to M_λ. This, combined with the observation preceding the proposition, proves part (i).

Since M is finitely generated, one can find finitely many weights $\lambda_1, \ldots, \lambda_s \in T$ such that all weights of M are contained in $(\lambda_1 \cdot \mathbf{q}^{\mathbf{Y}}) \cup \ldots \cup (\lambda_s \cdot \mathbf{q}^{\mathbf{Y}})$ and, moreover, $\lambda_i \neq \lambda_j \bmod \mathbf{q}^{\mathbf{Y}}$ whenever $i \neq j$. It follows, since all weights of M are in $(\lambda_1\cdot\mathbf{q}^{\mathbf{Y}})\cup\ldots\cup(\lambda_s\cdot\mathbf{q}^{\mathbf{Y}})$, that M is generated by the subspace $\bigoplus_{i=1}^s M(\lambda_i)$. Furthermore, the same calculation as in the first part implies that the $\mathcal{H}$-submodule in M generated by this subspace is isomorphic to $\bigoplus_{i=1}^s \mathsf{M}_{\lambda_i} \otimes M(\lambda_i)$. This proves part (ii).

To prove (iii), suppose $M \in \mathcal{M}(\mathcal{H}, \mathcal{A})$. We use induction on the minimal dimension d of an $\mathcal{A}$-invariant subspace $V \subset M$ which generates M over $\mathcal{H}$. It follows from the definitions that if $d = 1$ then $M \simeq \mathsf{M}_\lambda$ for some λ. If $d > 1$, choose a nonzero vector $v \in V$ of some $\mathcal{A}$-weight λ and note that such a choice induces a nonzero homomorphism of $\mathcal{H}$-modules $\mathsf{M}_\lambda \to M$. Since M_λ is simple, this homomorphism is necessarily injective. The quotient M/M_λ is generated by an $\mathcal{A}$-invariant subspace $V/\langle v\rangle$, hence we can apply the assumption of induction to this $\mathcal{H}$-module, and (iii) follows. □

3 $\mathcal{H}[W]$-modules

Let $\Delta \subset \mathfrak{h}$ be a finite reduced root system. Let W be the Weyl group of Δ, write $\mathfrak{h}^\vee$ for the dual of $\mathfrak{h}$, and let $\mathbf{X} \subset \mathfrak{h}^\vee$, and $\mathbf{Y} \subset \mathfrak{h}$ be the coroot and weight lattices, respectively. The group W acts naturally on $\mathbf{X}$ and on $\mathbf{Y}$. The diagonal W-action on $\mathbf{X} \oplus \mathbf{Y}$ makes $\mathcal{H} = \mathcal{H}(\mathbf{X} \oplus \mathbf{Y})$ a left W-module with W-action $w : h \mapsto {}^w h$, $h \in \mathcal{H}$. Write $\mathcal{H}^W$ for the subalgebra of W-invariants. Further, introduce a *twisted group algebra*, $\mathcal{H}[W]$, as the complex vector space $\mathcal{H} \otimes_{\mathbb{C}} \mathbb{C}[W]$ with multiplication:

$$(f \otimes w) \cdot (g \otimes y) = (f \cdot {}^w g) \otimes (w \cdot y) \quad f, g \in \mathcal{H},\ w, y \in W.$$

We use similar notation $\mathcal{H}[W']$ for any subgroup $W' \subset W$, and view $\mathbb{C}\mathbf{X}$, resp. $\mathbb{C}\mathbf{Y}$, as a commutative subalgebra of $\mathcal{H}[W']$ via the composition of imbeddings $\mathbb{C}\mathbf{X} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}[W']$.

The group W acts naturally on T and on $\Lambda = T/\mathbf{q}^{\mathbf{Y}}$. Given $\lambda \in T$, consider its image in Λ, and let $W^\lambda \subset W$ denote the isotropy group of the image of λ. The W^λ-action on T keeps the subset $\lambda \cdot \mathbf{q}^{\mathbf{Y}}$ stable, hence we may define W^λ-action on M_λ by the assignment $w : v_\mu \mapsto v_{w(\mu)}$. This way we make the twisted group algebra, $\mathcal{H}[W^\lambda]$, act on M_λ. Recall that earlier we denoted by $\mathcal{A}$ the subalgebra $\mathbb{C}\mathbf{X}$.

Theorem 3.1 (cf. [LS, 2.1]). *If $M \in \mathcal{M}^{ss}(\mathcal{H}, \mathcal{A})$, then the restriction of M to $\mathcal{H}^W$-module is semisimple, i.e., $M \in \mathcal{M}^{ss}(\mathcal{H}^W, \mathcal{A}^W)$. Furthermore,* $\operatorname{Ind}_{\mathcal{H}}^{\mathcal{H}[W]} M \in \mathcal{M}^{ss}(\mathcal{H}[W], \mathcal{A})$.

Proof. This follows from Proposition 2.5 and the twisted version of Maschke Theorem, see [M, Theorems 0.1 and 7.6(iv)]. □

Let $\mathcal{M}_\lambda(\mathcal{H}[W^\lambda], \mathcal{A})$ be the full subcategory of $\mathcal{M}^{ss}(\mathcal{H}[W^\lambda], \mathcal{A})$ formed by the modules M such that all the weights of the $\mathcal{A}$-action belong to the coset $\lambda \cdot \mathbf{q}^{\mathbf{Y}}$.

Let $M \in \mathcal{M}_\lambda(\mathcal{H}[W^\lambda], \mathcal{A})$. Note that the subgroup W^λ does not necessarily map the weight space $M(\lambda)$ into itself: if $w \in W^\lambda$ then, by definition of W^λ, we have $w(\lambda) \in \lambda \cdot \mathbf{q}^{\mathbf{Y}}$. Thus, it is possible that $w(\lambda) \neq \lambda$ so that, for $m \in M(\lambda)$, the element $w(m)$ is pushed out of the $M(\lambda)$. We define a "corrected" *dot-action* $w : m \mapsto w \cdot m$ of the group W^λ on the vector space $M(\lambda)$ as follows. As we have seen by definition, for any $w \in W^\lambda$, there exists a uniquely determined $y \in \mathbf{Y}$ such that $w(\lambda) = \lambda \cdot \mathbf{q}^y$. Then, for $m \in M(\lambda)$, put $w \cdot m = e^{-y} w(m)$. Here

$w(m) \in M$ stands for the result of w-action on m, and we claim that the element $e^{-y}w(m)$ belongs to $M(\lambda)$ (while $w(m)$ does not, in general).

Write $\mathcal{M}(W^\lambda)$ for the category of finite-dimensional $\mathbb{C}W^\lambda$-modules. With the dot-action of W^λ introduced above, we may now define a functor (cf. [LS, 2.2]) $\Phi : \mathcal{M}_\lambda(\mathcal{H}[W^\lambda], \mathcal{A}) \rightsquigarrow \mathcal{M}(W^\lambda)$ by the assignment $M \mapsto M(\lambda)$. On the other hand, given a representation N of W^λ one has an obvious $\mathcal{H}[W^\lambda]$-action on $\mathsf{M}_\lambda \otimes_{\mathbb{C}} N$ and this gives a functor $\Psi : \mathcal{M}(W^\lambda) \rightsquigarrow \mathcal{M}_\lambda(\mathcal{H}[W^\lambda], \mathcal{A})$.

Theorem 3.2. *The functors Ψ and Φ are mutually inverse equivalences.*

Proof. One has $\Phi\Psi(N) \simeq N$. If M is in $\mathcal{M}_\lambda(\mathcal{H}[W^\lambda], \mathcal{A})$ and $M(\lambda) = \Phi(M)$, then by Theorem 3.1 and Proposition 2.5, $M \simeq (\mathsf{M}_\lambda)^{\oplus m}$ as an $\mathcal{H}$-module and hence $M = \mathcal{H} \cdot M(\lambda)$. Thus, there is a morphism of $\mathcal{H}$-modules $\psi : \Psi(M(\lambda)) = \mathsf{M}_\lambda \otimes_{\mathbb{C}} M(\lambda) \to M$ given by $hv_\lambda \otimes m \mapsto h(m)$. The map ψ is injective since M_λ is simple over $\mathcal{H}$. One can easily check that ψ is actually an isomorphism of $\mathcal{H}[W^\lambda]$-modules. □

Since $\mathcal{H}$ is a subalgebra of $\mathcal{H}[W^\lambda]$ one may regard $\mathcal{H}[W^\lambda]$ as a *right* $\mathcal{H}$-module. Let $\widehat{W}^\lambda$ denote the set of isomorphism classes of simple W^λ-modules.

