UNRAMIFIED BRAUER GROUPS OF FINITE SIMPLE GROUPS OF LIE TYPE A_ℓ

by

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Abstract. — We study the subgroup $B_0(G)$ of $H^2(G, \mathbb{Q}/\mathbb{Z})$ consisting of all elements which have trivial restrictions to every Abelian subgroup of G. The group $B_0(G)$ serves as the simplest nontrivial obstruction to stable rationality of algebraic varieties V/G where V is a faithful complex linear representation of the group G. We prove that $B_0(G)$ is trivial for finite simple groups of Lie type A_ℓ.

Contents

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0. Introduction

In this article we answer a group-theoretical question related to the problem of stable rationality of quotient spaces V/G where G is a finite (algebraic) group and V is a faithful complex linear representation of G. In algebraic terms, rationality of V/G means that the field of invariants $\mathbb{C}(V)^G$ is a pure transcendental extension of the constant field $\mathbb{C}$, and stable rationality means that there exist a finite number of independent variables $y_1, \dots, y_k$ such that $\mathbb{C}(V)^G(y_1, \dots, y_k)$ becomes a pure transcendental extension of $\mathbb{C}$. Simple arguments (based on the geometric version of Hilbert-90 theorem) show that stable rationality of V/G does not depend on an individual faithful representation V of G but only on the group G itself. Explicit computations of invariants show that the varieties V/G are rational in some non-trivial cases like, for example, the standard representation of the symmetric group S_n.

However, this is not true in general. The first examples of nonrational and even nonstably rational varieties V/G were obtained by D. Saltman [**S84**]. These solutions of the so-called Noether problem were obtained by showing that some birational invariant, which we denote by $B_0(G)$, is nontrivial for some series of groups G. For any finite group G, $B_0(G)$ is the subgroup of $H^2(G, \mathbb{Q}/\mathbb{Z})$ consisting of all elements having trivial restriction on every Abelian subgroup of G. It was shown in [**Bo88, Bo90**] that $B_0(G)$ coincides with geometric birational invariant of a smooth projective model $\widetilde{V/G}$ for V/G, the so-called *unramified Brauer group*, introduced earlier by Artin and Mumford [**AM**]. Namely,

$$B_0(G) = \mathbf{Br}_{\mathrm{nr}}(V/G) = H^3(\widetilde{V/G}, \mathbb{Z})_{\mathrm{tors}}.$$

This fact reduces the computation of the Artin-Mumford invariant V/G to a purely group-theoretical question.

The group $B_0(G)$ is, in fact, the first of the series of birational invariants $H^i_{\mathrm{nr}}(G)$ of the varieties V/G constructed via group cohomology (see [**CO**] for general definitions and properties, and also [**Bo88, Bo93, S95**]). In this notation, $B_0(G) = H^2_{\mathrm{nr}}(G)$.

However, some results and conjectures in algebraic geometry indicate that for finite simple groups this kind of birational invariants must vanish. Our leading hypotheses are the following:

Hypothesis 0.1. — For any finite simple group G the quotient V/G is stably rational.

There is no much evidence to support this statement and, at the moment, there are no methods to approach this problem except for the group $A_5 = \mathrm{PGL}(2, F_4)$ where it holds. It follows from the fact that A_5 has a three-dimensional faithful linear representation V and the corresponding quotient has a natural structure of a principal $\mathbb{C}^*$-fibration over the unirational surface $\mathbb{P}^2(V)/A_5$ which, in fact, is rational by classification theory of algebraic surfaces (for example, see [**VP**], [**Vo**]).

Hypothesis 0.2. — For any finite simple group G the nonramified cohomology groups $H^i_{\mathrm{nr}}(G) = 0$.

In this article we test this general hypothesis, formulated in [**Bo93**], for the first nontrivial invariant $B_0(G) = H^2_{\mathrm{nr}}(G)$ in the case of finite simple groups G of Lie type A_ℓ. Every group of Lie type A_ℓ is isomorphic to the projective special linear group $\mathrm{PSL}(n, F_q)$ over a finite field F_q, and the corresponding second cohomology groups can be found in [**Go**]. The following theorem contains our main result

***Theorem 1* (Main)**. — *For every finite field F_q of order q and every integer $n \geqslant 2$,*

$$B_0(\mathrm{PSL}(n, F_q)) = 0.$$

Remark 1.1. — In his seminal paper [**Q**] D. Quillen obtained a general description of the cohomology of linear groups $\mathrm{GL}(n, F_q)$ which supports the Hypotheses 0.2 for higher cohomology groups of $\mathrm{PSL}(n, F_q)$.

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2. The group $B_0(G)$

In this section we briefly recall some basic properties of $B_0(G)$. Let G be a group and $H^2(G, \mathbb{Q}/\mathbb{Z})$ be the set of equivalence classes of central extensions

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{i} \tilde{G} \xrightarrow{j} G \longrightarrow 1$$

of G by $\mathbb{Q}/\mathbb{Z}$. Let $\mathcal{A}$ be the set of all Abelian subgroups of G.

Definition 2.1. — *The subgroup $B_0(G)$ of $H^2(G, \mathbb{Q}/\mathbb{Z})$ is defined by*

$$B_0(G) := \{\gamma \in H^2(G, \mathbb{Q}/\mathbb{Z}) \mid \gamma\,|_H = 0 \text{ for all } H \in \mathcal{A}\}. \tag{2.1}$$

Lemma 2.2. — *Let G be a group and*

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{i} \tilde{G} \xrightarrow{j} G \longrightarrow 1$$

be an extension of G by $\mathbb{Q}/\mathbb{Z}$. Let A be an Abelian subgroup of G. For the restriction

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{i} \tilde{A} \xrightarrow{j|_{\tilde{A}}} A \longrightarrow 1$$

to be trivial, it is necessary and sufficient that $\tilde{A}$ be an Abelian group.

