BRAUER GROUPS AND CREPANT RESOLUTIONS

by

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Abstract. — We suggest a twisted version of the categorical McKay correspondence and prove several results related to it.

Contents

1. Introduction

The original McKay correspondence starts with a finite subgroup $G \subset SL(2, \mathbb{C})$ and its natural linear action on $\mathbb{C}^2$. It turns out that the singular quotient $\mathbb{C}^2/G$ admits a unique resolution $X \to \mathbb{C}^2/G$ with trivial canonical class, and that the cohomology of X has a basis labeled by irreducible representations of G. Its generalization assumes that a finite group G acts on a smooth irreducible variety U over $\mathbb{C}$ in such a way that

(i) for any $g \in G$ the codimension of the fixed point set U^g is ≥ 2,
(ii) the G-action preserves the canonical bundle of U, and
(iii) the quotient U/G admits a *crepant* resolution X.

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Then the cohomology of X has the same dimension as the orbifold cohomology groups $H^\bullet_{orb}(U;G)$, see [CR] for definition of orbifold cohomology and [LP], [Y] for the proof of the assertion.

A more general *categorical* version of the McKay correspondence, still largely conjectural, states that in this situation there is an equivalence

$$D^b_G(U) \longrightarrow D^b(X)$$

of the bounded derived category of G-equivariant coherent sheaves on U and the bounded derived category of coherent sheaves on X. As explained in [Ba], once such equivalence is established, one can apply the cyclic homology construction and obtain an isomorphism of $\mathbb{Z}_2$-graded vector spaces

$$H^\bullet_{orb}(U;G) \simeq H^\bullet(X)$$

recovering the usual (i.e., homological) McKay correspondence.

The goal of this paper is to describe a conjectural “twisted" version of the categorical McKay correspondence.

- On one hand, given a class $\alpha \in H^2(G, \mathbb{C}^*)$ one can define the twisted equivariant derived category $D^b_{G,\alpha}(U)$ and the twisted orbifold cohomology $H^\bullet_{orb,\alpha}(U;G)$, cf. [AR], [VW] and Section 2 of this paper.
- On the other hand, if $\mathcal{A}$ is an Azumaya algebra on X, cf. [Gr], then we have the corresponding derived category $D^b(X, \mathcal{A})$ and its (co)homology theory $H^\bullet(X, \mathcal{A})$.

One might therefore ask the following

Question. In the above situation, when are the twisted derived categories equivalent (resp. their homology groups isomorphic)?

By a standard construction reviewed in Section 2 any class α does define a natural Azumaya algebra $\mathcal{A}^\alpha$ on a dense open subset $X_0 \subset X$. Our first result describes when the class of $\mathcal{A}^\alpha$ in the *cohomological* Brauer group

$$Br(X_0) := H^2_{\acute{e}t}(X_0, \mathcal{O}^*),$$

cf. [Gr], extends to X. After proving in Section 3 that $Br(X)$ is the same for all resolutions of U/G (not necessarily crepant) we explicitly describe a subgroup $B_G(U) \subset H^2(G, \mathbb{C}^*)$ (here we use the notation of [Bo2]) which fits into a commutative diagram

$$\begin{tikzcd}
Br(X) \arrow[r, hook] & Br(X_0) \\
B_G(U) \arrow[u, hook] \arrow[r, hook] & H^2(G, \mathbb{C}^*) \arrow[u, hook]
\end{tikzcd}$$

In the important case when U is a vector space with a linear G-action one has $B_G(U) = Br(X)$. Our proof follows rather closely the projective case considered in [Bo2]. Even though it is still an open question, see [Gr], whether or not the cohomological Brauer group describes Azumaya algebras up to equivalence, by Section 4 of [Ca] any class $\beta \in Br(X)$ still leads to a twisted derived category $D^b(X, \beta)$ (when β does come from an Azumaya algebra $\mathcal{A}$, this is equivalent to the category of $\mathcal{A}$-modules). This leads to the following

Conjecture. (Twisted McKay correspondence) In the situation desribed above, let $\alpha \in B_G(U)$. Then there exists a derived equivalence

$$D^b_{G,\alpha}(U) \longrightarrow D^b(X, \alpha).$$

When the G-action is free, we have $B_G(U) = H^2(G, \mathbb{C}^*)$ and the above equivalence immediately follows from definitions. In Section 4 we give an example of a less trivial case.

In Section 5 we consider the cohomological consequence of the twisted McKay correspondence. It turns out that, in characteristic zero, the homology of $D^b(X, \alpha)$ is simply $H^\bullet(X)$. For affine X this was proved essentially by Weibel and Cortiñas, cf. [CW], and in Theorem 4 we deduce the general case from their result. On the other hand, generalizing [Ba] we also prove in Theorem 5 that the homology of $D^b_{G,\alpha}(X)$ can be identified with the twisted orbifold homology $H^\bullet_\alpha(U; G)$ described in [VW]. Since the definition of $B_G(U)$ implies that for $\alpha \in B_G(U)$ one has a vector space isomorphism

$$H^\bullet_\alpha(U, G) \simeq H^\bullet(U; G),$$

the twisted McKay correspondence on homological level simply reduces to the untwisted version (not very exciting, but it is hard to expect anything else since the Brauer group captures only torsion information). It is quite possible that homology with finite coefficients can give something different in the twisted case, but we do not pursue this topic here.