Proposition 3.3 (cf. [LS, 2.4]). *There is an $\mathcal{H}[W^\lambda]$-module decomposition*

$$\mathcal{H}[W^\lambda] \otimes_{\mathcal{H}} \mathsf{M}_\lambda \cong \bigoplus\nolimits_{\chi \in \widehat{W}^\lambda} (\mathsf{M}_\lambda \otimes_{\mathbb{C}} \chi)^{\oplus d_\chi} \quad , \qquad d_\chi := \dim \chi .$$

Furthermore, the $\mathcal{H}[W^\lambda]$-modules $\{\mathsf{M}_\lambda \otimes_{\mathbb{C}} \chi\ ,\ \chi \in \widehat{W}^\lambda\}$ are simple and pairwise nonisomorphic.

Proof. $\Phi(\mathcal{H}[W^\lambda] \otimes_{\mathcal{H}} \mathsf{M}_\lambda)$ is the left regular representation of W^λ. □

For any $\chi \in \widehat{W}^\lambda$, put $V_\chi := \Psi(\chi) = \mathsf{M}_\lambda \otimes_{\mathbb{C}} \chi \ \in\ \mathcal{M}_\lambda(\mathcal{H}[W^\lambda], \mathcal{A})$. Set

$$Z_\chi \ :=\ \operatorname{Ind}_{\mathcal{H}[W^\lambda]}^{\mathcal{H}[W]} V_\chi \ =\ \mathcal{H}[W] \bigotimes\nolimits_{\mathcal{H}[W^\lambda]} V_\chi \ \in \mathcal{M}^{ss}(\mathcal{H}[W^\lambda], \mathcal{A}).$$

Theorem 3.4 (cf. [LS, 2.5]). *There is an $\mathcal{H}[W]$-module isomorphism*

$$\mathcal{H}[W] \otimes_{\mathcal{H}} \mathsf{M}_\lambda \cong \bigoplus\nolimits_{\chi \in \widehat{W}^\lambda} Z_\chi{}^{\oplus d_\chi} .$$

Furthermore, Z_χ are simple pairwise nonisomorphic $\mathcal{H}[W]$-modules.

Proof. We have an obvious isomorphism:

$$\mathcal{H}[W] \otimes_{\mathcal{H}} \mathsf{M}_\lambda \ \cong\ \mathcal{H}[W] \otimes_{\mathcal{H}[W^\lambda]} \mathcal{H}[W^\lambda] \otimes_{\mathcal{H}} \mathsf{M}_\lambda .$$

The decomposition of the theorem now follows from Proposition 3.3. To prove that Z_χ are simple $\mathcal{H}[W]$-modules we write an $\mathcal{H}[W]$-module direct sum decomposition:

$$Z_\chi \cong \bigoplus\nolimits_{j=1}^{s} w_j V_\chi \quad \text{and} \quad w_j V_\chi \cong (\mathsf{M}_{w_j(\lambda)})^{\oplus d_\chi} ,$$

where $w_1 = e, \ldots, w_s$, are representatives in W of the right cosets W/W^λ. Any simple $\mathcal{H}$-submodule of Z_χ is contained in some $w_j V_\chi$.

By Theorem 3.1, the $\mathcal{H}[W]$-module $\mathcal{H}[W] \otimes_{\mathcal{H}} \mathsf{M}_\lambda$ is semisimple. Therefore, Z_χ, being a direct summand of a semisimple module, is a semisimple $\mathcal{H}[W]$-module. Hence Z_χ contains a simple submodule M with a nonzero projection from M to $w_j V_\chi$. Viewing M as an $\mathcal{H}$-module we see that $M = \bigoplus_j (M \cap w_j V_\chi)$. Since V_χ is a simple $\mathcal{H}[W^\lambda]$-module, we have $V_\chi \subset M$ and therefore $\bigoplus_j w_j V_\chi \subset M$. Hence $Z_\chi = M$.

Finally, any isomorphism $\theta : Z_\chi \to Z_\psi$ for some $\chi \neq \psi$ maps V_χ to V_ψ (just view it as a morphism of $\mathcal{H}$-modules). This would contradict Proposition 3.3. □

Proposition 3.5. *Any simple $\mathcal{H}[W]$-module M such that the $\mathbb{C}\mathbf{X}^W$-action on M is locally finite is isomorphic to Z_χ, for a certain $\chi \in \widehat{W}^\lambda$, $\lambda \in \Lambda/W$.*

Proof. We have $Z_\chi = \mathrm{Ind}_{\mathcal{H}[W^\lambda]}^{\mathcal{H}[W]}(V_\chi)$. By the Schur lemma and Frobenius reciprocity $\mathrm{Hom}(A, \mathrm{Res}\, B) = \mathrm{Hom}(\mathrm{Ind}\, A, B)$, it then suffices to show that $\mathrm{Res}_{\mathcal{H}[W^\lambda]}^{\mathcal{H}[W]}(M)$ has a submodule isomorphic to V_χ. But the latter follows from the proof of Theorem 3.2. □

Thus, we have reduced the classification of simple $\mathcal{H}[W]$-modules to the classification of irreducible representations of the finite group W^λ. The latter group is *not* a Weyl group, however. Therefore its representation theory is not classically known in geometric terms. In Section 5 we will develop an analogue of "Springer theory" for W^λ, relating irreducible representations of W^λ to semistable G-bundles on the elliptic curve $\mathbb{C}^*/\mathbf{q}^{\mathbb{Z}}$.

Remark 3.6. Note that one has the following alternative definition of Z_χ:

$$Z_\chi := \mathrm{Ind}_{\mathcal{A}[W^\lambda]}^{\mathcal{H}[W]}(\lambda \otimes \chi) = \mathcal{H}[W] \bigotimes_{\mathcal{A}[W^\lambda]} (\lambda \otimes \chi),$$

where λ denotes the one-dimensional $\mathcal{A}[W^\lambda]$-module, in which the group $W^\lambda \subset \mathcal{A}[W^\lambda]$ acts via the dot-action.

4 Morita equivalence

The algebra $\mathcal{H}$ may be viewed either as an $(\mathcal{H}[W], \mathcal{H}^W)$ - bimodule, $\mathcal{H}^l$, or as an $(\mathcal{H}^W, \mathcal{H}[W])$-bimodule, $\mathcal{H}^r$.

Proposition 4.1 (cf. [LS, 3.1]). (i) $\mathcal{H}[W]$ *and* $\mathcal{H}^W$ *are simple rings. These rings are Morita equivalent via the following functors*:

$$\mathbf{F} : \mathcal{H}[W]\text{-mod} \rightsquigarrow \mathcal{H}^W\text{-mod} \quad , \quad M \mapsto \mathcal{H}^r \otimes_{\mathcal{H}[W]} M,$$

$$\mathbf{I} : \mathcal{H}^W\text{-mod} \rightsquigarrow \mathcal{H}[W]\text{-mod} \quad , \quad N \mapsto \mathcal{H}^l \otimes_{\mathcal{H}^W} N.$$

(ii) *There are functorial isomorphisms*: $\mathbf{F}(M) \cong \mathrm{Hom}_{\mathcal{H}[W]}(\mathcal{H}^l, M) \cong M^W$.

Proof. (i) See [M, Theorems 2.3 and 2.5(a)]. (ii) Exercise. □

Similar results hold for $\mathcal{H}^{W^\lambda}$- and $\mathcal{H}[W^\lambda]$-modules, respectively. We write $\mathbf{F}_\lambda$ and $\mathbf{I}_\lambda$ for the corresponding functors.