Proof. — This lemma is equivalent to saying that an element $\gamma' \in H^2(A, \mathbb{Q}/\mathbb{Z})$ is trivial if and only if the corresponding extension $\tilde{A}$ of A is an Abelian group. It is clear that the trivial extension is an Abelian group since A is a direct product of cyclic groups and $\mathbb{Q}/\mathbb{Z}$ is a divisible group. To prove the triviality of the extension $\tilde{A}$ it is sufficient to find a section $s : A \to \tilde{A}$. If $x \in A$ has order n, then there is an element $\tilde{x} \in \tilde{A}$, with $j(\tilde{x}) = x$, of order n. To find such an element first we take any y with $j(y) = x$. Then $y^n \in \mathbb{Q}/\mathbb{Z}$. We have the following possibilities

- If $y^n = 0$, then we set $\tilde{x} := y$;
- If $y^n \neq 0$, then $y^n = a^n$, where $a \in \mathbb{Q}/\mathbb{Z}$, since the group $\mathbb{Q}/\mathbb{Z}$ is infinitely divisible. Now we can take $\tilde{x} := ya^{-1}$.

Hence for any cyclic subgroup in A we can find a section s. Since A is a direct sum of cyclic groups, the sum of these sections provides a section for A. Therefore, $\tilde{A}$ is a semi-direct product of A and $\mathbb{Q}/\mathbb{Z}$, but since it is also a central extension, it is a direct product. This implies that $\tilde{A}$ is a trivial central extension of A if and only if $\tilde{A}$ is an Abelian group. Therefore, the fact that the preimage $\tilde{A} = j^{-1}(A)$ is an Abelian group is equivalent to the triviality of the restriction of $\gamma \in H^2(G, \mathbb{Q}/\mathbb{Z})$ on A. □

Corollary 2.3. — *Let G be a group and $\mathcal{X}$ be the set of subgroups of G generated by two elements whose commutator is the identity element of G. Then*

$$B_0(G) = \{\gamma \in H^2(G, \mathbb{Q}/\mathbb{Z}) \mid \gamma\,|_H = 0 \text{ for all } H \in \mathcal{X}\}.$$

This fact appeared previously in [Bo88] and [S90].

The description above also yields a simple criterion for a given element γ of $H^2(G, \mathbb{Q}/\mathbb{Z})$ not to lie in $B_0(G)$. Let $\tilde{G}$ be a central $\mathbb{Q}/\mathbb{Z}$-extension of G defined by $\gamma \in H^2(G, \mathbb{Q}/\mathbb{Z})$. Denote by $K_\gamma \subset \mathbb{Q}/\mathbb{Z}$ the subgroup in $\mathbb{Q}/\mathbb{Z}$ which lies in the kernel of every character $\tilde{G} \to \mathbb{Q}/\mathbb{Z}$. The group K_γ is always a finite cyclic group.

Corollary 2.4. — *An element γ of $H^2(G, \mathbb{Q}/\mathbb{Z})$ does not belong to $B_0(G)$ if and only if some nonidentity element of K_γ can be represented as a commutator of a pair of elements $a, b \in \tilde{G}$.*

Proof. — If $\gamma \in B_0(G)$ then the preimages of any commuting pair of elements in G commute in $\tilde{G}$ as well. Thus if a nonzero element $h \in K_\gamma$ can be represented as $h = aba^{-1}b^{-1}$, then the elements $j(a)$ and $j(b)$ commute, but a and b do not. This implies that γ is not in $B_0(G)$. Notice that any character of $\tilde{G}$ is trivial on $aba^{-1}b^{-1}$, and hence any element of the form $h = aba^{-1}b^{-1}$ always belongs to K_γ. □

Corollary 2.5. — *If the generator of K_γ can be represented as a commutator $aba^{-1}b^{-1}$, then any quotient of $\tilde{G}$ by a cyclic subgroup of $\mathbb{Q}/\mathbb{Z}$ not containing K_γ represents a central extension of G which is not in $B_0(G)$.*

Lemma 2.6. — *Let G be a finite group and*

$$H^2(G, \mathbb{Q}/\mathbb{Z}) = \bigoplus_p H^2(G, \mathbb{Q}/\mathbb{Z})_{(p)}$$

be the primary decomposition of $H^2(G, \mathbb{Q}/\mathbb{Z})$, where by $H^2(G, \mathbb{Q}/\mathbb{Z})_{(p)}$ we denote the p-primary component of $H^2(G, \mathbb{Q}/\mathbb{Z})$.

1. *We have*

$$B_0(G) = \bigoplus_p B_{0,p}(G),$$

where $B_{0,p}(G) := B_0(G) \cap H^2(G, \mathbb{Q}/\mathbb{Z})_{(p)}$.

2. *For every Sylow p-subgroup $\mathrm{Syl}_p(G)$ of G we have an embedding*

$$B_{0,p}(G) \subset B_0(\mathrm{Syl}_p(G)).$$

Proof. — The assertion (1) is a consequence of the fact that a central extension of a finite Abelian group is nilpotent and therefore decomposes into a direct product of p-groups.

