RESTRICTED SCHUR MULTIPLIERS AND THEIR APPLICATIONS

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Abstract. We compute the norm of the restriction of a Schur multiplier, arising from a multiplication operator, to a coordinate subspace. This result is used to generalize Wielandt's minimax inequality. Furthermore, we compute various s-numbers of an elementary Schur multiplier, and determine criteria for membership of such multipliers in certain operator ideals.

1. Introduction

Throughout this paper, we work with unconditional matrix spaces. A Banach space $\mathfrak{X}$ of $I \times I$ matrices (I is a finite or infinite index set) is called an *unconditional matrix space* if:

(1) All matrices with finitely many non-zero entries belong to $\mathfrak{X}$.
(2) For any matrix $(a_{ij}) \in \mathfrak{X}$, $\|(a_{ij})\| = \sup_{J \subset I, |J| < \infty} \|\mathcal{P}_J(a_{ij})\|$, where $\mathcal{P}_J$ is the truncation onto $J \times J$, that is, $(\mathcal{P}_J(a_{ij}))_{pq} = \left\{ \begin{array}{ll} a_{pq} & (p,q) \in J \times J \\ 0 & (p,q) \notin J \times J \end{array} \right.$.
(3) For any matrix $(a_{ij}) \in \mathfrak{X}$, and any to sequences (α_i), (β_j) of scalars of absolute value 1, $\|(a_{ij})\| = \|(\alpha_i \beta_j a_{ij})\|$.

Clearly, tensor products of Banach spaces with 1-unconditional bases are unconditional matrix spaces. Another important class of examples arises in the following way. Suppose $\mathcal{E}$ is a *symmetric sequence space*, that is, a Banach space of sequences such that, for any $(x_1, x_2, \ldots) \in \mathcal{E}$, we have:

(1) For any sequence (ω_i) in the unit ball of $\mathbb{C}$, and for any permutation $\pi : \mathbb{N} \to \mathbb{N}$ $(\omega_1 x_{\pi(1)}, \omega_2 x_{\pi(2)}, \ldots) \in \mathcal{E}$, and $\|(\omega_1 x_{\pi(1)}, \omega_2 x_{\pi(2)}, \ldots)\|_{\mathcal{E}} = \|(x_1, x_2, \ldots)\|_{\mathcal{E}}$.
(2) $\|(x_1, x_2, \ldots)\|_{\mathcal{E}} = \lim_n \|(x_1, \ldots, x_n)\|_{\mathcal{E}}$, where we define $\|(x_1, \ldots, x_n)\|_{\mathcal{E}}$ as $\|(x_1, \ldots, x_n, 0, 0, \ldots)\|_{\mathcal{E}}$.

Denote by $\mathcal{S}^{\mathcal{E}}$ the space of operators T on ℓ_2 whose sequences of singular values $(s_i(T))_{i \in \mathbb{N}}$ belong to $\mathcal{E}$ (see Section II.7 of [12] for the definition of singular numbers for non-compact operators). The norm on this space is given by $\|T\|_{\mathcal{E}} = \|(s_i(T))_{i \in \mathbb{N}}\|_{\mathcal{E}}$. By e.g. Chapters 1-2 of [28], or Chapter III of [12], this is an unconditional matrix

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space. Moreover, any separable symmetrically normed ideal is of the form $\mathcal{S}^{\mathcal{E}}$, where $\mathcal{E}$ is *regular*: $\lim_n \|(0,\ldots,0,x_n,x_{n+1},\ldots)\|_{\mathcal{E}} = 0$ for any $(x_1,x_2,\ldots) \in \mathcal{E}$.

For the sake of brevity, write $\mathcal{S}^p = \mathcal{S}^{\ell^p}$ for $1 \leqslant p < \infty$, and $K(\ell_2) = \mathcal{S}^\infty = \mathcal{S}^{c_0}$. The n-dimensional version is denoted by $\mathcal{S}_n^{\mathcal{E}}$.

The pairing between a matrix space and its conjugate arises from the *parallel duality*: for matrices (finite or infinite) $T = (t_{ij})$ and $S = (s_{ij})$, $\langle T, S\rangle = \mathrm{Tr}\, T\, {}^tS = \sum_{i,j} t_{ij}s_{ij}$ (${}^tS$ is the transposed of S).