Since $\mathcal{H}^W$ commutes with W^λ, we may regard M_λ as a left $\mathcal{H}^W \times W^\lambda$-module. Let $L_\chi = \mathrm{Hom}_{W^\lambda}(\chi^*, \mathsf{M}_\lambda)$ be the χ^*-isotypic component of the $\mathcal{H}[W^\lambda]$-module M_λ. Notice that by Proposition 4.1(ii) we have

$$L_\chi = (\mathsf{M}_\lambda \otimes_{\mathbb{C}} \chi)^{W^\lambda} = \mathbf{F}_\lambda(V_\chi) = \mathcal{H} \otimes_{\mathcal{H}[W^\lambda]} V_\chi = \mathbf{F}(Z_\chi).$$

Since $\mathsf{M}_\lambda \cong \bigoplus_{\chi \in \widehat{W}_\lambda} L_\chi \otimes V_\chi^*$ as $\mathcal{H}^W \times W^\lambda$-modules, we deduce an $\mathcal{H}[W]$-module decomposition:

$$\mathsf{M}_\lambda \cong \mathcal{H} \otimes_{\mathcal{H}[W]} \mathcal{H}[W] \otimes_{\mathcal{H}} \mathsf{M}_\lambda \cong \bigoplus\nolimits_\chi (\mathcal{H} \otimes_{\mathcal{H}[W]} Z_\chi)^{\oplus d_\chi} = \bigoplus\nolimits_\chi L_\chi^{\oplus d_\chi}. \quad (4.2)$$

Theorem 4.3 (cf. [LS, 3.4]). (i) *The $\mathcal{H}^W$-modules $\{L_\chi,\ \chi \in \widehat{W^\lambda}\}$ are simple and pairwise nonisomorphic.*

(ii) *Every simple object of $\mathcal{M}(\mathcal{H}^W, \mathcal{A}^W)$ is isomorphic to L_χ, for some $\chi \in \widehat{W}^\lambda$.*

Proof. (i) Follows from Theorem 3.4 and Morita equivalence. (ii) Follows from Proposition 3.5 and Morita equvalence. □

Proposition 4.4 (cf. [LS, 3.6]). *If M_λ and M_μ have a simple $\mathcal{H}^W$-submodule in common then $\mu \in W \cdot \lambda$, in which case $\mathsf{M}_\lambda \cong \mathsf{M}_\mu$.*

Proof. By Morita equivalence and the identity $\mathsf{M}_\lambda \cong \mathcal{H} \otimes_{\mathcal{H}[W]} \mathcal{H}[W] \otimes_{\mathcal{H}} \mathsf{M}_\lambda$, it is enough to consider the $\mathcal{H}[W]$-modules $\mathcal{H}[W] \otimes_{\mathcal{H}} \mathsf{M}_\lambda$ and $\mathcal{H}[W] \otimes_{\mathcal{H}} \mathsf{M}_\mu$. Now consider these modules as $\mathcal{H}$-modules and apply the decomposition $Z_\chi \cong \bigoplus_{j=1}^s w_j V_\chi$ from the proof of Theorem 3.4. □

5 Representations and G-bundles on elliptic curves

In this section we fix G, a connected and simply-connected complex semisimple group. We write $\mathbb{T}$ for the *abstract* Cartan subgroup of G, that is, $\mathbb{T} := B/[B, B]$, for an arbitrary Borel subgroup $B \subset G$, see [CG, ch.3]. Let $\mathbb{W}$ denote the *abstract* Weyl group, the group acting on $\mathbb{T}$ and generated by the given set of simple reflections. We also fix $\mathbf{q} \in \mathbb{C}^*$ such that $|\mathbf{q}| < 1$, and set $\mathcal{E} = \mathbb{C}^*/\mathbf{q}^{\mathbb{Z}}$.

For any complex reductive group H we let $\mathfrak{M}(\mathcal{E}, H)$ denote the moduli space of topologically trivial semistable H-bundles on $\mathcal{E}$.

Definition 5.1. A G-bundle $P \in \mathfrak{M}(\mathcal{E}, G)$ is called "semisimple" if any of the following 3 equivalent conditions hold:

(i) The structure group of P can be reduced from G to a maximal torus $T \subset G$;

(ii) The automorphism group $\mathrm{Aut} P$ is reductive;

(iii) The *substack* corresponding to the isomorphism class of P is closed in the *stack* of all G-bunles on $\mathcal{E}$.

We write $\mathfrak{M}(\mathcal{E}, G)^{ss}$ for the subspace in $\mathfrak{M}(\mathcal{E}, G)$ formed by semisimple G-bundles. To each G-bundle $P \in \mathfrak{M}(\mathcal{E}, G)$ one can assign its *semisimplification*, $P^s \in \mathfrak{M}(\mathcal{E}, G)^{ss}$. By definition, P^s corresponds to the unique closed isomorphism class in the stack of G-bundles on $\mathcal{E}$ which is contained in the closure of the isomorphism class of P. This gives the semisimplification morphism $ss : \mathfrak{M}(\mathcal{E}, G) \to \mathfrak{M}(\mathcal{E}, G)^{ss}$. It is known further (cf. for example [La]) that there are natural isomorphisms of algebraic varieties:

$$\mathfrak{M}^\circ(\mathcal{E}, \mathbb{T}) \simeq X_*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathcal{E} \quad \textit{and} \quad \mathfrak{M}(\mathcal{E}, G)^{ss} \simeq \big(X_*(\mathbb{T}) \otimes_{\mathbb{Z}} \mathcal{E}\big)/W , \tag{5.2}$$

where $\mathfrak{M}^\circ(\mathcal{E}, \mathbb{T})$ stands for the connected component of the trivial representation in $\mathfrak{M}(\mathcal{E}, \mathbb{T})$ and $X_*(\mathbb{T}) = \mathrm{Hom}_{\text{alg group}}(\mathbb{C}^*, \mathbb{T})$. Moreover, the connected components of $\mathfrak{M}(\mathcal{E}, \mathbb{T})$ are labelled by the lattice $X_*(\mathbb{T})$, and are all isomorphic to each other.

By a B-structure on a G-bundle P we mean a reduction of its structure group from G to a Borel subgroup of G. Let $\mathcal{B}(\mathcal{E}, G)$ denote the moduli space of pairs: $\{G\textit{-bundle } P \in \mathfrak{M}(\mathcal{E}, G)\ ,\ B\textit{-structure on } P\}$. Forgetting the B-structure gives a canonical morphism $\pi : \mathcal{B}(\mathcal{E}, G) \longrightarrow \mathfrak{M}(\mathcal{E}, G)$. On the other hand, given a B-structure on P one gets a B-bundle P_B, and push-out via the homomorphism: $B \twoheadrightarrow B/[B, B] = \mathbb{T}$ gives a $\mathbb{T}$-bundle on $\mathcal{E}$. Thus, there is a well-defined morphism of algebraic varieties $\nu : \mathcal{B}(\mathcal{E}, G) \longrightarrow \mathfrak{M}(\mathcal{E}, \mathbb{T})$. Further, set

$$\widetilde{G} = \{(x, B) \ \mid \ B \ \textit{ is Borel subgroup in } \ G\ ,\ x \in B\},$$

and let $\pi : \widetilde{G} \to G$ be the first projection.

We have the following two commutative diagrams, where the one on the left is the Grothendieck–Springer “universal resolution” diagram, cf. e.g., [CG, ch. 3], and the one on the right is its “analogue” for bundles on the elliptic curve $\mathcal{E}$:

$$\begin{tikzcd} & \widetilde{G} \arrow[dl, "\pi"'] \arrow[dr, "\nu"] & \\ G \arrow[dr] & & \mathbb{T} \arrow[dl] \\ & \mathtt{Spec}(\mathbb{C}[G]^G) \simeq \mathbb{T}/\mathbb{W} & \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{B}(\mathcal{E}, G) \arrow[dl, "\pi"'] \arrow[dr, "\nu"] & \\ \mathfrak{M}(\mathcal{E}, G) \arrow[dr, "ss"'] & & \mathfrak{M}(\mathcal{E}, \mathbb{T}) \arrow[dl] \\ & \mathfrak{M}(\mathcal{E}, G)^{ss} \simeq \mathfrak{M}(\mathcal{E}, \mathbb{T})^\circ/\mathbb{W} & \end{tikzcd} \tag{5.3}$$

Observe that for any $P \in \mathfrak{M}(\mathcal{E}, G)$, the group $\mathrm{Aut} P$ acts naturally on the set $\mathcal{B}(\mathcal{E}, G)_P := \pi^{-1}(P)$ of all B-structures on P. This induces an action of

Aut $P/\mathrm{Aut}^\circ P$, the (finite) group of connected components, on the complex top homology group: $H_{top}(\mathcal{B}(\mathcal{E}, G)_P, \mathbb{C})$.

Definition 5.4. An irreducible representation of the group Aut $P/\mathrm{Aut}^\circ P$ is called "admissible" if it occurs in $H_{top}(\mathcal{B}(\mathcal{E}, G)_P, \mathbb{C})$ with nonzero multiplicity.