The statement in (2) follows from the fact that we have an embedding from $H^2(G, \mathbb{Q}/\mathbb{Z})_{(p)}$ into $H^2(\mathrm{Syl}_p(G), \mathbb{Q}/\mathbb{Z})$. □

Corollary 2.7. — *Let G be a finite group and $\mathrm{Syl}_p(G)$ be a Sylow p-subgroup of G. If $\mathrm{Syl}_p(G)$ is an Abelian group, then*

$$B_{0,p}(G) = 0.$$

Proof. — We have $B_{0,p}(G) \subset B_0(\mathrm{Syl}_p(G))$ but the latter is zero since $\mathrm{Syl}_p(G)$ is Abelian. □

We will also use several results concerning p-groups from [**Bo88**].

Let G be a central extension of an Abelian group A and C be the center of G. There is a natural map $\bigwedge^2 A \to C$ where

$$\bigwedge^2 A := \{x \wedge y \mid x, y \in A\}.$$

The group $\bigwedge^2 A$ is the quotient of $A \otimes_{\mathbb{Z}} A$ by the subgroup generated by all elements $x \otimes x$. In particular, the wedge product $x \wedge y$ is bilinear, and for any $x \in A, x \wedge x = 0$. The image of $x \wedge y \in \bigwedge^2 A$ under this map is equal to $\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$ where $\tilde{x}$ and $\tilde{y}$ are any two preimages of x and y, respectively, in G. We denote by S the kernel of the map $\bigwedge^2 A \to C$. Obviously, this is a subgroup of $\bigwedge^2 A$. Consider the elements of the form $x \wedge y \in S$ and denote by $S_\Lambda \subset S$ the subgroup in S generated by these elements. The element $x \wedge y \in S_\Lambda$ corresponds to a pair of elements $x, y \in A$ which lifts to a commuting pair of elements in G.

Lemma 2.8. — *Let $f : G \to A$ be a central extension of an Abelian group A by C. Then*

1. *The group $B_0(G)$ is contained in the image*
$$f^* H^2(A, \mathbb{Q}/\mathbb{Z}) \subset H^2(G, \mathbb{Q}/\mathbb{Z}).$$
2. *There is an isomorphism*
$$f^* H^2(A, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(S, \mathbb{Q}/\mathbb{Z}).$$
3. *The is an isomorphism*
$$B_0(G) \cong \mathrm{Hom}(S/S_\Lambda, \mathbb{Q}/\mathbb{Z}).$$

Proof. — See [**Bo88**], Lemma 5.1. □

There are also several results for the more general case of meta-Abelian groups.

Lemma 2.9. — *Let G be a finite group containing a normal Abelian subgroup A such that the quotient G/A is a cyclic group. Then $B_0(G) = 0$.*

Proof. — See [**Bo88**], Lemma 4.9. □

This result serves as a tool to compute $B_0(G)$ for some meta-Abelian groups.

Let G be a finite meta-Abelian group, $A = G/[G,G]$, and $f : G \to A$ be the canonical projection. Denote by

$$G_c := G/[G,[G,G]].$$

Let S be the kernel of the linear map $\bigwedge^2 A \to G_c$ and S'_Λ be the subspace in S generated by elements $x \wedge y$, where $x, y \in A$, such that there is a pair $x', y' \in G$ with $f(x') = x, f(y') = y$, and $[x', y'] = 1$.

The exact sequence of groups

$$1 \longrightarrow [G,G] \longrightarrow G \xrightarrow{f} A \longrightarrow 1$$

defines the spectral sequence

$$E_2^{p,q}(G) := H^p(A, H^q([G,G], \mathbb{Q}/\mathbb{Z}))$$

which computes the group $H^2(G, \mathbb{Q}/\mathbb{Z})$. Moreover, this spectral sequence provides $H^2(G, \mathbb{Q}/\mathbb{Z})$ with a filtration

$$f^*H^2(A, \mathbb{Q}/\mathbb{Z}) \subset K \subset H^2(G, \mathbb{Q}/\mathbb{Z}),$$

so that K is the kernel of the restriction map:

$$H^2(G, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^2([G,G], \mathbb{Q}/\mathbb{Z}). \tag{2.2}$$

Therefore, the quotient $H^2(G, \mathbb{Q}/\mathbb{Z})/K$ has a nontrivial restriction on $[G,G]$ and $K/f^*H^2(A, \mathbb{Q}/\mathbb{Z})$ admits an injective map

$$r : K/f^*H^2(A, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^1(A, \mathrm{Hom}([G,G], \mathbb{Q}/\mathbb{Z})). \tag{2.3}$$

This filtration is functorial with respect to the group homomorphisms. Namely, if $A' \subset A$ is a subgroup then denote the subgroup

$$i : f^{-1}A' \hookrightarrow G,$$

as G'. Then we have an embedding $G' \hookrightarrow G$, and a natural cohomology map

$$i^* : H^2(G, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^2(G', \mathbb{Q}/\mathbb{Z}).$$

This map is a map of filtered groups. Namely, if

$$1 \longrightarrow [G,G] \longrightarrow G' \xrightarrow{f_1} A' \longrightarrow 1$$

is the exact sequence of groups for G', induced from the corresponding sequence for G, then we obtain a natural homomorphism of spectral sequences

$$E_2^{p,q}(G) \longrightarrow E_2^{p,q}(G'),$$

which is the restriction map

$$H^p(A, H^q([G,G], \mathbb{Q}/\mathbb{Z})) \longrightarrow H^p(A', H^q([G,G], \mathbb{Q}/\mathbb{Z})).$$

This provides the map between cohomology with an invariant filtration. Namely, if

$$f_1^* H^2(A', \mathbb{Q}/\mathbb{Z}) \subset K' \subset H^2(G', \mathbb{Q}/\mathbb{Z})$$

is the induced filtration on $H^2(G', \mathbb{Q}/\mathbb{Z})$, then i^* maps K to K', and $f^* H^2(A, \mathbb{Q}/\mathbb{Z})$ to $f_1^* H^2(A', \mathbb{Q}/\mathbb{Z})$.