For a matrix $\phi = (\phi_{ij})_{i,j\in I}$, we define a Schur multiplier $\mathbb{S}_\phi$: for any matrix $T = (t_{ij}) \in \mathfrak{X}$ ($\mathfrak{X}$ is an unconditional matrix space), $\mathbb{S}_\phi T = \phi \diamond T = (\phi_{ij}t_{ij})$. The matrix ϕ is called the *symbol* of the Schur multiplier. If $\mathbb{S}_\phi$ is viewed as an operator from $\mathfrak{X}$ to $\mathfrak{Y}$, we use the notation $\mathbb{S}_\phi^{\mathfrak{X},\mathfrak{Y}}$. If $\mathfrak{X} = \mathfrak{Y}$, we write $\mathbb{S}_\phi^{\mathfrak{X}}$. If $\mathfrak{X} = \mathcal{S}^{\mathcal{E}}$ and $\mathfrak{Y} = \mathcal{S}^{\mathcal{F}}$, we write $\mathbb{S}_\phi^{\mathcal{E},\mathcal{F}}$. Furthermore, if $\mathcal{E} = \ell_p$ and $\mathcal{F} = \ell_q$, write $\mathbb{S}_\phi^{p,q}$.

For $\mathbf{S} \subset I \times I$, denote by $\mathbb{S}_{\phi,\mathbf{S}}$ the restriction of $\mathbb{S}_\phi$ to $\mathrm{span}[E_{ij} : (i,j) \in \mathbf{S}]$. Here, $(E_{ij})_{i,j\in I}$ are *matrix units*, that is, the matrices with 1 in the (i,j) position, and zeroes elsewhere. To denote restricted Schur multipliers between specific spaces, we use the notation $\mathbb{S}_{\phi,\mathbf{S}}^{\mathfrak{X},\mathfrak{Y}}$ etc.. We denote by $\mathfrak{X}_{\mathbf{S}}$ the set of all $T = (t_{ij}) \in \mathfrak{X}$, for which $t_{ij} = 0$ when $(i,j) \notin \mathbf{S}$.

Throughout this paper, we are primarily concerned with *elementary Schur multipliers* – that is, those for which the symbol ϕ satisfies $\phi_{ij} = a_ib_j$, for sequences $a = (a_i), b = (b_j) \in \ell_\infty$. In this case, we write $\phi = a \cdot b$.

Theorem 1.1. *Suppose $\mathfrak{X}$ is an unconditional matrix space, $\mathbf{S}$ is a subset of $\mathbb{N}\times\mathbb{N}$, and $a = (a_i)$, $b = (b_j)$ are sequences of scalars. Then $\|\mathbb{S}_{a\cdot b,\mathbf{S}}^{\mathfrak{X}}\| = \sup_{(i,j)\in\mathbf{S}} |a_ib_j|$.*

Note that, in particular, $\|\mathbb{S}_{a\cdot b}\| = \sup_i |a_i| \sup_j |b_j|$ – in fact, this is evident from the unconditionality of our matrix spaces. In general, computing, or even estimating, the norms of Schur multipliers – to say nothing of their restrictions – is extremely tough. Perhaps the best explored case is that of multipliers on $B(\ell_2)$. There, the importance of elementary multipliers is apparent: by Grothendieck's Theorem, a Schur multiplier is bounded iff it lies in the pointwise closure of the convex hull of the elementary ones (see e.g. Chapter 5 of [25]). Surprisingly, the norm of a "random" Schur multipliers is much easier to handle [24]. The same techniques were further refined in [7]. This paper characterizes the matrices A with the property that any "A-dominated" Schur multiplier is bounded. Compactness of Schur multipliers on $B(\ell_2)$ was described in [15].

Much less is known about Schur multipliers on the Schatten spaces $\mathcal{S}^p$. It was shown in [13] that, for any even integer p and any $q > p$, there exists a matrix ϕ s.t. $\mathbb{S}_\phi$ is bounded on $\mathcal{S}^p$, but not on $\mathcal{S}^q$. The boundedness of various Schur multipliers was established in [10, 14], while [4] contains lower estimates on the norms of Schur idempotents. A special class of Schur multipliers was recently investigated in

[27]. The norm estimates obtained there were used to show that Lipschitz functions operate continuously on self-adjoint elements of $\mathcal{S}^p$ $(1 < p < \infty)$.