One of the main results of this paper is the following.

Theorem 5.5. *There exists a bijection between the set of (isomorphism classes of) simple objects of $\mathcal{M}(\mathcal{H}[W], \mathcal{A})$ and the set of (isomorphism classes of) pairs (P, α), where $P \in \mathfrak{M}(\mathcal{E}, G)$, and α is an* admissible *representation of the group* Aut $P/(\mathrm{Aut}\ P)^\circ$.

The rest of this section is devoted to the proof of the theorem. As a first approximation, recall Proposition 2.5, which states that simple objects of the category $\mathcal{M}(\mathcal{H}, \mathcal{A})$ are in one-to-one correspondence with the points of the abelian variety $\Lambda = T/\mathbf{q}^{\mathbb{Z}}$ which is, by (5.2), nothing but $\mathfrak{M}^\circ(\mathcal{E}, \mathbb{T})$. In the same spirit, it turns out that replacing algebra $\mathcal{H}$ by $\mathcal{H}[W]$ leads to the replacement of $\mathfrak{M}(\mathcal{E}, \mathbb{T})$ to $\mathfrak{M}(\mathcal{E}, G)$, as a parameter space for simple modules. Specifically, the transition from Proposition 3.5 to G-bundles will be carried out in two steps. In the first step, we reinterpret the data involved in Proposition 3.5 in terms of loop groups, and in the second step we pass from loop groups to G-bundles.

We need some notation regarding formal loop groups. Let $\mathbb{C}((z))$, $\mathbb{C}[[z]]$, $\mathbb{C}[z]$ be the field of formal Laurent series, the ring of formal Taylor series and the ring of polynomials, respectively. Let $G((z))$ be the group of all $\mathbb{C}((z))$-rational points of G, and similarly for $G[[z]]$, $G[z]$. We consider $\mathbf{q}$-conjugacy classes in $G((z))$, i.e., $G((z))$-orbits on itself under $\mathbf{q}$-conjugation: $g(z) : h(z) \mapsto g(\mathbf{q}z)h(z)g(z)^{-1}$. A $\mathbf{q}$-conjugacy class, is said to be *integral* if it contains at least one element of $G[[z]]$.

Fix a Borel subgroup $B = T \cdot U \subset G$, where T is a maximal torus of G and U is the unipotent radical of B. By [BG, Lemma 2.2] we have the following:

Jordan q-normal form for G[[z]]. *Any element $h \in G[[z]]$ is $\mathbf{q}$-conjugate to a product $s \cdot b(z)$, where $s \in T$ is a constant loop, and $b \in U[z]$ are such that:*

(J1) $b(\mathbf{q}z) \cdot s = s \cdot b(z)$,

(J2) $\mathrm{Ad}\, s(v) = \mathbf{q}^m v$, *for some* $v \in \mathrm{Lie}\, G$, $m > 0 \implies v \in \mathrm{Lie}\, U$.

For any group M, we write M° for the identity connected component of M, and $Z_M(x)$ for the centralizer of an element x in M. Given $h \in G((z))$ we write $G_{\mathbf{q},h}$ for the $\mathbf{q}$-centralizer of $h(z)$ in $G((z))$:

$$G_{\mathbf{q},h} := \{g(z) \in G((z)) \quad | \quad g(\mathbf{q}z)h(z)g(z)^{-1} = h(z)\}.$$

Let $W_G = N_G(T)/T$ be the Weyl group of (G, T). Given $s \in T$, write $\lambda(s)$ for its image in $\Lambda = T/\mathbf{q}^{\mathbb{Z}}$, and let $W^{\lambda(s)}$ denote the isotropy group of the point $\lambda(s) \in T/\mathbf{q}^{\mathbb{Z}}$ under the natural W-action.

Theorem 5.6. *Let $h = s \cdot b \in G[[z]]$ be written in its $\mathbf{q}$-normal form. Then we have $G_{\mathbf{q},h} = Z_{G_{\mathbf{q},s}}(b)$. Furthermore,*

(i) *$G_{\mathbf{q},s}$ is a finite-dimensional reductive group isomorphic to a (not necessarily connected) subgroup $H \subset G$ containing the maximal torus T.*

(ii) *There exists a unipotent element $u \in H$, uniquely determined up to conjugacy in H, such that under the isomorphism in* (i) *we have $G_{\mathbf{q},h} = Z_{G_{\mathbf{q},s}}(b) \xrightarrow{\sim} Z_H(u)$.*

(iii) *The group $W^{\lambda(s)}$ is isomorphic to $W_H := N_H(T)/T$, the "Weyl group" of the disconnected group H.*

The proof of the theorem will follow from Lemma 5.11 and Proposition 5.13 given later in this section.

From loop group to G-bundles. In [BG] we have constructed a bijection:

$$\mathfrak{M}(\mathcal{E}, G) \quad \stackrel{\Theta}{\longleftrightarrow} \quad \textit{integral } \mathbf{q}\textit{-conjugacy classes in } G((z)) \ . \tag{5.7}$$

Let $P = \Theta(h)$ be the G-bundle corresponding to a $\mathbf{q}$-conjugacy class of $h \in G((z))$, and $P^s = ss(P)$ its semisimplification. Without loss of generality we may assume that h is written in its $\mathbf{q}$-normal form $h = s \cdot b$. Using Theorem 5.6 it is easy to verify that under the bijection (5.7) we have:

- $P^s = \Theta(s)$ and $\operatorname{Aut} P^s \simeq G_{\mathbf{q},s} \simeq H \subset G$. (5.8.1)
- $\operatorname{Aut} P \simeq G_{\mathbf{q},h} \simeq Z_H(u)$. (5.8.2)

Further, recall the variety $\mathcal{B}(\mathcal{E}, G)_P$ of all B-structures on P, see (5.3). Let $\mathcal{B}(\mathcal{E}, G)_P^\circ$ denote a connected component of $\mathcal{B}(\mathcal{E}, G)_P$. Write $\mathcal{B}(H)$ for the flag variety of the group H, and $\mathcal{B}(H)_u$ for the Springer fiber over u, the u-fixed point set in $\mathcal{B}(H)$. Then we have:

- $\mathcal{B}(\mathcal{E}, G)_P^\circ \simeq \mathcal{B}(H)_u$. (5.8.3)

Furthermore, the natural $Z_{H^\circ}(u)$-action on $\mathcal{B}(H)_u$ goes under the isomorphism above and the imbedding: $Z_{H^\circ}(u) \hookrightarrow Z_H(u) = \operatorname{Aut} P$ to the natural $\operatorname{Aut} P$-action on $\mathcal{B}(P)$.

By isomorphism (5.8.3), one identifies the action of the finite group $Z_{H^\circ}(u)/Z_H^\circ(u)$ on $H_{top}(\mathcal{B}(H)_u, \mathbb{C})$, the top homology, with the action of the corresponding subgroup of $\operatorname{Aut} P/\operatorname{Aut}^\circ P$ on $H_{top}(\mathcal{B}(\mathcal{E}, G)_P^\circ, \mathbb{C})$. It follows that an irreducible representation of $\operatorname{Aut} P/\operatorname{Aut}^\circ P$ is admissible in the sense of Definition 5.4 if and only if the restriction of the corresponding representation of $Z_H(u)/Z_H^\circ(u)$ to the subgroup $Z_{H^\circ}(u)/Z_H^\circ(u) \subset Z_H(u)/Z_H^\circ(u)$ is isomorphic to a direct sum of irreducible representations which have nonzero multiplicity in the $Z_{H^\circ}(u)/Z_H^\circ(u)$-module $H_{top}(\mathcal{B}(H)_u, \mathbb{C})$.

Finally, we observe that the isotropy group $W^{\lambda(s)}$ occurring in part (iii) of Theorem 5.6 is exactly the group whose irreducible representations label the simple objects of the category $\mathcal{M}(\mathcal{H}[W], \mathcal{A})$, see Proposition 3.5. Thus, according to the isomorphism $W^{\lambda(s)} \simeq W_H$ of Theorem 5.6 (iii), we are interested in a parametrisation of irreducible representations of the group W_H. Such a parametrisation is provided by a version of the Springer correspondence for disconnected reductive groups, developed in the last section (Appendix) of this paper. This concludes an outline of the proof of Theorem 5.6.