Consider the case of the cyclic subgroup $A' = \mathbb{Z}_m^{(i)} \subset A$ where $i \in I$ runs over all cyclic subgroups of A. Denote by σ_i the natural restriction map

$$\sigma_i : H^1(A, \mathrm{Hom}([G,G], \mathbb{Q}/\mathbb{Z})) \longrightarrow H^1(\mathbb{Z}_m^{(i)}, \mathrm{Hom}([G,G], \mathbb{Q}/\mathbb{Z})),$$

and by σ the sum of such maps over all cyclic subgroups in A. The next result is a weaker version of Theorem 4.2 in [**Bo88**] which is better adapted to our setting.

Lemma 2.10. — *Consider the map*

$$\sigma : H^1(A, \mathrm{Hom}([G,G], \mathbb{Q}/\mathbb{Z})) \longrightarrow \sum_i H^1(\mathbb{Z}_m^{(i)}, \mathrm{Hom}([G,G], \mathbb{Q}/\mathbb{Z})) \qquad (2.4)$$

where $\mathbb{Z}_m^{(i)} \subset A$ runs through all cyclic subgroups of A. If σ is an embedding, then the group $B_0(G) \subset f^* H^2(G, \mathbb{Q}/\mathbb{Z})$ and $B_0(G) \cong \mathrm{Hom}(S/S_\Lambda, \mathbb{Q}/\mathbb{Z})$.

Proof. — The group $B_0(G)$ is contained in K since, by definition, every element of $B_0(G)$ restricts trivially on the Abelian group $[G,G]$. We want to show that $B_0(G)$ belongs to the kernel of the map r on K which coincides with $f^* H^2(A, \mathbb{Q}/\mathbb{Z})$.

Assume the contrary, that $\gamma \in K, \gamma \in B_0(G)$, and $r(\gamma) \neq 0$. If $\gamma \notin \mathrm{Ker}(r)$ then, by assumption on the injectivity of σ, there is a cyclic subgroup $\mathbb{Z}_m^{(i)} \subset A$ such that $(\sigma_i \circ r)(\gamma) \neq 0$.

This implies that the restriction of γ on $G' = f^{-1}(\mathbb{Z}_m^{(i)}) \subset G$ is nontrivial. It means that the restriction of the cocycle $\gamma \in H^2(G, \mathbb{Q}/\mathbb{Z})$ to $H^2(G', \mathbb{Q}/\mathbb{Z})$ is nontrivial as well. Notice that G' contains a normal Abelian subgroup $[G, G]$ so that $G'/[G, G]$ is a cyclic group. By Lemma 2.9, the group $B_0(G')$ is trivial and hence the restriction of γ in $H^2(G', \mathbb{Q}/\mathbb{Z})$ does not belong to $B_0(G') = 0$. Hence $\gamma \notin B_0(G)$ from the very beginning and we obtain a contradiction.

Therefore, if $\gamma \in B_0(G)$ then $i^*(\gamma) = 0$ for any cyclic subgroup in A. Hence, by our assumption on the A-module $\mathrm{Hom}([G, G], \mathbb{Q}/\mathbb{Z})$, we obtain that $B_0(G) \subset \mathrm{Ker}(r)$ and therefore it belongs to $f^*H^2(A, \mathbb{Q}/\mathbb{Z})$.

The image $f^*H^2(A, \mathbb{Q}/\mathbb{Z})$ in $H^2(G, \mathbb{Q}/\mathbb{Z})$ can be identified with $\mathrm{Hom}(S, \mathbb{Q}/\mathbb{Z})$. Indeed, since $A = G/[G, G]$ any element of $H^2(A, \mathbb{Q}/\mathbb{Z})$ which has a trivial image in $H^2(G, \mathbb{Q}/\mathbb{Z})$ becomes trivial already in $H^2(G/[[G, G], G], \mathbb{Q}/\mathbb{Z})$. Thus the image $f^*H^2(A, \mathbb{Q}/\mathbb{Z}) \subset H^2(G, \mathbb{Q}/\mathbb{Z})$ coincides with $\mathrm{Hom}(S, \mathbb{Q}/\mathbb{Z})$ by Lemma 2.8.

By definition, an element $f^*(a)$ belongs to $B_0(G)$ if for every Abelian subgroup $B \subset G$ with two generators the restriction $f^*(a) \mid_B = 0$. We have to check this property only for subgroups B which project onto subgroups $f(B)$ with two generators in A. If a', b' are two commuting elements in G generating a subgroup B then they define an element $f(a') \wedge f(b') \in S$ and the restriction of $\gamma \in f^*H^2(A, \mathbb{Q}/\mathbb{Z})$ on B coincides with the value of γ on $f(a') \wedge f(b')$ where we identified $f^*H^2(A, \mathbb{Q}/\mathbb{Z})$ with $\mathrm{Hom}(S, \mathbb{Q}/\mathbb{Z})$ and $H^2(B, \mathbb{Q}/\mathbb{Z})$ with the cyclic group $\mathrm{Hom}(\bigwedge^2 B, \mathbb{Q}/\mathbb{Z})$.