Theorem 1.1 yields a generalization of the Wielandt minimax principle.

Theorem 1.2. *Suppose $\mathcal{E}$ is a symmetric sequence space, H_1 and H_2 are Hilbert spaces, and $A \in K(H_1, H_2)$ has singular values $a_1 \geqslant a_2 \geqslant \ldots \geqslant 0$. Suppose, furthermore, that $n \in \mathbb{N}$, and $i_1 < i_2 < \ldots < i_n$ are positive integers. Then:*

(1)
$$\|(a_{i_1}, \ldots, a_{i_n})\|_{\mathcal{E}} = \sup_{(E_k)} \inf_{(e_k)} \|A|_{\mathrm{span}[e_1,\ldots,e_n]}\|_{\mathcal{E}}.$$
Here, $\sup_{(E_k)}$ is taken over all families of subspaces of E with $E_1 \hookrightarrow E_2 \hookrightarrow \ldots \hookrightarrow E_n$, with $\dim E_k = i_k$. $\inf_{(e_k)}$ is taken over all the orthonormal systems $(e_k)_{k=1}^n$, with $e_k \in E_k$ for every k.

(2)
$$\|(a_{i_1}, \ldots, a_{i_n})\|_{\mathcal{E}} = \inf_{(E_k)} \sup_{(e_k)} \|A|_{\mathrm{span}[e_1,\ldots,e_n]}\|_{\mathcal{E}}.$$
Here, $\inf_{(E_k)}$ is taken over all families of subspaces of E with $E_n \hookrightarrow E_{n-1} \hookrightarrow \ldots \hookrightarrow E_1$, with $\mathrm{codim}\, E_k = i_k - 1$. $\sup_{(e_k)}$ is taken over all the orthonormal systems $(e_k)_{k=1}^n$, with $e_k \in E_k$ for every k.

If $\mathcal{E}$ is 2-convex (for instance, if $\mathcal{E} = \ell_p$, with $p \geqslant 2$), the above result follows from Wielandt-style inequalities on eigenvalues (see e.g. Section III.3 of [5]).

Theorem 1.1 also allows us to compute various s-numbers of elementary Schur multipliers on $B(\ell_2)$. We refer the reader to [23] for general information about s-numbers, and to [17] (and references therein) for their non-commutative versions. General theory of operator spaces can be found in [11, 18, 26]. Below, we recall some necessary definitions.

Suppose T is a linear operator between Banach (or operator) spaces X and Y. We define the approximation numbers a_n, Kolmogorov numbers d_n, and Gelfand numbers c_n:

$$\begin{aligned} a_n(T) &= \inf\{\|T - S\| : S \in B(X,Y),\ \mathrm{rank}\, S < n\} \\ d_n(T) &= \inf\{\|qT\| : q : Y \to Y/F \text{ quotient map},\ \dim F < n\} \\ c_n(T) &= \inf\{\|T|_E\| : E \hookrightarrow X,\ \mathrm{codim}\, E < n\}. \end{aligned} \tag{1.1}$$

The non-commutative versions of these sequences are defined similarly, with $a_n^{(cb)}(T) = \inf\{\|T - S\|_{cb} : S \in B(X,Y),\ \mathrm{rank}\, S < n\}$ etc..

It is known that $\lim_n d_n(T) = 0$ iff $\lim_n c_n(T) = 0$ iff T is compact. For this reason, we are mostly interested in non-zero compact elementary Schur multipliers. It is easy to see that $\mathbb{S}_{a \cdot b} \neq 0$ is compact iff $a, b \in c_0$. In this case, denote by $\widetilde{[ab]}$ the non-increasing rearrangement of the double sequence $(a_i b_j)$.

Theorem 1.3. *For $a, b \in c_0$, consider $\mathbb{S} = \mathbb{S}_{a\cdot b}$ as an operator on an unconditional matrix space $\mathfrak{X}$. Then $a_n(\mathbb{S}) = c_n(\mathbb{S}) = d_n(\mathbb{S}) = \widetilde{[ab]}_n$ for any $n \in \mathbb{N}$. If $\mathbb{S}$ acts on $B(\ell_2)$ or $\mathcal{S}^p$ ($1 \leqslant p \leqslant \infty$), then, in addition, $a_n^{(cb)}(\mathbb{S}) = c_n^{(cb)}(\mathbb{S}) = d_n^{(cb)}(\mathbb{S}) = \widetilde{[ab]}_n$.*

The information about the behavior of s-numbers of an elementary Schur multiplier can be used to determine its membership in various operator ideals (see e.g. [22, 23] for the definitions and basic properties). Some results of this type are contained in Section 5.