We now begin a detailed exposition, and recall the Bruhat decomposition for the group $G[z, z^{-1}]$. Let $G_1[z] \subset G[z]$ denote the subgroup of loops equal to $e \in G$ at $z = 0$ and denote by $\mathcal{U}^+$ the subgroup $U \cdot G_1[z]$. Similarly, $\mathcal{U}^-$ will denote $U^- \cdot G_1[z^{-1}]$ where $U^- \subset G$ is the unipotent subgroup opposite to U and $G_1[z^{-1}]$ is the kernel of evaluation map $G[z^{-1}] \to G$ at $z = \infty$.

Proposition 5.9 (cf. [PS, Chapter 8]). *Any element of $g(z) \in G[z, z^{-1}]$ admits a* unique *representation of the form*

$$g(z) = u_1(z) \cdot \lambda(z) \cdot n_w \cdot t \cdot u_2(z)$$

where $u_1(z), u_2(z) \in \mathcal{U}^+$, $\lambda(z) \in \mathbf{Y} = Hom_{alg}(\mathbb{C}^*, T)$, $t \in T$, $w \in W$ and $u_2(z)$, in addition, satisfies $[\lambda(z)n_w] \cdot u_2(z) \cdot [\lambda(z)n_w]^{-1} \in \mathcal{U}^-$.

Corollary 5.10. *The* $\mathbf{q}$*-conjugacy classes that intersect $T \subset G((z))$ are parametrized by Λ/W.*

Proof. Suppose that $s \in T$ is $\mathbf{q}$-conjugate to $s' \in T$ by an element $g(z) \in G((z))$. Rewriting this in the form $g(\mathbf{q}z)s = s'g(z)$, then using the above decomposition and its uniqueness, we obtain $s' = w(s) \cdot \lambda(\mathbf{q})$. Conversely, for any $w \in W$ and $\lambda \in \mathbf{Y}$, the element s is conjugate to $w(s) \cdot \lambda(\mathbf{q})$ by the element $g(z) = \lambda(z) \cdot n_w$. □

Uniqueness of the $\mathbf{q}$-normal Jordan form follows from

Lemma 5.11. *Suppose that two loops $s \cdot b(z)$ and $s' \cdot b'(z)$ satisfy the Jordan form conditions (J1)-(J2), and that $f(\mathbf{q}z)(s \cdot b(z))f(z)^{-1} = s' \cdot b'(z)$ for some $f(z) \in G((z))$. Then $f(\mathbf{q}z) \cdot s \cdot f(z)^{-1} = s'$ and $f(z)b(z)f(z)^{-1} = b'(z)$.*

Proof. Choose a faithful representation: $G \to GL(V)$, and a basis in V such that U maps to upper-triangular matrices and T maps to diagonal matrices. We may assume without loss of generality that the loops $s \cdot b(z)$ and $s' \cdot b'(z)$ are both maped into upper-triangular matrices, $A(z)$ and $A'(z)$, resp.

First we consider the case when all diagonal entries of $A(z)$ (resp. $A'(z)$) differ only by powers of $\mathbf{q}$, i.e., when they are of the form $a\mathbf{q}^{m_1}, \ldots, a\mathbf{q}^{m_k}$, where k is the dimension of V and $m_1 \geq m_2 \geq \ldots \geq m_k$, due to Jordan form condition (J2). Further, by the Jordan form condition (J1) all entries A_{ij} above the diagonal are of the form $\alpha_{ij} z^{m_i - m_j}$, $i < j$, $\alpha_{ij} \in \mathbb{C}$. Also let $a'\mathbf{q}^{n_1}, \ldots, a'\mathbf{q}^{n_k}$ be the diagonal entries of $A'(z)$ and let $F(z)$ be the matrix corresponding to $f(z)$.

We prove by descending induction on $i - j$ that $F_{i,j} = cz^l$, for an appropriate constant c and an integer l, depending on i, j. Our proof is based on the simple observation that, for any constant B and any integer l, the equation $x(\mathbf{q}z) = \mathbf{q}^l x(z) + Bz^l$ admits a solution in $\mathbb{C}((z))$ if and only if $B = 0$, in which case the solution has to be $x(z) = cz^l$, $c \in \mathbb{C}$.

The largest value of $i - j$, attained for $i = k$, $j = 1$, corresponds to the lower left corner element $F_{k,1}(z)$. From the equation $F(\mathbf{q}z)A(z) = A'(z)F(z)$ one has $F_{k,1}(\mathbf{q}z)a\mathbf{q}^{m_1} = F_{k,1}(z)a'\mathbf{q}^{n_k}$. If the ratio a/a' is not a power of $\mathbf{q}$, this equation, as well as other equations considered below, has only zero solution (which gives a noninvertible matrix $F(z)$). Hence we can assume that $a' = a\mathbf{q}^r$ for some integer r. Then the above equation for $F_{k,1}(z)$ implies that $F_{k,1}(z) = \phi_{k,1} z^{n_k - m_1 + r}$ for some constant $\phi_{k,1}$.

We use this expression for $F_{k,1}$ to write the equations for $F_{k-1,1}$ and $F_{k,2}$, then write the equations for $F_{k-2,1}$, $F_{k-1,2}$, $F_{k,3}$, etc. In general, by descending induction on $i - j$ (ranging from $i - j = k - 1$ to $i - j = -k + 1$) one obtains equations of the type

$$F_{i,j}(\mathbf{q}z) = \mathbf{q}^{n_i - m_j + r} g_{i,j}(z) + C z^{n_i - m_j + r}$$

for some constant C depending on i, j and the previously computed values of $g_{s,t}$. As before, this leads to

$$C = 0 \quad \text{and} \quad F_{i,j}(z) = \phi_{i,j} z^{n_i - m_j + r}, \quad \phi_{ij} \in \mathbb{C}.$$

This equation implies that $f(\mathbf{q}z) \cdot s \cdot f(z)^{-1} = s'$, and $f(z)b(z)f(z)^{-1} = b'(z)$ is an immediate consequence.

In the general case, by Jordan form condition (J1) one can choose a basis of V so that $A(z)$, $A'(z)$ will have square blocks as in the first part of the proof ("$\mathbf{q}$-Jordan blocks") along the main diagonal, and zeros everywhere else. We can assume that any two diagonal entries which differ by a power of $\mathbf{q}$, belong to the same block. A direct computation shows that, up to permutation of blocks in $A(z)$ and $A'(z)$, the conjugating matrix $F(z)$ also has square blocks along the main diagonal and zeros everywhere else. Now we apply the above argument to each individual block to obtain the result in the general case. □

Corollary 5.12. (i) *The assignment: $s \cdot b(z) \mapsto s$ descends to a well-defined map* $\Phi : \ \{$*integral* $\mathbf{q}$ *− conjugacy classes in* $G((z))\} \longrightarrow \Lambda/W$.

(ii) *Let $s \cdot b(z)$ be a Jordan $\mathbf{q}$-normal form, and $\lambda \in \Lambda/W$ the image of $s \in T$ in Λ/W. Then the set $\Phi^{-1}(\lambda)$ can be identified with those (ordinary) conjugacy classes in $G_{\mathbf{q},s}$, the $\mathbf{q}$-centralizer of s in $G((z))$, which have nontrivial intersection with $U[z]$.*

Remark. We will see below that $G_{\mathbf{q},s}$ is a finite-dimensional reductive group and that $\Phi^{-1}(\lambda)$ is nothing but the set of unipotent conjugacy classes in this reductive group.

Now we begin to study the automorphism group of the G-bundle P^s associated to $s \in T$. To describe $G_{\mathbf{q},s} \simeq \mathrm{Aut}\, P^s$ first recall that by [BG, Lemma 2.5], $G_{\mathbf{q},s}$ consists of polynomial loops, i.e., $G_{\mathbf{q},s} \subset G[z, z^{-1}]$. Thus, there is a well-defined evaluation map $ev_{z=1} : G_{\mathbf{q},s} \to G$ sending a polynomial loop to its value at $z = 1$. Let $H \subset G$ be the image of $G_{\mathbf{q},s}$. Write $N_H(T)$ for the normalizer of T in H, and $W_H := N_H(T)/T$ for the "Weyl group" of the (generally *diconnected*) group H.