Therefore,

$$B_0(G) \cong \mathrm{Hom}(S/S_\Lambda, \mathbb{Q}/\mathbb{Z}) \subset \mathrm{Hom}(S, \mathbb{Q}/\mathbb{Z}) \cong f^*H^2(A, \mathbb{Q}/\mathbb{Z}).$$

The lemma follows. □

From Lemma 5.6 in [Bo88] we also easily obtain

Corollary 2.11. — *Any p-group G with $B_0(G) \neq 0$ has order $\geq p^6$.*

3. Simple groups and $H^2(G, \mathbb{Q}/\mathbb{Z})$

The invariant $B_0(G) \subset H^2(G, \mathbb{Q}/\mathbb{Z})$ has an analogue, denoted by $H^i_{\mathrm{nr}}(G)$, for group cohomology in any dimension. For more details we refer to the articles [CO],[C], [Bo88], [Bo93], [S95] (though, the definitions are different). In this notation, $B_0(G) = H^2_{\mathrm{nr}}(G)$. According to the Bloch-Kato conjecture [BK] all the cohomology of Galois groups of algebraic closures of function fields are, roughly speaking, induced from the Abelian quotients of these Galois groups. A more geometric version of these conjectures (for example, see [**Bo88**, **Bo93**]) is that, in fact, most of the finite birationally invariant classes in the cohomology of algebraic varieties are induced from the similar birational classes of special p-groups. Somehow it looks like finite simple groups do not produce nontrivial birational invariants of algebraic varieties which leads to the general hypothesis that all nonramified cohomology of a finite simple group are trivial (see [**Bo88**]).

Unfortunately, the computation of general nonramified cohomology groups is a highly nontrivial task. However, there is a description of all groups $H^2(G, \mathbb{Q}/\mathbb{Z})$ for all finite simple groups G (see [**Go**], [**Gr**]). This fact, together with the rather well understood structure of Sylow subgroups of finite simple groups, leads to the following conjecture, which is a particular case of the general hypotheses stated above:

Conjecture 3.1 **(Bogomolov [Bo93]).** — If G is a finite simple group, then

$$H^2_{\mathrm{nr}}(G) = B_0(G) = 0. \tag{3.1}$$

By now the complete list of finite simple groups G is well known. It is contained, for example, in [**Go**]. Apart from a finite number of sporadic groups all the other are of Lie type. They are parameterized by a finite number of infinite series depending on the rank of the corresponding Lie algebra and on the finite base field F_q.

Thus any of the simple groups of Lie type has a naturally attached prime number. The advantage of using a simple group G is that there is a uniquely defined universal covering group $\tilde{G}$. The group $\tilde{G}$ is a universal central extension of G with the group $H^2(G, \mathbb{Q}/\mathbb{Z})$ as a center, so that any central extension G' of G with $[G', G'] = G'$ is a quotient of $\tilde{G}$.

All the groups $H^2(G, \mathbb{Q}/\mathbb{Z})$ are also listed (see [**Go**]). The general feature of the corresponding list is that most of the groups in $H^2(G, \mathbb{Q}/\mathbb{Z})$ which appear for a given group G in the series are related to a central extension of the corresponding algebraic Lie groups. The relevant central extensions have a very simple description. It is easy to show that they define trivial elements in $B_0(LG)$ for a Lie group LG over complex numbers (see [**Bo88**], [**Bo90**]). Similarly, in each Lie series of finite groups, apart from a finite number of cases, we have a description of the corresponding extension in terms of Lie groups.

In this article we prove this result for finite simple groups of Lie type A_ℓ where the groups are $\mathrm{PSL}(n, F_q)$. The group $\mathrm{SL}(n, F_q)$ is a covering group for all $\mathrm{PSL}(n, F_q)$

except for the following cases:

$$(n,q) = (2,4), (2,9), (3,2), (3,4), (4,2). \tag{3.2}$$

Therefore, our strategy will be the following:

- First, we consider the general case when $\mathrm{SL}(n, F_q)$ is the universal cyclic extension of $\mathrm{PSL}(n, F_q)$ by the cyclic group $\mathbb{Z}_{\gcd(n,q-1)}$ where the group $\mathbb{Z}_{\gcd(n,q-1)}$ is represented by diagonal matrices. To prove the result in this case it is sufficient to represent the generator μ_p of any Sylow subgroup $\mathbb{Z}_{p^n}$, where p is a prime, of diagonal $\mathbb{Z}_{\gcd(n,q-1)}$ matrices as a commutator $ABA^{-1}B^{-1}$, where $A, B \in \mathrm{SL}(n, F_q)$.
- Second, we consider the above five exceptional pairs (3.2). Here we do not know the exact description of the corresponding central extension. We prove the theorem by establishing a stronger result that $B_0(\mathrm{Syl}_p(G)) = 0$ for all prime p dividing the order of $H^2(G, \mathbb{Q}/\mathbb{Z})$.

4. The group $\mathrm{SL}\,(n, F)$ as a central extension

Let F be a field and let μ be a primitive p^n-th root of unity in F.

Lemma 4.1. — *Assume that m is divisible by p^n where p is prime. The scalar matrix μI_m, which belongs to the center of $\mathrm{SL}(m, F)$, can be written in a commutator form $[A_1, B_1] = A_1B_1A_1^{-1}B_1^{-1}$, where $A_1, B_1 \in \mathrm{SL}(m, F)$.*

Proof. — We have the following possibilities:

1. Let p be an odd prime and A, B be two square matrices of order p^n over the field F such that $A = (a_{i,j}) = (\delta_{i,j}\mu^i)$, and $B = (b_{i,j})$ with
   - $b_{i,i+1} = b_{p^n,1} = 1$, and
   - $b_{i,j} = 0$ otherwise.