We prove Theorem 1.1 in Section 2. Section 3 is devoted to establishing Theorem 1.2. The proof of Theorem 1.3 is given in Section 4.

2. Proof of Theorem 1.1

Suppose $\mathfrak{X}$ is an unconditional matrix space on $I \times I$. For $J_1, J_2 \subset I$, define the truncation map $\mathcal{P}_{J_1,J_2}$ by setting, for $(a_{ij}) \in \mathfrak{X}$,

$$\big(\mathcal{P}_{J_1,J_2}(a_{ij})\big)_{pq} = \begin{cases} a_{pq} & (p,q) \in J_1 \times J_2 \\ 0 & (p,q) \notin J_1 \times J_2 \end{cases} .$$

Lemma 2.1. *Suppose $\mathfrak{X}$ is an unconditional matrix space on $I \times I$, $J_1, J_2 \subset I$, and $T \in \mathfrak{X}$ satisfies $\mathcal{P}_{J_1^c,J_2^c}T = 0$. Then, for $\lambda \in (0,1)$, $\|\mathcal{P}_{J_1,J_2^c}T + \mathcal{P}_{J_1^c,J_2}T + \lambda\mathcal{P}_{J_1,J_2}T\|_{\mathfrak{X}} \leqslant \|T\|_{\mathfrak{X}}$.*

Proof. Write $T = (t_{ij})$, and let

$$T_1 = \mathcal{P}_{J_1,J_2^c}T + \mathcal{P}_{J_1^c,J_2}T = \frac{T + T_0}{2}, \text{ where } (T_0)_{ij} = (-1)^{\chi_{J_1}(i)}(-1)^{\chi_{J_2}(j)}t_{ij}.$$

As $\mathfrak{X}$ is an unconditional matrix space, $\|T_1\| \leqslant \|T\| = \|T_0\|$. Note that $T = T_1 + PTQ$, hence

$$\|T_1 + \lambda PTQ\|_{\mathfrak{X}} = \|\lambda T + (1-\lambda)T_1\|_{\mathfrak{X}} \leqslant \|T\|_{\mathfrak{X}} = 1,$$

by the triangle inequality. ∎

Proof of Theorem 1.1. One direction is clear:

$$\|\mathbb{S}^{\mathfrak{X}}_{a\cdot b,\mathbf{S}}\| \geqslant \sup_{(i,j)\in\mathbf{S}} \|\mathbb{S}^{\mathfrak{X}}_{a\cdot b,\mathbf{S}}E_{ij}\| = \sup_{(i,j)\in\mathbf{S}} |a_i b_j|.$$

To establish the opposite inequality, we can assume that $\mathbf{S}$ is a subset of $\{1,\ldots,N\} \times \{1,\ldots,N\}$, and that $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_N > 0$, and $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_N > 0$. By re-scaling, and enlarging the set $\mathbf{S}$ if necessary, we can assume that $\mathbf{S} = \{(i,j) : a_i b_j \leqslant 1\}$. It suffices to show that, for such $\mathbf{S}$,

$$\|\mathbb{S}^{\mathfrak{X}}_{a\cdot b,\mathbf{S}}\| \leqslant 1 \tag{2.1}$$

By increasing N and extending the sequences a and b, we can assume that the points $(1,N)$ and $(N,1)$ belong to $\mathbf{S}$. Clearly, there exists $n < N$ and sequences $1 \leqslant i_1 < i_2 < \ldots < i_n < N$, $1 \leqslant j_1 < j_2 < \ldots < j_n < N$, so that $\mathbf{S}^c = \cup_{s=1}^n [1, i_s] \times$

$[1, j_{n+1-s}]$. Moreover, such sequences (and $n = n(\mathbf{S})$) are uniquely determined by the set $\mathbf{S}$. For notational simplicity, let $i_0 = j_0 = 0$, and $i_{n+1} = j_{n+1} = N$.