Proposition 5.13. (i) *The evaluation map $ev_{z=1} : G_{\mathbf{q},s} \to G$ is injective;*

(ii) *The idenity component H° of H equals the connected reductive subgroup of G corresponding to the root subsystem $\Delta_{\mathbf{q},s} \subset \Delta$ of all roots $\alpha \in \Delta \subset Hom(T, \mathbb{C}^*)$ for which $\alpha(s)$ is an integral power of $\mathbf{q}$;*

(iii) *The group W_H is isomorphic to the subgroup $W^\lambda \subset W$ of all $w \in W$ which fix $\lambda \in \Lambda = \mathbb{C}^*/\mathbf{q}^{\mathbb{Z}}$, the image of $s \in T$.*

Proof. The equation $g(\mathbf{q}z) \cdot s \cdot g(z)^{-1} = s$ can be rewritten as $g(\mathbf{q}z) = s \cdot g(z) \cdot s^{-1}$. We decompose $g(z)$ as in Proposition 5.9 and, using the uniqueness of this decomposition, obtain

$$u_1(\mathbf{q}z) = su_1(z)s^{-1}, \quad u_2(\mathbf{q}z) = su_2(z)s^{-1}, \quad s = w(s) \cdot \lambda(\mathbf{q}).$$

Rewrite $u_1(z) \in \mathcal{U}^+$ as $exp\big(\sum_{k=0}^{\infty} g_k z^k\big)$; then $\mathrm{Ad}\, s(g_k) = \mathbf{q}^k$ due to the first equation. In particular, only finitely many of g_k are nonzero and by the Jordan form condition (J2), $g_k \in \mathrm{Lie}\,(U)$. Hence $u_1(z) \in U[z]$ and, since different eigenspaces of $\mathrm{Ad}\, s$ on $\mathrm{Lie}\,(U)$ have zero intersection, $u_1(z)$ is uniquely determined by $u_1(1) = exp(\sum_{k=0}^{N} g_k)$. The same argument applies to $u_2(z)$. Moreover, since $u_2' = \big[\lambda(z)n_w\big]u_2(z)\big[\lambda(z)n_w\big]^{-1} \in U^- \cdot G_1[z^{-1}]$ and $\mathrm{Ad}\, su_2'(z) = u_2'(\mathbf{q}z)$, we can repeat the argument once more and conclude that $u_2'(1) = n_w u_2(1) n_w^{-1} \in U^-$.

Now we can show that $g(z)$ is determined by $g(1) = u(1)n_w t u_2(1)$. In fact, since $u_1(1), u_2(1) \in U$ and $n_w u_2(1) n_w^{-1} \in U^-$, the usual Bruhat decomposition for $g(1) \in G$ implies that n_w, $u_1(1)$ and $u_2(1)$ are uniquely determined by $g(1)$, hence $u_1(z)$ and $u_2(z)$ are uniquely determined by $g(1)$. The element $\lambda(z)$ can be reconstructed from a and w since $s = w(s) \cdot \lambda(\mathbf{q})$ and $\mathbf{q}$ is not a root of unity. The proposition follows. □

Example. The following example, showing that the component group H/H° can in fact be nontrivial, was kindly communicated to us by D. Vogan.

Recall that for the root system of type D_4, the coroot lattice $\mathbf{Y}$ can be identified with the subgroup of the standard Eucledian lattice $L_4 = \langle e_1, e_2, e_3, e_4 \rangle$, $(e_i, e_j) = \delta_{ij}$ formed by all vectors in L_4 with even sum of coordinates. Then the set of coroots is identified with $\pm e_i \pm e_j$, $i \neq j$, and the Weyl group W acts by permuting the e_i, and changing the sign of any even number of the basis vectors e_i. The choice of the simple coroots $\alpha_1^\vee = e_1 - e_2$, $\alpha_2^\vee = e_2 - e_3$, $\alpha_3^\vee = e_3 - e_4$, $\alpha_4^\vee = e_3 + e_4$ identifies $T = \mathbf{Y} \otimes_{\mathbb{Z}} \mathbb{C}^*$ with $(\mathbb{C}^*)^4$. Now consider the element $s = (-1, \sqrt{\mathbf{q}}, -1, -\sqrt{\mathbf{q}}) \in (\mathbb{C}^*)^4 \simeq T$. A straightforward calculation shows that, in the notation of the above proposition, $W_H = \{\pm 1\}$ while $\Delta_{\mathbf{q},s}$ is empty.

End of proof of Theorem 5.5. By Proposition 3.5 we have to establish the correspondence between the set of pairs (λ, χ) where $\lambda \in \Lambda$, $\chi \in \widehat{W^\lambda}$, and the set of pairs (P, α) as in the statement of Theorem 5.5. Take any lift $s \in T$ of the element $\lambda \in \Lambda$ and consider the G-bundle P^s corresponding to s, together with its automorphism group H. By Proposition 5.13 (iii) and the Springer Correspondence (see Appendix), the representation χ defines a unipotent H-orbit together with the admissible representation α of the centralizer of any point u in this orbit. The element $u \in H$ corresponds via Proposition 5.13 (i) to a certain loop $b(z)$, such that $s \cdot b(z)$ is a $\mathbf{q}$-normal form. The bundle P corresponds to the $\mathbf{q}$-conjugacy class of $s \cdot b(z)$. $\square$

It will be convenient for us in the next section to reinterpret the parameters (P, α) entering Theorem 5.5 in a different way as follows. First, giving $P \in \mathfrak{M}(\mathcal{E}, G)$ is equivalent, according to (5.7), to giving the $\mathbf{q}$-conjugacy class of an element $h(z) \in G[[z]]$. Using the Jordan $\mathbf{q}$-normal form, write $h(z) = s \cdot b(z)$, where $s \in T$, is a semisimple element in G, the subgroup of constant loops. Furthermore, by Theorem 5.6 we have $\operatorname{Aut} P/\operatorname{Aut}^\circ P = Z_{G_{\mathbf{q},s}}(b)/Z^\circ_{G_{\mathbf{q},s}}(b)$.

Let $Q = ss(P)$ be the semisimplification of P. By (5.8.1), this is the G-bundle on $\mathcal{E}$ corresponding, under the bijection (5.7), to the constant loop s. Let G_Q denote the associated vector bundle on $\mathcal{E}$ corresponding to the principal G-bundle Q and the adjoint representation of the group G. By construction, $b(z)$ is a polynomial loop with unipotent values that $\mathbf{q}$-commutes with s. Hence $b(z)$ gives rise to a unipotent automorphism $\hat{b} \in \operatorname{Aut} Q$. This way one obtains a bijection:

$$\mathfrak{M}(\mathcal{E}, G) \quad \longleftrightarrow \quad \left\{ \begin{matrix} \text{semisimple } G\text{-bundle } Q \in \mathfrak{M}(\mathcal{E}, G)^{ss} \\ \text{and a unipotent element } u \in \operatorname{Aut} Q \end{matrix} \right\}. \tag{5.14}$$

It is not difficult to show that the set $\mathcal{B}(\mathcal{E}, G)_P$, see (5.8.3) is identified, under the bijection above, with the set of u-stable B-structures on the G-bundle $ss(P)$.

Fix $\mathbf{q} \in \mathbb{C}^*$, which is not a root of unity. An element of the group $G((z))$ will be called $\mathbf{q}$-*semisimple*, resp. $\mathbf{q}$-*unipotent*, if it is $\mathbf{q}$-conjugate to a constant semisimple loop, resp. conjugate (in the ordinary sense) to an element of $U[z]$. Write $G((z))^{\mathsf{q-ss}}$ and $G((z))^{\mathsf{q-uni}}$ for the sets of $\mathbf{q}$-semisimple and $\mathbf{q}$-unipotent elements, respectively. Given $h(z) \in G((z))$, recall the notation $G_{\mathbf{q},h}$ for the $\mathbf{q}$-centralizer of h in $G((z))$, and for any $u(z) \in G((z))$, put

$$Z_{\mathbf{q},h}(u) = \{g(z) \in G((z)) \;\Big|\; g(\mathbf{q}z)h(z) = h(z)g(z) \;\;\&\;\; g(z)u(z) = u(z)g(z)\}$$

a simultaneous "centralizer" of $h(z)$ and $u(z)$. If h is $\mathbf{q}$-semisimple and u $\mathbf{q}$-commutes with h, then the group $Z_{\mathbf{q},h}(u)$ acts on $\mathcal{B}(G_{\mathbf{q},h})_u$, the u-fixed point set in the flag variety of the finite-dimensional reductive group $G_{\mathbf{q},h}$, see Theorem 5.6 (i). This gives a $Z_{\mathbf{q},h}(u)/Z^\circ_{\mathbf{q},h}(u)$-action on $H_*(\mathcal{B}(G_{\mathbf{q},h})_u)$, the *total* homology. An irreducible representation of the component group $Z_{\mathbf{q},h}(u)/Z^\circ_{\mathbf{q},h}(u)$ is said to be *admissible* if it occurs in $H_*(\mathcal{B}(Z_{\mathbf{q},h})_u)$ with nonzero multiplicity. We let $Z_{\mathbf{q},h}\widehat{(u)/Z^\circ_{\mathbf{q},h}}(u)$ denote the set of admissible $Z_{\mathbf{q},h}(u)/Z^\circ_{\mathbf{q},h}(u)$-modules (cf. Definition 5.2 and the paragraph below formula (5.8.3)).