Finally, in Section 6 we discuss some related open problems.

Remark. Perhaps it is appropriate to mention here two more versions of the cohomological Brauer group:

(i) the analytic Brauer group $Br_{an}(X) = H^2_{an}(X, \mathcal{O}^*_{an})$, and
(ii) the topological Brauer group $Br_{top}(X) = H^3(X, \mathbb{Z})_{tors}$.

One can show that $Br(X) = Br_{an}(X)_{tors}$ for all X, and that $Br(X) \simeq Br_{top}(X)$ whenever $H^2_{an}(X, \mathcal{O}_{an}) = 0$.

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2. Projective cocycles and twisted derived categories

A finite abelian group A which can be generated by (at most) two elements is called *bicyclic*. Thus, either A is itself cyclic, or it is isomorphic to a product of two cyclic groups.

The next theorem deals with the Schur multiplier $H^2(A, \mathbb{C}^*)$ of A in the second case. We assume that 2-cocycles are normalized: $c(1, g) = c(g, 1) = 1$.

Theorem 1. — *Let $A \simeq C_1 \times C_2$ with C_1, C_2 cyclic. Then*
(a) $H^2(A, \mathbb{C}^*) = Hom(C_1 \otimes_{\mathbb{Z}} C_2, \mathbb{C}^*)$;
(b) A 2-cocycle $c : A \times A \to \mathbb{C}^*$ is a coboundary iff $c(g, h) = c(h, g)$ for all $g, h \in A$.

Proof. — Part (a) follows from the general results in in [Ka]. The “only if" part in (b) follows from the definition of a coboundary and the fact that A is abelian. To prove the “if" part note that the symmetry condition is preserved if we adjust a cocycle by a coboundary, and by part (a) this adjustment can be made in such a way that the value of $c(g, h)$ will depend only on the image of g in C_1 and the image of h in C_2. By symmetry such a cocycle is trivial. □

Let G be a finite group acting on an affine variety $U = Spec(R)$. Fix a 2-cocycle

$$c : G \times G \longrightarrow \mathbb{C}^*$$

representing a class in $H^2(G, \mathbb{C}^*)$. The *twisted group algebra* $R^c[G]$ is the set of all linear combinations $\sum_{g \in G} r_g \cdot g$ with the multiplication rule

$$(r_1 \cdot g_1) * (r_2 \cdot g_2) = c(g_1, g_2)(r_1 g_1(r_2) \cdot g_1 g_2)$$

The cocycle condition for c is equivalent to associativity of $R^c[G]$. Up to isomorphism, $R^c[G]$ depends only on the class α of c in $H^2(G, \mathbb{C}^*)$, hence we can (and will) denote it by $R^\alpha[G]$. Since c is normalized, $1 \in R$ gives a unity in $R^c[G]$.

Note further that $R^c[G]$ is naturally an algebra over the ring of invariants R^G. Moreover, if the G-action is free, $R^c[G]$ gives an Azumaya algebra over R^G. Localizing this construction, for any G acting freely on a quasiprojective variety U, and any class $\alpha \in H^2(G, \mathbb{C}^*)$ we get an Azumaya algebra $\mathcal{A}^\alpha$ on U/G (defined up to isomorphism).

In general, let $U_0 \subset U$ be the open subset on which the action is free. For any resolution of singularities $X \longrightarrow U/G$ denote by X_0 the preimage of U_0/G. Then by pullback our construction gives an Azumaya algebra $\mathcal{A}^\alpha$ on X_0 for any $\alpha \in H^2(G, \mathbb{C}^*)$.

3. The Brauer group of a resolution

In this paper a *valuation* will always mean a discrete rank one valuation. All varieties are over the field of complex numbers $\mathbb{C}$. Let Y be a reduced irreducible variety with field of rational functions K, and denote by $S(Y)$ be set of all valuations of K which become divisorial on *some* resolution $Z \longrightarrow Y$ (i.e., the corresponding map $v : K^* \longrightarrow \mathbb{Z}$ simply computes the order of a rational function along a fixed prime divisor on Z). If Y is proper $S(Y)$ is the set of all valuations, and when $Y = Spec\ R$ is affine and normal $S(Y)$ is the set of valuations v such that $R \subset \mathcal{O}_v \subset K$, where $\mathcal{O}_v = v^{-1}(\mathbb{Z}_{\geq 0})$. The next result clarifies the role of $S(Y)$ in the computation of the cohomological Brauer group $Br(X) = H^2_{\acute{e}t}(X, \mathcal{O}^*)$. We send the interested reader to [**Gr**] for the relation of $Br(X)$ and the group of equivalence classes of Azumaya algebras.
Remark. Note that in [**Gr**] the group $Br(X)$ is denoted by $Br'(X)$.