   Then A and B belong to $\mathrm{SL}(p^n, F)$, and $[A, B] = ABA^{-1}B^{-1} = \mu I_{p^n}$.
2. If $p = 2$ and $n = 1$, then the matrices $A = \left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right)$, where $a^2 + b^2 = -1$, and $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ satisfy the required condition. For a finite field F_q, with q odd, such a and b always exist. If $F = F_q$ and $q = 4k + 1$ then -1 is a square and we can take $a = i$, where $i^2 = -1$, and $b = 0$. If $q = 4k + 3$ then the set of all elements of the form $x^2 + y^2$ coincides with F_q. Indeed, it contains all quadratic residues x^2, and it is invariant under the multiplication by an arbitrary element z^2, where $z \in F_q$. Therefore, if it contains at least

one non-quadratic element, then it coincides with F_q. If not, then quadratic residues form an additive subgroup in F_q. However, F_q does not have an additive subgroup of index 2 for odd q. Therefore, F_q is the set of elements of the form $x^2 + y^2$. In particular, there are a and b such that $a^2 + b^2 = -1$, and they provide entries for A.

3. If $p = 2$ and $n > 1$, we select $A = \sigma_1 X$ and $B = Y\sigma_2$ where X and Y are diagonal matrices and σ_1, σ_2 are commuting permutation matrices. In this case we have an equality

$$ABA^{-1}B^{-1} = \sigma_1 XY\sigma_2 X^{-1}\sigma_2^{-1}\sigma_1^{-1}Y^{-1}$$

since σ_1 and σ_2 commute.

Thus the equation

$$ABA^{-1}B^{-1} = \mu I \tag{4.1}$$

is equivalent to

$$XY(\sigma_2 X^{-1}\sigma_2^{-1})(\sigma_1^{-1}Y^{-1}\sigma_1) = \mu I \tag{4.2}$$

where all the matrices $X, Y, (\sigma_2 X^{-1}\sigma_2^{-1})$, and $(\sigma_1^{-1}Y^{-1}\sigma_1)$ are diagonal. Assume that σ_1 has order $2^k, k \geq 1$ and σ_2 has order $2^{n-k}, n-k \geq 1$. Then the corresponding linear space has a special coordinate system $z_{i,j}, 1 \leq i \leq 2^k, 1 \leq j \leq 2^{n-k}$ with the property that σ_1 cyclically permutes coordinates $z_{i,j}$ with the same index i and σ_2 cyclically permutes coordinates $z_{i,j}$ with the same parameter j.

Therefore, if we denote by $x_{i,j}$ and $y_{i,j}$ diagonal elements of the matrices X and Y above, then the equation (4.2) becomes equivalent to a series of equations

$$x_{i,j}y_{i,j}x^{-1}_{i+1 (\mathrm{mod}\ 2^k),j}\, y^{-1}_{i,j+1 (\mathrm{mod}\ 2^{n-k})} = \mu \tag{4.3}$$

for the diagonal elements. If we denote by

- $u_{i,j} := x_{i,j}x^{-1}_{i+1 (\mathrm{mod}\ 2^k),j}$, and
- $v_{i,j} := y_{i,j}y^{-1}_{i,j+1 (\mathrm{mod}\ 2^{n-k})}$,

then

$$\prod_i u_{i,j} = \prod_j v_{i,j} = 1.$$

The equation (4.3) above becomes $u_{i,j} = \mu v_{i,j}^{-1}$ and hence we have obtained equations only for the parameters $u_{i,j}$.

Thus our initial matrix equation (4.1) was reduced to equations for $u_{i,j}$:

$$\prod_i u_{i,j} = 1, \ \prod_j u_{i,j} = \mu^{2^{n-k}}.$$

These parameters $u_{i,j}$ define the complementary set of parameters $v_{i,j}$. Notice that for any $u_{i,j}, v_{i,j}$ satisfying $\prod u_{i,j} = \prod v_{i,j} = 1$ we can find $x_{i,j}, y_{i,j}$ so that

$$u_{i,j} = x_{i,j} x^{-1}_{i+1 (\mathrm{mod}\ 2^k), j}, \ v_{i,j} = y_{i,j} y^{-1}_{i,j+1 (\mathrm{mod}\ 2^{n-k})},$$

and hence we can obtain solutions of the equation (4.1).

Therefore, there are many matrix pairs A, B that satisfy (4.1). For example, we can decompose 2^n coordinates into two groups of order 2^{n-1} and take the diagonal matrix X to have $[1, 1, \dots, i], [1, 1, \dots, -i]$ on the main diagonal, where brackets show the boundaries of each block. The element i, with $i^2 = -1$, is contained in F_q since, by our assumption, 4 divides $q-1$. Similarly, take Y to have $[1, \mu, \dots, \mu^{2^{n-1}-1}], [\mu, \mu^2, \dots, \mu^{2^{n-1}}]$ on the main diagonal. The permutation σ_2 has order 2 and interchanges these two blocks of variables, and σ_1 permutes variables cyclically within each block. Recall that $\mu^{2^{n-1}} = -1$. Then $X(\sigma_2 X^{-1} \sigma_2^{-1}) = [-1, 1, \dots, 1][-1, 1, \dots, 1]$ and $Y(\sigma_1^{-1} Y^{-1} \sigma_1) = [-\mu, \mu, \dots, \mu][-\mu, \mu, \dots, \mu]$ and hence

$$XY(\sigma_2 X^{-1} \sigma_2^{-1})(\sigma_1^{-1} Y^{-1} \sigma_1) = \mu I.$$

If we take $A = \sigma_1 X, B = Y \sigma_2 \in \mathrm{SL}_{2^n}(F)$, then $[A, B] = ABA^{-1}B^{-1} = \mu I_{2^n}$.