Note that $\mathbb{S}_{a'a''\cdot b''b'} = \mathbb{S}_{a'\cdot b'}\mathbb{S}_{a''\cdot b''}$ (the product of sequences a' and a'' is the pointwise one), and that $\mathbb{S}_{a'\cdot b'}$ is a contraction whenever a' and b' belong to the unit ball of ℓ_∞. Therefore, by increasing (some of) the a_i's and b_j's, we can assume that $a_k = a_{i_s+1} = \alpha_s$ when $i_s < k \leqslant i_{s+1}$, and $b_k = b_{j_s+1} = \beta_s$ when $j_s < k \leqslant j_{s+1}$. Furthermore, we can assume that $\alpha_s\beta_{n-s} = 1$ for $0 \leqslant s \leqslant n$. We are going to establish (2.1) under these assumptions, by using induction on $n = n(\mathbf{S})$.

The case of $n = 0$ is trivial: then $\mathbf{S}^c = \emptyset$, hence $|a_1 b_1| \leqslant 1$, and $\mathbb{S}_{a\cdot b}$ is contractive. Next consider the case of $n = 1$. Then $\mathbf{S}^c = [1, i_1] \times [1, j_1]$, $a_k = \alpha_0$ for $i_0 + 1 = 1 \leqslant k \leqslant i_1$, $a_k = \alpha_1$ for $i_1 + 1 \leqslant k \leqslant i_2 = N$, $b_k = \beta_0 = 1/\alpha_1$ for $j_0 + 1 = 1 \leqslant k \leqslant j_1$, and $b_k = \beta_1 = 1/\alpha_0$ for $j_1 + 1 \leqslant k \leqslant j_2 = N$. Suppose $T = \sum_{(i,j)\in\mathbf{S}} t_{ij}E_{ij}$ satisfies $\|T\|_{\mathfrak{X}} \leqslant 1$, and show that $\|\mathbb{S}_{a\cdot b}T\|_{\mathfrak{X}} \leqslant 1$. Indeed, write $T = T_1 + T_2$, where

$$T_1 = \sum_{(i,j)\in[1,i_1]\times[j_1+1,N]\cup[i_1+1,N]\times[1,j_1]} t_{ij}E_{ij} \text{ and } T_2 = \sum_{(i,j)\in[i_1+1,N]\times[j_1+1,N]} t_{ij}E_{ij}.$$

In other words, $T_1 = \mathcal{P}_{J_1,J_2^c}T + \mathcal{P}_{J_1^c,J_2}T$ and $T_2 = \mathcal{P}_{J_1,J_2}T$, where $J_1 = \{k : k > i_1\}$, and $J_2 = \{k : k > j_1\}$. Note that $\mathcal{P}_{J_1^c,J_2^c}T = 0$. Then $\mathbb{S}_{a\cdot b}T = T_1 + \lambda T_2$, with $\lambda = \alpha_1/\alpha_0 \in (0, 1)$. By Lemma 2.1, $\|\mathbb{S}_{a\cdot b}T\|_{\mathfrak{X}} \leqslant 1$.

Now consider the general $n > 1$. As before, we are assuming that $a_k = \alpha_s$ for $i_s < k \leqslant i_{s+1}$, and $b_k = \beta_s = 1/\alpha_{n-s}$ for $j_s < k \leqslant j_{s+1}$. Modify these sequences: let

$$a'_k = \begin{cases} a_k & 1 \leqslant k \leqslant i_n \\ \alpha_{n-1} & i_n + 1 \leqslant k \leqslant N \end{cases}, \quad b'_k = \begin{cases} b_k & j_1 + 1 \leqslant k \leqslant N \\ \beta_1 & 1 \leqslant k \leqslant j_1 \end{cases}.$$

Note that $a'_k = \alpha_{n-1}$ iff $i_{n-1} < k \leqslant i_{n+1} = N$, $b'_k = \beta_1$ iff $0 = i_0 < k \leqslant j_2$, and

$$a'_k b'_\ell \leqslant 1 \text{ iff } (k, \ell) \in \mathbf{S}' = \left(\cup_{s=1}^{n-1} [1, i_s] \times [1, j_{n+1-s}]\right)^c \supset \mathbf{S}.$$