We now consider the following set:

$$\mathbf{M} = \left\{ (s, u, \chi) \;\middle|\; \begin{matrix} s \in G((z))^{\mathsf{q-ss}}\,,\; u \in G((z))^{\mathsf{q-uni}} \\ s(z)u(z)s(z)^{-1} = u(\mathbf{q}z)\,,\; \chi \in Z_{\mathbf{q},s}\widehat{(u)/Z^\circ_{\mathbf{q},s}}(u) \end{matrix} \right\}. \tag{5.15}$$

Thus, we can reformulate Theorem 5.5 as follows:

Theorem 5.16. *There exists a natural bijection between the set of isomorphism classes of simple objects of* $\mathcal{M}(\mathcal{H}[W], \mathcal{A})$ *and the set of* $\mathbf{q}$*-conjugacy classes in* $\mathbf{M}$.

Springer correspondence for disconnected groups

In this appendix we show how to extend the classical Springer Correspondence to the case of not necessarily connected reductive groups. First we recall briefly (see [CG, Chapter 3] for details) the situation for a general *connected* reductive group, such as the group H° of §5.

Let $\mathcal{N} \subset H^\circ$ be the subset of unipotent elements and $\mathcal{B}$ the variety of Borel subgroups in H°. The subvariety $\widetilde{\mathcal{N}} \subset \mathcal{B} \times \mathcal{N}$ of all pairs $\{(B_H, u) \mid u \in B_H\}$, provides an H°-equivariant smooth resolution $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$, called the *Springer resolution*.

Denote by $\mathcal{Z}$ the fiber product $\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$, which can also be indentified with (cf. [CG]) the subvariety in $T^*(\mathcal{B} \times \mathcal{B})$ given by the union of conormal bundles to the H°-orbits on $\mathcal{B} \times \mathcal{B}$ (with respect to the diagonal action). The top Borel–Moore homology group $H(\mathcal{Z})$ is endowed with a structure of an associative algebra via the convolution product (see [CG]). Moreover, the set $\mathbb{W} \subset H(\mathcal{Z})$ of fundamental classes of irreducible components of $\mathcal{Z}$ forms a group with respect to the convolution product, called the *abstract Weyl group*, and $H(\mathcal{Z})$ can be identified with the group algebra of $\mathbb{W}$. A particular choice of a Borel subgroup $B_H \supset T$ identifies the usual Weyl group $W^\circ = N_{H^\circ}(T)/T$ with $\mathbb{W}$ by sending the class of $n_w \in N_{H^\circ}$ to the fundamental class of the conormal bundle to the H°-orbit of $(B_H, n_w B_H n_w^{-1}) \in \mathcal{B} \times \mathcal{B}$.

Consider a unipotent orbit $\mathcal{O} \subset \mathcal{N}$. The top Borel–Moore homology groups of the fibers of $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ over $\mathcal{O}$ form an irreducible local system $L_{\mathcal{O}}$ on $\mathcal{O}$ which is equivariant with respect to $\mathbb{W} \times H^\circ$ (the action of $\mathbb{W}$ in the fibers of $L_{\mathcal{O}}$ comes from the convolution construction, cf. [CG], and the action of H° from the H°-equivariance of π). Decompose $L_{\mathcal{O}}$ into a direct sum of irreducible $\mathbb{W} \times H^\circ$-equivariant local systems $L_1, \dots, L_k$. For any representation ϕ of $\mathbb{W}$ we can consider the local system I_i formed by the $\mathbb{W}$-invariants of the tensor product $\phi^\vee \otimes L_i$. It turns our that, for any irreducible representation ϕ, there exists a unique orbit $\mathcal{O}_\phi$ and a unique $L_\phi \in \{L_1, \dots, L_k\}$ for which the local system I_ϕ, constructed from ϕ as above, is nonzero. Moreover, such I_ϕ is an irreducible H°-equivariant local system associated to an "admissible representation" of the component group of the centralizer Z_u of a point $u \in \mathcal{O}$. Recall (see Definition 5.4 and the discussion before Proposition 5.9) that the representation of $Z_H(u)/Z_H^\circ(u)$ is called *admissible* if its restriction to the subgroup $Z_{H^\circ}(u)/Z_H^\circ(u) \subset Z_H(u)/Z_H^\circ(u)$ is isomorphic to a direct sum of irreducible representations which have nonzero multiplicity in the $Z_{H^\circ}(u)/Z_H^\circ(u)$-module $H_{top}(\mathcal{B}(H)_u, \mathbb{C})$, where $\mathcal{B}(H)_u$ is the fiber $\pi^{-1}(u)$ over any point of the orbit $\mathcal{O}$.

Below we will use the language of equivariant local systems (which is equivalent to the language of admissible representations).

The *Springer correspondence* $\phi \mapsto (\mathcal{O}_\phi, I_\phi)$ gives a bijection between the set of irreducible representations of its Weyl group $\mathbb{W}$ and the set of pairs $(\mathcal{O}, I)$ where $\mathcal{O}$ is a unipotent orbit of H° and I is a certain H°-equivariant irreducible local system on $\mathcal{O}$ coming from an admissible representation of Z_u/Z_u°.

We proceed to representation theory of the "Weyl group" $W_H = N_H(T)/T$ of a *disconnected* reductive group H (see Proposition 5.13).

Lemma A.1. *A choice of a Borel subgroup $B_H \supset T$ in H° identifies the Weyl group W_H with the semidirect product $W^\circ \rtimes (W_H/W^\circ)$. Moreover, one has a canonical isomorphism $H/H^\circ \simeq W_H/W_H^\circ$.*

Proof. Consider the subgroup $N'(T) := N_H(B_H) \cap N_H(T)$. Then the embedding $N'(T) \subset H$ induces the isomorphisms

$$H/H^\circ \simeq N'(T)/T \simeq W_H/W^\circ.$$

Since $N'(T)$ is a subgroup of $N_H(T)$, we obtain an embedding $W_H/W^\circ \subset W_H$. Now the assertion of the lemma follows. □

Remark. A different choice of a Borel subgroup B'_H containing T gives a conjugate embedding $w(W_H/W^\circ)w^{-1} \subset W_H$, where $w \in W^\circ$ is the unique element which conjugates B_H into B'_H.

Note that H acts on $\mathcal{N}$ and on $\widetilde{\mathcal{N}}$. In particular H permutes the irreducible components of $\mathcal{Z}$. This induces an H/H°-action on $\mathbb{W}$ by group automorphisms.

Proposition A.2. *The isomorphism $\mathbb{W} = W^\circ$ (depending on the choice of B_H) and the canonical isomorphism $H/H^\circ \simeq W_H/W^\circ$ identify the above action of H/H° on $\mathbb{W}$ with the conjugation action of W_H/W° on W° arising from Lemma A.1.*

Proof. It suffices to replace the pair (H, H°) by $(N'(T), T)$. Let n_w be a lift to N_{H° of a certain element $w \in W$, and let $\mathcal{Z}_w$ be the cotangent bundle to the orbit of $(B_H, n_w B_H n_w^{-1}) \in \mathcal{B} \times \mathcal{B}$. Similarly, let $n_\sigma \in N'(T)$ be a lift of an element $\sigma \in W_H/W^\circ$. Denote $\sigma w \sigma^{-1} \in W^\circ \subset W_H$ by w^σ, then $n_{w^\sigma} = n_\sigma n_w n_\sigma^{-1}$ is a lift of w^σ to $N_{H^\circ}(T)$.