4. If m is divisible by p^n, then matrices A_1 and B_1, consisting of m/p^n diagonal blocks of matrices A and B respectively, also satisfy the relation $[A_1, B_1] = A_1 B_1 A_1^{-1} B_1^{-1} = \mu I_m$.

The lemma follows. □

Corollary 4.2. — *Any element of the group $H^2(\mathrm{PSL}(n, F), \mathbb{Q}/\mathbb{Z})$ that defines a central extension of $\mathrm{PSL}(n, F)$ which is isomorphic to the quotient $\mathrm{SL}(n, F)/\mathbb{Z}_h$, where $\mathbb{Z}_h$ is a central subgroup, does not belong to $B_0(\mathrm{PSL}(n, F))$ (see Corollary 2.4). In particular, $B_0(\mathrm{PSL}(n, F)) = 0$ unless the pair (n, q) is one of the five exceptional cases in (3.2).*

This finishes the proof of our theorem in the general case.

5. Special simple groups of Lie type A_ℓ

Consider the remaining cases. These are the following groups:

1. $G_1 = \mathrm{PSL}(2, F_4)$, $H^2(G_1, \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}_2$;
2. $G_2 = \mathrm{PSL}(2, F_9)$, $H^2(G_2, \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_3$;
3. $G_3 = \mathrm{PSL}(3, F_2)$, $H^2(G_3, \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}_2$;
4. $G_4 = \mathrm{PSL}(3, F_4)$, $H^2(G_4, \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$;
5. $G_5 = \mathrm{PSL}(4, F_2)$, $H^2(G_5, \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}_2$.

Lemma 5.1. — *In the cases* $(1) - (3)$ *above the group* $B_0(G) = 0$.

Proof. — In the cases $(1) - (3)$ the relevant Sylow p-groups have order $< p^6$, and therefore, by Corollary 2.11, $B_0(G) = 0$. ☐

Lemma 5.2. — *The group* $B_0(G_4) = 0$.

Proof. — Consider first the element of order 3 in $H^2(G_4, \mathbb{Q}/\mathbb{Z})$. The corresponding central extension of $G_4 = \mathrm{PSL}(3, F_4)$ is the group $\mathrm{SL}(3, F_4)$, and hence, by Lemma 4.1, the element of order 3 does not belong to $B_0(G_4)$. The similar result for the elements of order 2 in $H^2(G_4, \mathbb{Q}/\mathbb{Z})$ follows from the following more general result. ☐

Lemma 5.3. — *The group* $B_0(\mathrm{Syl}_2(G_4)) = 0$.

Proof. — The group $\mathrm{Syl}_2(G_4)$ is a central extension of the Abelian group $F_4^2 = \mathbb{Z}_2^4$ by $F_4 = \mathbb{Z}_2^2$. Thus we again can use the general formula $B_0(\mathrm{Syl}_2(G_4)) = (S/S_\Lambda)^*$. We have two generators $\langle x, y\rangle$ over F_4, which is linearly generated over $\mathbb{Z}_2$ by $\langle 1, s\rangle$ with $s^2 + s + 1 = 0$. The commutator map $\bigwedge^2 \mathbb{Z}_2^4 \to F_4$ coincides with the map between the second exterior power of $\mathbb{Z}_2^4$, considered as $\mathbb{Z}_2$-space, and the second exterior power of the same space, considered as F_4-space. Thus S_Λ is generated by elements $[u, su]$ where $u \in \mathbb{Z}_2^4 = F_4^2$. The space S has dimension 4 over $\mathbb{Z}_2$ and a basis $\langle x, y, sx, sy\rangle$. To finish the proof we will show that the elements

$$[x, sx],\ [y, sy],\ [x + y, s(x + y)],\ [x + sy, s(x + sy)]$$

are linearly independent over $\mathbb{Z}_2$. Indeed, this set is linearly equivalent to

$$[x, sx],\ [y, sy],\ ([x, sy] + [y, sx]),\ ([x, y] + [x, sy] + [sy, sx]),$$

and the last one is linearly independent over $\mathbb{Z}_2$. Therefore, $S_\Lambda = S$ and the group $B_0(\mathrm{Syl}_2(G_4)) = 0$, which implies that $B_0(G_4) = 0$. ☐

Theorem 5.4. — *The group* $B_0(\mathrm{Syl}_2(G_5)) = 0$.

Proof. — The group $\mathrm{Syl}_2(G_5)$ is a meta-Abelian 2-group. We are going to use Lemma 2.10 in order to compute the group $B_0(\mathrm{Syl}_2(G_5))$.

The group $\mathrm{Syl}_2(G_5)$ is the group of upper-triangular $\mathbb{Z}_2$-matrices. It is generated by the elements $\langle a_{i,i+1}\rangle$, with $i = 1, 2, 3$. Here $(a_{i,j})$ denotes the matrix having 1's on the main diagonal and 0's everywhere else apart from the (i, j)'th entry, which is equal to 1. Denote the group $\mathrm{Syl}_2(G_5)$ by G to simplify the notations. The group $[G, G] = \mathbb{Z}_2^3$ is generated by the elements $\langle a_{1,3}, a_{1,4}, a_{2,4}\rangle$.

Lemma 5.5. — *The conditions of Lemma 2.10 are satisfied for the group* $G = \mathrm{Syl}_2(G_5)$.

Proof. — The module $\mathrm{Hom}([G, G], \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}_2^3$. For simplicity, we will denote it by W. Denote the generators of the quotient $A = G/[G, G]$ by $a_{i,i+1}, i = 1, 2, 3$, and the generators of W by $a^*_{1,3}, a^*_{1,4}, a^*_{2,4}$, respectively.