Thus, $n(\mathbf{S}') = n - 1$, hence, by the induction hypothesis, $\mathbb{S}_{a'\cdot b'}$ determines a contraction on $\mathfrak{X}_{\mathbf{S}'}$. Furthermore, for $(k, \ell) \in \mathbf{S}$,

$$\frac{a_k b_\ell}{a'_k b'_\ell} = \begin{cases} 1 & (k, \ell) \notin [i_n + 1, N] \times [j_1 + 1, N] \\ \lambda = \alpha_n/\alpha_{n-1} & (k, \ell) \in [i_n + 1, N] \times [j_1 + 1, N] \end{cases}. \tag{2.2}$$

Suppose $T = (t_{ij})_{(i,j)\in\mathbf{S}}$ satisfies $\|T\|_{\mathfrak{X}} \leqslant 1$, and show that $\|\mathbb{S}_{a\cdot b}T\|_{\mathfrak{X}} \leqslant 1$. Let $T' = \mathbb{S}_{a'\cdot b'}T$ ($T' = (t'_{ij})$, with $t'_{ij} = a'_i b'_j t_{ij}$). By the induction hypothesis, $\|T'\|_{\mathfrak{X}} \leqslant 1$. We have to show that $\|\mathbb{S}_{a\cdot b}T\|_{\mathfrak{X}} \leqslant \|T'\|_{\mathfrak{X}}$. To this end, write $T' = T_1 + T_2$, where

$$T_1 = \sum_{(i,j)\in[1,i_n]\times[j_1+1,N]\cup[i_n+1,N]\times[1,j_1]} t'_{ij}E_{ij} \text{ and } T_2 = \sum_{(i,j)\in[i_n+1,N]\times[j_1+1,N]} t'_{ij}E_{ij}.$$

In other words, $T_1 = \mathcal{P}_{J_1,J_2^c}T + \mathcal{P}_{J_1^c,J_2}T$ and $T_2 = \mathcal{P}_{J_1,J_2}T$, where $J_1 = \{k : k > i_n\}$, and $J_2 = \{k : k > j_1\}$. By (2.2), $\mathbb{S}_{a\cdot b}T = T_1 + \lambda T_2$. Thus, by Lemma 2.1, $\|\mathbb{S}_{a,b}T\|_{\mathfrak{X}} \leqslant \|T'\|_{\mathfrak{X}} \leqslant \|T\|_{\mathfrak{X}}$. This proves (2.1), and therefore, Theorem 1.1. ∎

Corollary 2.2. *Suppose $\mathfrak{X}$ is an unconditional matrix space on $I \times I$, $\mathbf{S}$ is a subset of $I \times I$, and (a_i), (b_j) are sequences of scalars. Then, for any $x \in \mathfrak{X}_{\mathbf{S}}$,*

$$\|\mathbb{S}_{a \cdot b} x\|_{\mathfrak{X}} \geqslant \inf_{(i,j) \in \mathbf{S}} |a_i b_j| \|x\|_{\mathfrak{X}}.$$

Proof. We can assume that the set $\mathbf{S}$ is finite, and $\inf_{(i,j) \in \mathbf{S}} |a_i b_j| = c > 0$. Truncating, we can assume that the operator $\mathbb{S}_{a \cdot b}$ acts on the space of $N \times N$ matrices, for some N, and that $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_N > 0$, $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_N > 0$. Consider the operator $\mathbb{S}_{a' \cdot b'}$, where $a'_i = 1/a_i$, and $b'_j = 1/b_j$. By Theorem 1.1, $\|\mathbb{S}^{\mathfrak{X}}_{a' \cdot b', \mathbf{S}}\| = \sup_{(i,j) \in \mathbf{S}} a'_i b'_j = 1/c$. As $\mathbb{S}_{a' \cdot b'} \mathbb{S}_{a \cdot b} = I$, we are done. ■

3. Proof of Theorem 1.2

Proof. Without loss of generality, assume that $H_1 = H_2 = H$, and the operator A is positive and diagonal. By a small perturbation argument, we can assume that $H = \ell_2^N$ for some $N \geqslant i_n$, and $A\delta_s = a_s \delta_s$ for $1 \leqslant s \leqslant N$ ($(\delta_s)_{s=1}^N$ stands for the canonical basis for ℓ_2^N). Furthermore, assume that $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n > 0$.