Theorem 2. — *If $X \longrightarrow Y$ is a resolution of singularities, then*

$$Br(X) = \bigcap_{v \in S(Y)} Br(\mathcal{O}_v) \subset Br(K).$$

In particular, the Brauer group does not depend on the choice of X. Moreover, once X is fixed, in the above intersection it suffices to consider only the divisorial valuations of X.

Proof. — Let $\alpha \in Br(X)$ and let $D \subset Z$ be a prime divisor on some resolution Z, giving a valuation v. After removing a codimension 2 subset $Z' \subset Z$ we can construct a regular birational map $Z \setminus Z' \longrightarrow X$. The pullback of α gives a class in $Br(Z \setminus Z')$. Localizing at D we get $\alpha \in Br(\mathcal{O}_v)$.

Now let α be a class in the right hand side of the formula. There exists an affine $U_0 \subset X$ such that $\alpha \subset Br(U_0)$. Let $D_1, \ldots, D_r$ be the irreducible components of $X \setminus U_0$ and $v_1, \ldots, v_r$ the corresponding valuations. Since $\alpha \in Br(\mathcal{O}_{v_i})$ for all i, there exist affine open subsets U_i such that $U_i \cap D_i \neq \varnothing$ and $\alpha \in Br(U_i)$. Therefore $\alpha \in Br(\bigcup_{i=0}^r U_i)$ which is equal to $Br(X)$ by the Purity Theorem, cf. [**Gr**], since $X \setminus (\bigcup_{i=0}^r U_i)$ has codimension at most 2 in X. The same argument shows that the divisorial valuations of X are sufficient to define $Br(X)$. $\square$

Let G be a finite group acting on a smooth variety U almost freely (i.e., the action is free on some open dense subset $U_0 \subset U$). If $L = \mathbb{C}(U)$ is the field of rational functions on U, then $K = \mathbb{C}(U/G)$ can be canonically identified with L^G.

By Hilbert Theorem 90 we have an exact sequence

$$1 \longrightarrow H^2(G, \mathbb{C}^*) \longrightarrow Br(K) \longrightarrow Br(L).$$

In terms of the previous section, a class $\alpha \in H^2(G, \mathbb{C}^*)$ gives an Azumaya K-algebra $L^\alpha[G]$, which has a class in the cohomological Brauer group $Br(K)$. Since the Brauer group of U_0/G

(resp. U_0) is a subgroup of $Br(K)$ (resp. $Br(L)$) we actually have an exact sequence

$$1 \longrightarrow H^2(G, \mathbb{C}^*) \longrightarrow Br(U_0/G) \longrightarrow Br(U_0).$$

Again, a class α maps to the class representing the Azumaya algebra $\mathcal{A}^\alpha$ defined in the previous section. By the previous result, the Brauer group $Br(X)$ of a resolution $X \longrightarrow U/G$ does not depend on the choice of X. We can assume that $X \longrightarrow U/G$ is an isomorphism over U_0/G, then $Br(X)$ naturally becomes a subgroup of $Br(U_0/G)$. Denote

$$B_G(U) = B(X) \cap H^2(G, \mathbb{C}^*).$$

The next theorem gives a direct computation of $B_G(U)$ in terms of fixed point subvarieties of G in U. Its proof is an adaptation of [Bo2] to our (possibly) non-compact case. We say that a bicyclic subgroup $A \subset G$ acts *cyclically* on a subvariety $U' \subset U$ if U' is A-invariant and A acts on U' via some cyclic quotient of A.

Theorem 3. — *Let G be a finite with an almost free action on a smooth variety U*

$$B_G(U) = \bigcap_{A \subset G} Ker(H^2(G, \mathbb{C}^*) \longrightarrow H^2(A, \mathbb{C}^*))$$

where the intersection is taken over all bicyclic subgroups A which act cyclically on a closed irreducible subvariety $U' \subset U$.

Proof. — Let $v \in S(U/G)$ be a divisorial valuation of K. Associated to v, and the extension $K = L^G \subset L$, is the decomposition subgroup $D_v \subset G$ and its inertia subgroup I_v, cf. [Se]. In characteristic zero I_v is cyclic and central in D_v. Take $\alpha \in H^2(G, \mathbb{C}^*) \subset Br(K)$, then $\alpha \in Br(\mathcal{O}_v)$ iff the restriction $\alpha|_{D_v}$ is induced from the quotient $G_v = D_v/I_v$ (cf. proof of Theorem 1.3' in [Bo2]). To restate this condition note that $H^2(I_v, \mathbb{C}^*) = 0$ since I_v is cyclic; so by Hochschild-Serre we have an exact sequence

$$H^2(G_v, \mathbb{C}^*) \longrightarrow H^2(D_v, \mathbb{C}^*) \longrightarrow Hom(G_v, Hom(I_v, \mathbb{C}^*))$$

(any cocycle $D_v \times D_v \longrightarrow \mathbb{C}^*$ after possible adjustment by a coboundary descends to $G_v \times D_v \longrightarrow \mathbb{C}^*$, and then the second arrow restricts it to $G_v \times I_v$).