The group A acts on W. The action of A is trivial on $V = \langle a^*_{1,3}, a^*_{1,4}\rangle \subset W$, and on the quotient $W/V = C$. We have an exact sequence

$$H^0(A, C) \xrightarrow{\delta} H^1(A, V) \longrightarrow H^1(A, W) \longrightarrow H^1(A, C).$$

Since the action of A is trivial on V and C, any element in $H^1(A, V)$ and $H^1(A, C)$ restricts nontrivially on some cyclic subgroup of A.

Now we have to prove the same for $H^1(A, V)/\delta H^0(A, C)$. The cohomology group $H^0(A, C) = \mathbb{Z}_2$, and we denote the only nontrivial element in this group by w. Denote the corresponding nontrivial element of $H^1(A, V)$ by δw. We have that $A = \mathbb{Z}_2^3, V = \mathbb{Z}_2^2$ and $H^1(A, V) = A^* \otimes V$. Thus we can select generators $\langle x_1, x_2, x_3\rangle$ of A and $\langle v_1, v_2\rangle$ of V so that either

- $\delta w = x^*_1 \otimes v_1$, or
- $\delta w = x^*_1 \otimes v_1 + x^*_2 \otimes v_2$.

Here $\langle x^*_1, x^*_2, x^*_3\rangle$ is a basis for A^*, such that $x^*_i(x_j) = \delta_{i,j}$.

In order to finish the proof, we need the following lemma

Lemma 5.6. — *Let* $y \in A^* \otimes V$ *be a nonzero element such that* $y \neq \delta w$. *Then there is a projection*

$$p_y : A^* \longrightarrow \mathbb{Z}_2, \tag{5.1}$$

such that the images of the elements δw and y under the induced projection $A^* \otimes V \to \mathbb{Z}_2 \otimes V$ are different, and the image of y is nonzero.

Proof. — Let $y = \sum b_{i,j} x_i^* \otimes v_j$.

– If $b_{i,j} \neq 0$ for $(i,j) \neq (1,1)$, then the projection of y on one of the groups $\mathbb{Z}_2 x_1^* \otimes V, \mathbb{Z}_2 x_2^* \otimes V$, or $\mathbb{Z}_2 x_3^* \otimes V$ is nonzero and different from δw. Otherwise, $y = \delta w$ which proves the result in the first case.
– If $b_{3,j} \neq 0$ for some j, then the restriction of y on $\mathbb{Z}_2 x_3^* \otimes V$ is nontrivial and nonequal to w.
– If either $b_{1,2}$ or $b_{2,1}$ is nonzero, then the restriction of y on either $\mathbb{Z}_2 x_1^* \otimes V$ or $\mathbb{Z}_2 x_2^* \otimes V$ is nonzero and different from the corresponding restriction of δw.

Therefore, we can assume that either

– $y = x_1^* \otimes v_1$, or
– $y = x_2^* \otimes v_2$.

In both cases, the restriction of y on $\mathbb{Z}_2(x_1^* + x_2^*)$ will be nonzero and different from the restriction of δw. □

Thus if we take the subgroup $p_y^* : \mathbb{Z}_2 \hookrightarrow A$, where p_y^* is dual to the map in (5.1), then the restriction of y on $p_y^* \mathbb{Z}_2$ is nonzero in $H^1(\mathbb{Z}_2, W)$. This finishes the proof of Lemma 5.5. □

Corollary 5.7. — *By Lemma 2.10, we obtain that $B_0(\mathrm{Syl}_2(G_5))$ is contained in the image $f^* H^2(A, \mathbb{Q}/\mathbb{Z})$ where A is the maximal Abelian quotient of $\mathrm{Syl}_2(G_5)$.*

Now, in order to finish the proof of the triviality of $B_0(G)$ we have to compute the groups $f^* H^2(A, \mathbb{Q}/\mathbb{Z})$ and S in the above case. Denote the basis $\mathbb{Z}_2$-characters of the group $\mathrm{Syl}_2(G_5)$ by $\langle a_{i,i+1}^* \rangle$, $i = 1, 2, 3$.

Lemma 5.8. — *The group $f^* H^2(A, \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}_2$ and it is generated by an element $a_{1,2}^* \wedge a_{3,4}^*$ which is nontrivial on the commutative group generated by $\langle a_{1,2}, a_{3,4} \rangle$.*

Proof. — The group $H^2(A, \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}_2^3$. Let us show that both elements $a_{1,2}^* \wedge a_{2,3}^*$ and $a_{2,3}^* \wedge a_{3,4}^*$ belong to the kernel of f^*. Consider two projections of $\mathrm{Syl}_2(G_5)$ onto a central extension of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, which is the upper-triangular group of 3×3 matrices. The first one is given by deleting the fourth column, and the second one by deleting the first row. This shows that the elements $a_{1,2}^* \wedge a_{2,3}^*$ and $a_{2,3}^* \wedge a_{3,4}^*$ are trivial already on the corresponding triangular groups, and hence they are trivial on $\mathrm{Syl}_2(G_5)$. The group $\langle a_{1,2}, a_{3,4} \rangle$ is an Abelian $B = \mathbb{Z}_2 \oplus \mathbb{Z}_2$-subgroup of $\mathrm{Syl}_2(G_5)$.

The element $f^*(a^*_{1,2} \wedge a^*_{3,4})$ is nontrivial on B which shows that $a^*_{1,2} \wedge a^*_{3,4} \neq 0$, and it does not belong to $B_0(\mathrm{Syl}_2(G_5))$. $\square$

Thus we proved in particular that $B_0(G_5) = 0$ which finishes the proof of Theorem 5.4. $\square$

Simultaneously we finished the proof of our main Theorem 1, stated in the introduction.

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