By definition of $N'(T)$ the element n_σ normalizes B_H. Hence n_w sends $(B_H, n_w B n_w^{-1})$ to $(B_H, n_{w^\sigma} B_H n_{w^\sigma}^{-1})$. Thus, $n_\sigma \cdot \mathcal{Z}_w = \mathcal{Z}_{w^\sigma}$. □

Now we recall the basic facts of Clifford theory (cf. [Hu]) which apply to any finite group W_H and its normal subroup W°, not necessarily arising as Weyl groups.

The group W_H acts by conjugation on the set $\widehat{W}^\circ$ of irreducible representations of W°. Let $\mathcal{V}_1 \ldots \mathcal{V}_k$ be the orbits of its action. For any irreducible representation $\psi \in \widehat{W}_H$ we can find an orbit $\mathcal{V}_{i(\psi)}$ and a positive integer e, such that the restriction of ψ to W° is isomorphic to a multiple of the orbit sum:

$$\psi|_{W_0} \simeq e \cdot \left(\sum\nolimits_{\phi \in \mathcal{V}_{i(\psi)}} \phi\right).$$

Fix $\phi \in \mathcal{V}_{i(\psi)}$ and consider the subset $(\widehat{W}_H)_\phi \subset \widehat{W}_H$ of all representations whose restriction to W° contains an isotypical component isomorphic to ϕ (and hence automatically all representations in the orbit of ϕ). Obviously, $\widehat{W}_H$ is a disjoint union of $(\widehat{W}_H)_{\phi_i}$, where $\phi_i \in \mathcal{V}_i$ is any representative of the orbit $\mathcal{V}_i$.

To study $(\widehat{W}_H)_\phi$ we consider the stabilizer $W^\phi \subset W_H$ of $\phi \in \widehat{W}^\circ$. Then by Clifford theory (cf. [Hu]), the induction from W^ϕ to W_H establishes a bijection between $(\widehat{W}^\phi)_\phi$ and $(\widehat{W}_H)_\phi$. Moreover, any linear representation $\chi \in (\widehat{W}^\phi)_\phi$ is isomorphic to the tensor product $p_1 \otimes p_2$ of two *projective* representations p_1 and p_2 (cf. [Hu]) such that

(i) $p_1(x) = \phi(x)$, $p_2(x) = 1$ if $x \in W^\circ$

(ii) $p_1(gx) = p_1(g)\phi(x)$ and $p_1(xg) = \phi(x)p_1(g)$ if $x \in W^\circ$, $g \in W_H$.

Thus, p_2 is a projective representation of W_H/W° which plays the role of the multiplicity space of dimension e in terms of the above formula for $\psi|_{W^\circ}$. The second condition means that the projective cocycle of p_1 is in fact lifted from W^ϕ/W°. From now on we will fix the decomposition $\chi = p_1 \otimes p_2$.

Let $\widehat{W}_H$ denote the set of isomorphism classes of irreducible representations of W_H. Further, for any unipotent conjugacy class $\mathcal{O} \subset H$, let $\mathsf{Admiss}(\mathcal{O})$ stand for the set of (isomorphism classes of) irreducible admissible (in the sense specified before Lemma A.1) H-equivariant local systems on $\mathcal{O}$.

Theorem A.3. *There exists a bijection between the following sets*:

$$\widehat{W}_H \quad \longleftrightarrow \quad \left\{ \begin{matrix} \text{unipotent conjugacy class} \\ \mathcal{O} \subset H \text{ and } \alpha \in \mathsf{Admiss}(\mathcal{O}) \end{matrix} \right\}.$$

Proof. Take an irreducible representation ρ of W_H and let $\phi \in \widehat{W}^\circ$ be an irreducible subrepresentation of $\rho|_{W^\circ}$. By Clifford theory ρ is induced from a certain representation $\chi \in (\widehat{W}^\phi)_\phi$ as above.

Recall that by Springer Correspondence for W° the irreducible representation ϕ gives rise to a unipotent H° orbit $\mathcal{O}_\phi$ together with an H°-equivariant local system I_ϕ. We will show how to construct from $\chi = p_1 \otimes p_2$ the corresponding local system for H.

As a first step, we will construct a cerain local system $\tilde{I}_\chi$ on $\mathcal{O}_\phi$. This local system is equivariant with respect to the subgroup $H^\phi \subset H$ which corresponds to $W^\phi \subset W_H$ via the isomorphism $H/H^\circ = W_H/W^\circ$ of Lemma A.1 (it is easy to prove using Proposition A.2 that H^ϕ is the subgroup of all elements in H which preserve the orbit $\mathcal{O}$ and the local system I_ϕ). Then, imitating the induction map $(\widehat{W}^\phi)_\phi \to (\widehat{W}_H)_\phi$ we will obtain an H-equivariant local system on the unique unipotent H-orbit which contains $\mathcal{O}_\phi$ as its connected component.

We fix the choice of Borel subgroup B_H and, in particular, the isomorphisms: $\mathbb{W} \simeq W^\circ$ and $W_H \simeq W^\circ \rtimes (W_H/W^\circ)$.

Step 1. Recall that the Springer resolution $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is H-equivariant. It follows from the definitions that there exists an action of H^ϕ on the total space of L_ϕ, which extends the natural action of H°. However, the extended action does not commute with the W°-action any more. Instead, it satisfies the identity $h(w \cdot s) = h(w) \cdot h(s)$ where $h \in H^\phi$, $w \in W^\circ$ and s is a local section of L_ϕ. We would like to use the formula $I_\phi = (\phi^\vee \otimes L_\phi)^{W^\circ}$ to define the H^ϕ-equivariant structure on I_ϕ. To that end, we have to construct an action of H^ϕ on $\phi^\vee$ which agrees with the W°-action in the same way as before. This is done by using the composition $H^\phi \to H^\phi/H^\circ = W^\phi/W^\circ \hookrightarrow W_H$ and the projective action of W_H on $p_1^\vee$ coming from Clifford theory (recall that $p_1^\vee$ extends the W°-action on the vector space of $\phi^\vee$). By Proposition A.2 the two actions of H^ϕ on W° coincide, hence the projective action of H^ϕ on $p_1^\vee \otimes L_\phi$ indeed satisfies $h(w \cdot s) = h(w) \cdot h(s)$. Consequently, the local system $I_\phi = (p_1^\vee \otimes L_\phi)^{W^\circ}$ carries a projective action of H^ϕ.

Now we tensor the local system I_ϕ with the vector space of the projective representation $p_2^\vee$. Since H^ϕ acts on the vector space of $p_2^\vee$ via the same composition $H^\phi \to H^\phi/H^\circ \simeq W^\phi/W^\circ \hookrightarrow W_H$, the tensor product $p_2^\vee \otimes I_\phi$ carries an *a priori* projective action of H^ϕ. However, since the projective cocyles of p_1 and p_2, well-defined as functions on W^ϕ/W°, are mutually inverse, the same can be said about the projective cocycles of the H^ϕ-actions on I_ϕ and $p_2^\vee$. Therefore, these cocycles cancel out giving a *linear* action on the tensor product. This means that $p_2^\vee \otimes I_\phi$ is given the structure of an H^ϕ-equivariant local system, to be denoted by $\tilde{I}_\chi$.

Step 2. Next we consider a larger subgroup $\widehat{H}^\phi$ which preserves the unipotent orbit $\mathcal{O}_\phi$, but not necessarily the local system I_ϕ. It is easy to check that the composition $\widehat{H}^\phi \times_{H^\phi} \tilde{I}_\phi \to \widehat{H}^\phi \times_{H^\phi} \mathcal{O}_\phi \to \mathcal{O}_\phi$ defines an $\widehat{H}^\phi$-equivariant local system $\hat{I}_\chi$ over $\mathcal{O}_\phi$. Finally, $I_\chi = H \times_{\widehat{H}^\phi} \hat{I}_\chi$ is an H-equivariant local system over $H \times_{\widehat{H}^\phi} \mathcal{O}_\phi$. Note that the latter space is nothing but the union of those unipotent H°-orbits which are conjugate to each other with respect to the larger group H, i.e., a single unipotent H-orbit. It is easy to check that the assignment: $\chi \mapsto I_\chi$, $\chi \in (\widehat{W}^\phi)_\phi$ together with the decomposition: $\widehat{W}_H = \bigcup_{\phi_i \in \mathcal{V}_i} (\widehat{W}_H)_\phi$ and the induction isomorphisms: $(\widehat{W}^\phi)_\phi \xrightarrow{\sim} (\widehat{W}_H)_\phi$ yields the correspondence of the theorem. $\square$

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