Next we reduce to the case when G and A are p-groups. Suppose the assertion is known for all Sylow subgroups $G_p \subset G$. Given a bicyclic subgroup A as in the theorem and an element $\alpha \in B_G(U)$ we can find a diagram of resolutions

$$\begin{tikzcd}
X_p \arrow[r] \arrow[d] & X \arrow[d] \\
U/G_p \arrow[r] & U/G
\end{tikzcd}$$

and deduce that $\alpha|_{G_p} \in B_{G_p}(U)$ for all p. Writing $A = \bigoplus_p A_p$ we immediately conclude that $\alpha|_{A_p} = 0$ because all A_p also act cyclically and each A_p is conjugate to a subgroup of G_p (recall that conjugation acts trivially on cohomology). Since

$$H^2(A, \mathbb{C}^*) = \bigoplus_p H^2(A_p, \mathbb{C}^*),$$

this means that $\alpha|_A = 0$, as required.

In the other direction, if $\alpha \notin B_G(U)$ there is a valuation $v \in S(U/G)$ such that α gives a nonzero element of $Hom(G_v, Hom(I_v, \mathbb{C}^*))$. Taking the p-components, we find a Sylow subgroup $G_p \subset G$ and an extension v_p of v to L^{G_p} such that $\alpha|_{G_p} \notin Br(\mathcal{O}_{v_p})$, thus $\alpha \notin B_{G_p}(U)$. If the theorem is known for p-groups, there exists a bicyclic p-group $A_p = A$ acting cyclically on some U', for which $\alpha|_A \neq 0$.

Next we show that

$$\alpha \in B_G(U) \Leftrightarrow \alpha|_A = 0 \text{ for all } A \in Bic(G, U)$$

where $Bic(G, U)$ is the set of all bicyclic $A \subset G$, such that for some $v \in S(U/G)$ one has $A \subset D_v$ and the image of A in $G_v = D_v/I_v$ is cyclic. (At this step we will not use the p-group assumption.) If $\alpha|_{D_v}$ maps to zero in

$$Hom(G_v, Hom(I_v, \mathbb{C}^*)),$$

then $\alpha|_A$ maps to zero in

$$Hom(A/A \cap I_v, Hom(A \cap I_v, \mathbb{C}^*)).$$

Since both $A \cap I_v$ and $A/A \cap I_v$ are cyclic, $H^2(A, \mathbb{C}^*)$ is a subgroup of the latter group, thus $\alpha|_A = 0$. On the other hand, if $\alpha \notin Br(\mathcal{O}_v)$, then we find a cyclic subgroup $C \subset G_v$ which has nonzero image in $Hom(I_v, \mathbb{C}^*)$. The preimage of C in D_v is a bicyclic subgroup $A \subset D_v$ satisfying $\alpha|_A \neq 0$.

It remains to show that for a bicylic p-subgroup A of a p-group G the following conditions are equivalent:

(i) A acts cyclically on a closed irreducible subvariety $U' \subset U$, and
(ii) for some $v \in S(U/G)$ we have $A \subset D_v \subset G$ and $A/A \cap I_v$ is cyclic.

To prove (ii) ⇒ (i) choose a resolution $Z \longrightarrow U/G$ and a prime divisor D corresponding to v. There exists a G-equivariant birational map $Y \longrightarrow U$ with smooth Y, and a commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \\ U \arrow[r] & U/G \end{tikzcd}$$

Let D' be the preimage of the divisor D and $D'' \subset Y$ an irreducible component which dominates D. Then D'' is A-invariant (since $A \subset D_v$) and A acts on D'' via a cyclic quotient (since $A/A \cap I_v$ is cyclic). The image U' of D'' in U is a closed irreducible subvariety on which A acts cyclically.

To prove (i) $\Rightarrow$ (ii) first assume that $U' \subset U$ has codimension 1. Let D be the image of U' in U/G. Since U/G is non-singular in codimension 1, we can find a resolution $X' \longrightarrow U/G$ such that the preimage D' of D in X' is an irreducible divisor. The valuation v corresponding to D' clearly satisfies the conditions of (ii).

In general, choose a locally closed smooth G-invariant subvariety $V \subset U$ such that

$$V' = U' \cap V \subset V$$

is irreducible of codimension 1 and the generic orbit of A on V is free. By the earlier part of this proof and the codimension 1 case we get $B_A(V) = 0$. Finding a diagram of resolutions

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ V/A \arrow[r, hook] & U/A \end{tikzcd}$$

we conclude that every non-zero

$$\gamma \in H^2(A, \mathbb{C}^*) \subset Br(K)$$

is not in the image of $Br(X) \longrightarrow Br(K)$, otherwise by applying pullback $Br(X) \longrightarrow Br(Z)$ we would get a contradiction with $\gamma \notin B_A(U)$. Therefore $B_A(X) = 0$. In particular, take γ to be an element of order p in

$$H^2(A, \mathbb{C}^*) \simeq \mathbb{Z}/p^k\mathbb{Z}.$$

By the earlier part of the proof, there is a bicylic subgroup $A' \subset A$ and a valuation v of L^A such that $A' \subset D_v$, and $A'/I_v \subset A'$ is cyclic while $\gamma|_{A'} \neq 0$. However, since γ vanishes when restricted to any proper subgroup of A (this is where we finally use the p-group assumption), we must have $A' = D_v = A$. Restricting v from L^A to L^G (and possibly dividing $v|_{L^G}$:

$(L^G)^* \longrightarrow \mathbb{Z}$ by an integer to make it surjective), we get a valuation satisfying (ii), finishing the proof. ☐

4. An example

Consider an almost free action of G on a vector space V and assume that for all $g \in G$ we have $codim\ V^g \geq 2$. Take U to be the complement of a G-invariant closed subset Z of codimension ≥ 2. Then $Pic(U) = 0$, $Br(U) = 0$, and hence the Brauer group of any resolution $X \longrightarrow U/G$ is equal to the subgroup $B_G(U) \subset H^2(G, \mathbb{C}^*)$. We can further identify this as follows:

$$B_G(U) = \{\alpha \in H^2(G, \mathbb{C}^*) |\ \alpha(g, h) = \alpha(h, g) \text{ whenever } U^g \neq \varnothing \text{ and } gh = hg\}.$$

Note that the condition on the right hand side is preserved when a cocycle is multiplied by a coboundary. For instance, when $U = U_0$ is the subset of all vectors in V with trivial stabilizers, we have $B_G(U) = H^2(G, \mathbb{C}^*)$; when $U = V$ we get the subgroup $B_0(G)$ of classes in $H^2(G, \mathbb{C}^*)$ which restrict to zero on any abelian subgroup of G. This subgroup, known as *unramified cohomology of* G, was studied extensively in [Bo1]. Observe, that $B_0(G)$ does not depend on the choice of V - it is simply the group formed by classes which vanish when restricted to any abelian subgroup.

Groups with $B_0(G) \neq 0$ are relatively rare and the condition that V/G admits a *crepant* resolution puts a further restriction on the pair (V, G) (see the last section of this paper). However, it is relatively easy to find a a group G with $B_0(G) \neq 0$ and an open subset U in a representation V such that U/G is not smooth but admits a crepant resolution (we will automatically have $B_G(U) \neq 0$ since it contains $B_0(G) \neq 0$ as a subgroup). We now proceed to describe such an example.

Let p be a prime and consider a central extension of the form

$$1 \longrightarrow \mathbb{Z}_p^3 \longrightarrow G \longrightarrow \mathbb{Z}_p^4 \longrightarrow 1$$

If (a, b, c) is the basis of $\mathbb{Z}_p^3$ and (x_1, x_2, x_3, x_4) a lift of a basis from $\mathbb{Z}_p^4$ to G, it was proved in [Bo1] (cf. Example 3 before Lemma 5.5) that the relations

$$[x_1, x_2] = [x_3, x_4] = a; \quad [x_1, x_3] = [x_1, x_4] = 1; \quad [x_2, x_4] = b; \quad [x_2, x_3] = c$$

(where $[x, y] = xyx^{-1}y^{-1}$), imply that $B_0(G) \simeq \mathbb{Z}_p$. To describe an exact representation of G let $\varepsilon = exp(\frac{2\pi i}{p})$ and choose a pair of $p \times p$ matrices P, Q such that $[P, G] = \varepsilon I$. For $p = 2$ we can take the Pauli matrices

$$P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix};$$

for odd p we can take P to be the operator which permutes the basis vectors $(v_1, \ldots, v_p)$ cyclically: $v_i \mapsto v_{i+1}$ for $i = 1, \ldots, p-1$, and $v_p \mapsto v_1$; while Q is the diagonal matrix $diag(1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{p-1})$.

Let $V \simeq \mathbb{C}^{p^2+2p} = (\mathbb{C}^p \otimes \mathbb{C}^p) \oplus \mathbb{C}^p \oplus \mathbb{C}^p$ be the representation given by

$$x_1 \longmapsto (P \otimes I) \oplus I \oplus I; \qquad x_2 \longmapsto (Q \otimes I) \oplus P \oplus P$$

$$x_3 \longmapsto (I \otimes P) \oplus I \oplus Q; \qquad x_4 \longmapsto (1 \otimes Q) \oplus Q \oplus I$$

$$a \longmapsto \varepsilon(I \otimes I) \oplus I \oplus I; \qquad b \longmapsto (I \otimes I) \oplus \varepsilon I \oplus I; \qquad c \longmapsto (I \otimes I) \oplus I \oplus \varepsilon I$$

One can check directly, that non-scalar elements in the group H_1 of order p^3 generated by (P, Q) all have p distict eigenvectors with eigenvalues $1, \varepsilon, \ldots, \varepsilon^{p-1}$. For each of these eigenvectors, the stabilizer in H_1 is isomorphic to $\mathbb{Z}_p$.

Simlarly, all non-scalar elements H_2 in the group of order p^5 generated by

$$P \otimes 1,\ Q \otimes 1,\ 1 \otimes P,\ 1 \otimes Q$$

have p eigenspaces of dimension p, with the same eigenvalues. Again, each of the eigenspaces has stabilizer in H_2 which is isomorphic to $\mathbb{Z}_p$.

It follows, that for each $g \in G$ the fixed point subspace V^g has codimension $\geq p$ and for odd p the codimension p fixed subspaces are precisely

- $V^b = (\mathbb{C}^p \otimes \mathbb{C}^p) \oplus \mathbb{C}^p \oplus 0$, and
- $V^c = (\mathbb{C}^p \otimes \mathbb{C}^p) \oplus 0 \oplus \mathbb{C}^p$.

For $p = 2$ in addition to V^b and V^c one also has the fixed point subspace

$$V^{x_1} = V' \oplus \mathbb{C}^2 \oplus \mathbb{C}^2$$

where V' is the $(+1)$-eigenspace of $P \otimes 1$.

To describe a G-invariant open subset $U \subset V$ let Z be the union of those fixed point subspaces V^g which have codimension $\geq (p + 1)$. Define $U = V \subset Z$, then the singularities of U/G are the images of V^b, V^c (and V^{x_1} if $p = 2$). A single canonical blowup gives a crepant resolution $X \longrightarrow U/G$. By the Purity Theorem, cf. [Gr], and $codim\ Z \geq 2$ we conclude that $Br(X) = B_G(U)$ and this group is non-zero since it contains the subgroup $B_0(G) \simeq \mathbb{Z}_p$.

Further similar examples can be obtained with other finite p-groups listed in [Bo1].

5. Homology of categories

Even if the G-action on U is not free, for every G-invariant affine open subset $U' \subset U$ with algebra of functions R we can still consider $R^\alpha[G]$, cf. Section 2, and modules over this algebra. Localizing at G-invariant affine open subsets of U we get a notion of an α-twisted equivariant sheaf $\mathcal{F}$: this is a sheaf of $\mathcal{O}$-modules on U such that for any G-invariant open subset $V \subset U$, the group $\mathcal{F}(V)$ is equipped with an $\mathcal{O}(V)^\alpha[G]$-module structure, and for different invariant open subsets such structures agree with restriction of sections. Morphisms of α-twisted equivariant sheaves are given by those morphisms of $\mathcal{O}$-modules which commute with the $\mathcal{O}(V)^\alpha[G]$-action for every G-invariant V. Considering the bounded complexes of coherent α-twisted equivariant sheaves and localizing at quasi-isomorphisms we get the bounded derived category $D^b_{G,\alpha}(U)$ of α-twisted G-equivariant sheaves on U.

Alternatively, denote by the same letter α a cocycle representing the cohomology class. There exists a central group extension

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}_n \arrow[r] & \widetilde{G} \arrow[r, "\pi"] & G \arrow[r] & 1 \end{tikzcd}$$

and a character $\psi : \mathbb{Z}_n \longrightarrow \mathbb{C}^*$ such that $\alpha(g,h) = \psi(\widetilde{g}\widetilde{h}\widetilde{gh}^{-1})$, where $\widetilde{g}$, etc. denotes some lift of $g \in G$ to $\widetilde{G}$. The group $\widetilde{G}$ acts on U via its homomorphism to G. Since the subgroup $\mathbb{Z}_n \subset \widetilde{G}$ acts trivially on U, the stalk of a $\widetilde{G}$-equivariant sheaf on U at any point has a natural structure of a $\mathbb{Z}_n$-module. The derived category of equivariant sheaves $D^b_{\widetilde{G}}(U)$ splits into orthogonal direct sum of subcategories corresponding to different characters of $\mathbb{Z}_n$. It follows from the above definition that $D^b_{G,\alpha}(U)$ is equivalent to the subcategory of $D^b_{\widetilde{G}}(U)$ corresponding to the character ψ. For affine U this reduces to the statement that $R^\alpha[G]$ is isomorphic to a quotient of $R[\widetilde{G}]$ by the ideal J generated by $(t - \psi(t)1)$ with $t \in \mathbb{Z}_n$.

If now $\alpha \in Br(X)$ and α corresponds to an Azumaya algebra $\mathcal{A}$ on X we define $D^b(X,\alpha)$ to be the bounded derived category of finitely generated modules over $\mathcal{A}$. If α does not come from an Azumaya algebra (which should never happen, by a conjecture due to Grothendieck), we can apply the construction in Section 4 of [Ca] and still get a derived category $D^b(X,\alpha)$.

Suppose that we have a derived equivalence

$$D^b_{G,\alpha}(U) \simeq D^b(X,\alpha) \tag{5.1}$$

By a construction explained in [Ke] both derived categories have a series of homological invariants, including

(i) Hochschild homology HH_*,
(ii) cyclic homology HC_*,
(iii) periodic cyclic homology HP_*, and
(iv) negative cyclic homology HN_*.

We denote by H any of these homology theories.

Since the derived equivalence induced an isomorphism of homological invariants, cf. [Ke], (under an additional assumption, always satisfied in a geometric situation such as ours), the above equivalence (5.1) should imply an isomorphism of homology.

Let $H^\alpha(X)$ be the homology of $D^b(X,\alpha)$ and $H^\alpha_G(U)$ the homology of $D^b_{G,\alpha}(U)$. For $\alpha = 0$ we drop the superscript α. First, we show that the definiton of $H^\alpha(X)$ does not give anything new.

Theorem 4. — *The natural inclusion of algebras* $\mathcal{O} \longrightarrow \mathcal{A}$ *induces an isomorphism* $H(X) \simeq H^\alpha(X)$.

Proof. — In suffices to prove the claim for Hochschid homology ($H = HH_*$), the other cases being a consequence by Proposition 2.4 of [GJ].

In the affine case the derived category homology coincides with the usual Hochschild homology of rings, hence the result is proved in [CW].

In general, we cover X with affine open subsets $\{U_i\}_{i\in I}$ and recall that by a result of Gabber $\alpha|_{U_i}$ does come from an Azumaya algebra. Therefore, applying the Mayer-Vietoris sequence and Noetherian induction we finish as in Proposition 3.3 in [Ba]. □

The computation of $H^\alpha_G(U)$ is given by a theorem parallel to Theorem 7.4 in [AR]. For any $g \in G$ denote by Z_g the centralizer of g and observe that the fixed point subvariety U^g is Z_g-invariant. Following [AR] we denote by L^α_g the one dimensional representation of Z_g on which $h \in Z_g$ acts by $\alpha(g,h)\alpha(h,g)^{-1}$.

Theorem 5. — *Let U be a smooth complex variety with an action of a finite group G and let $\alpha \in H^2(G,\mathbb{C}^*)$. Then*

$$H^\alpha_G(U) = \bigoplus_{(g)} \Big(H(U^g)\otimes L^\alpha_g\Big)^{Z_g}$$

where the sum is taken over all conjugacy classes of G.

Proof. — Let $\widetilde{G}$, π and ψ be as in the beginning of this section. By the main result of [Ba] the homology of $D^b_{\widetilde{G}}(U)$ can be identified with

$$\big(\bigoplus_{f\in\widetilde{G}} H(U^f)\big)^{\widetilde{G}}$$

where an element $t \in \widetilde{G}$ sends

$$U^f \longmapsto U^{tft^{-1}}$$

inducing an action on homology. Since the derived category $D^b_{\widetilde{G}}(U)$ splits into orthogonal direct sum of subcategories labeled by characters of $\mathbb{Z}_n \subset \widetilde{G}$, we just have to extract from the above expression the component corresponding to ψ.

It follows from Step 2 after the proof of Proposition 3.2 in [Ba], that the induced $\mathbb{Z}_n$-action on $\bigoplus_{f\in\widetilde{G}} H(U^f)$ can be describe as follows: an element $h \in \mathbb{Z}_n$ sends $H(U^f)$ to $H(U^{hf})$ (both fixed point spaces are the same, but h permutes different copies of the same homology group in the direct sum). Since $\mathbb{Z}_n$ is central in $\widetilde{G}$ and acts trivially on U, this action commutes with the earlier $\widetilde{G}$-action.

To compute the component of ψ in $\big(\bigoplus_{f\in\widetilde{G}} H(U^f)\big)^{\widetilde{G}}$ we split the direct sum by grouping together those f which map to the same conjugacy class in G. For a conjugacy class $C \subset G$

consider

$$W_C = \bigoplus_{\pi(f)\in C} H(U^f)$$

Denote $\widetilde{Z}_g = \pi^{-1}(Z_g) \subset \widetilde{G}$, then the $\widetilde{G}$-module W_C is induced from the $\widetilde{Z}_g$-module

$$W_g = \bigoplus_{\pi(f)=g} H(U^f).$$

As a $\mathbb{Z}_n$-module the latter space is just a multiple of the regular representation of $\mathbb{Z}_n$. By definition of $\widetilde{G}$ the component of ψ in the latter sum, viewed as a Z_g-module, is simply $H(U^g) \otimes L_g^\alpha$. Taking the invariants and summing over all conjugacy classes of G we obtain the right hand side of the formula stated in the Theorem. □

In our last result we specialize to periodic cyclic homology, which is equal to the usual topological cohomology by a result of Feigin-Tsygan, cf. [**FT**]. This result provides an indirect confirmation of the twisted McKay correspondence conjectured in this paper.

Corollary 6. — *Let $X \longrightarrow U/G$ be a crepant resolution. For any $\alpha \in B_G(U)$ the derived categories $D^b(X,\alpha)$ and $D^b_{G,\alpha}$ have periodic cyclic homology of the same dimension.*

Proof. One one hand, any homology theory of $D^b(X,\alpha)$ is isomorphic to that of $D^b(X)$. On the other hand, by definition of $B_G(U)$ the character L_g^α vanishes whenever U^g is non-empty. Therefore the previous theorem implies that also $D^b_{G,\alpha}(U)$ and $D^b_G(U)$ have the same cyclic homology theories. Applying periodic cyclic homology to $D^b_G(U)$, resp. $D^b(X)$, we get orbifold cohomology of U, resp. usual cohomology of X. But these have the same dimension by [**LP**], [**Y**]. □

6. Open problems

In conclusion we state the following open problems:

(i) If would be interesting to construct an example of a finite G with a linear action on a vector space V, such that $B_G(V) = B_0(V) \neq 0$ and V/G admits a crepant resolution. Such examples should be relatively rare; for instance the standard symplectic example $V = W \oplus W^*$ will definitely not work, for in this case V/G admits a crepant resolution iff G acts on W by complex reflections which implies $B_0(G) = 0$ (this is because $B_0(G)$ does not depend on the choice of V and W/G is isomorphic to an affine space).

(ii) The second problem refers to the subgroup $B_G(U) \subset H^2(G,\mathbb{C}^*)$. Assume for simplicity that $Br(U) = 0$ then $Br(X) = B_G(U)$ for any resolution $X \longrightarrow U/G$. Is it possible, however, to define a “derived Brauer group" purely in terms of the (enhanced) derived category $D^b(X)$, which would give the full Schur multiplier $H^2(G,\mathbb{C}^*)$ in this case, and in general contain $Br(X)$ as a subgroup? We ask this question by analogy with the derived

Picard group, which is a natural extension of the usual Picard group. The Merkurjev-Suslin Theorem suggests that some answer may perhaps be obtained from K_2 but for practical purposes it should be more computable than K_2.

(iii) The third problem is related to the above two. Suppose we have an action of G on a vector space V and V/G admits a crepant resolution X. As we have seen in this paper, not all Brauer classes of $\mathbb{C}(V)^G$ extend to X. For example, when $G = S_N$ is the symmetric group acting of $V = (\mathbb{C}^2)^{\oplus n}$, the Hilbert scheme $Hilb^n(\mathbb{C}^2)$ of points on $\mathbb{C}^2$ provides a crepant resolution of V/G and it is easy to check that $Br(Hilb^n(\mathbb{C}^2)) = 0$. In general, if $\alpha \in H^2(G, \mathbb{C}^*) \setminus B_0(G)$ it would be interesting to find an interpretation of the orbifold cohomology $H^*_{G,\alpha}(V)$ in terms of X. To restate the same question: what type of geometric objects on X will correspond to projective representations of G?

References

[AR] Adem, A.; Ruan, Y.: Twisted orbifold K-theory. *Comm. Math. Phys.* **237** (2003), no. 3, 533–556.

[Ba] Baranovsky, V.: Orbifold cohomology as periodic cyclic homology. *Internat. J. Math.* **14** (2003), no. 8, 791–812.

[Bo1] Bogomolov, F.: The Brauer group of quotient spaces of linear representations. *Math. USSR-Izv.* **30** (1988), no. 3, 455–485.

[Bo2] Bogomolov, F.: Brauer groups of the fields of invariants of algebraic groups. *Math. USSR-Sb.* **66** (1990), no. 1, 285–299 .

[Ca] Caldararu, A.: Derived categories of twisted sheaves on elliptic threefolds. *J. Reine Angew. Math.* **544** (2002), 161–179.

[CR] Chen, W.; Ruan, Y.: A new cohomology theory of orbifold. *Comm. Math. Phys.* **248** (2004), no. 1, 1–31.

[CW] Cortiñas, G.; Weibel, C.: Homology of Azumaya algebras. *Proc. Amer. Math. Soc.* **121** (1994), no. 1, 53–55.

[FT] Feigin, B.; Tsygan, B.: Additive K-theory. *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, 67–209, Lecture Notes in Math., 1289, *Springer, Berlin*, 1987.

[GJ] Getzler, E.; Jones, J. D. S.: A_∞-algebras and the cyclic bar complex. *Illinois J. Math.* **34** (1990), no. 2, 256–283.

[Gr] Grothendieck, A: Le groupe de Brauer. I, II, III in *Dix Exposés sur la Cohomologie des Schémas* 1968 North-Holland, Amsterdam; Masson, Paris

[LP] Lupercio, E.; Poddar, M.: The global McKay-Ruan correspondence via motivic integration. *Bull. London Math. Soc.***36** (2004), no. 4, 509–515.

[Ka] Karpilovsky, G.: The Schur multiplier. London Mathematical Society Monographs. New Series, 2. *The Clarendon Press, Oxford University Press, New York*, 1987.

[Ke] Keller, B.: On the cyclic homology of exact categories. *J. Pure Appl. Algebra* **136** (1999), no. 1, 1–56.

[Se] Serre, J.-P.: Local fields. *Graduate Texts in Mathematics*, **67**. *Springer-Verlag, New York-Berlin*, 1979.

[VW] Vafa, C.; Witten, E.: On orbifolds with discrete torsion. *J. Geom. Phys.* **15** (1995), no. 3, 189–214.

[Y] Yasuda, T.: Twisted jets, motivic measures and orbifold cohomology. *Compos. Math.* **140** (2004), no. 2, 396–422.

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