(1) By taking $e_k = \delta_{i_k}$, we obtain

$$\|(a_{i_1}, \ldots, a_{i_n})\|_{\mathcal{E}} \leqslant \sup_{(E_k)} \inf_{(e_k)} \|A|_{\mathrm{span}[e_1, \ldots, e_n]}\|_{\mathcal{E}}.$$

To prove, the opposite inequality, suppose $E_1 \hookrightarrow \ldots \hookrightarrow E_n$ and $e_k \in E_k$ are as in the statement of the theorem. Let $F_k = \mathrm{span}[\delta_s : s \geqslant i_k]$. By Theorem III.3.2 and Exercise III.3.3 of [5], we can find an orthonormal system (f_k) s.t. $f_k \in F_k$ for each k, and $\mathrm{span}[e_1, \ldots, e_n] = \mathrm{span}[f_1, \ldots, f_n]$. Denote this space by F. It remains to show that

$$\|A|_F\|_{\mathcal{E}} \leqslant \|(a_{i_1}, \ldots, a_{i_n})\|_{\mathcal{E}} = \|A|_E\|_{\mathcal{E}}, \tag{3.1}$$

where $E = \mathrm{span}[\delta_{i_1}, \ldots, \delta_{i_n}]$. To this end, define an isometry $U : F \to E : f_k \mapsto \delta_{i_k}$, and an operator $T : E \to A(F) : \delta_{i_k} \mapsto A f_k / a_{i_k}$. Then $A|_F = T \circ A|_E \circ U$. By the ideal property of $\| \cdot \|_{\mathcal{E}}$, it suffices to show that $\|T\| \leqslant 1$.

Write $f_k = \sum_{j=i_k}^N \gamma_{kj} \delta_j$. Then $T\delta_{i_k} = \sum_{j=i_k}^N \gamma_{kj} a_j a_{i_k}^{-1} \delta_j$. But $a_j \leqslant a_{i_k}^{-1}$ whenever $\gamma_{kj} \neq 0$, hence, by Theorem 1.1,

$$\|(\gamma_{kj} a_j a_{i_k}^{-1})_{1 \leqslant k \leqslant n, 1 \leqslant j \leqslant N}\| \leqslant \|(\gamma_{kj})_{1 \leqslant k \leqslant n, 1 \leqslant j \leqslant N}\| = 1$$

(the latter is true, since the rows of the matrix (γ_{kj}) are orthonormal vectors). This establishes (3.1), thus completing the proof of the theorem.

(2) is established in a similar fashion, so we only sketch the proof. The inequality

$$\|(a_{i_1}, \ldots, a_{i_n})\|_{\mathcal{E}} \geqslant \inf_{(E_k)} \sup_{(e_k)} \|A|_{\mathrm{span}[e_1, \ldots, e_n]}\|_{\mathcal{E}}$$

is obvious. To establish the converse, consider (E_k) and (e_k) as in the statement of the theorem. Let $F = \mathrm{span}[e_1, \ldots, e_n]$, and $F_k = \mathrm{span}[\delta_s : s \leqslant i_k]$ $(1 \leqslant k \leqslant n)$. As in part (1), we show the existence of an orthonormal system (f_k) s.t. $f_k \in F_k$

for each k, and $F = \mathrm{span}[f_1, \ldots, f_n]$. To show that $\|A|_F\|_{\mathcal{E}} \geqslant \|(a_{i_1}, \ldots, a_{i_n})\|_{\mathcal{E}}$, we proceed as in part (1), except that now we have to show that $T : \delta_{i_k} \mapsto Af_k/a_{i_k}$ has a contractive inverse. To this end, use Corollary 2.2. ∎

Remark 3.1. Alternatively, one can deduce part (1) from Theorem 2.15 of [1].

4. Proof of Theorem 1.3

We can assume that $I = \mathbb{N}$, $a_1 \geqslant a_2 \geqslant \ldots \geqslant 0$, and $b_1 \geqslant b_2 \geqslant \ldots \geqslant 0$. For notational simplicity, we let $\lambda = \widetilde{[ab]}$ – that is, λ_m is the m-th largest element of $(a_i b_j)_{i,j \in \mathbb{N}}$. Denote $\mathbb{S}_{a \cdot b}$ (acting on $\mathfrak{X}$) simply by $\mathbb{S}$.

Recall two little known s-sequences (see [20]): for $T \in B(X, Y)$, define the *n-th Bernstein number* by

$$u_n(T) = \sup_{E \hookrightarrow X, \dim E \geqslant n} \inf_{x \in E, \|x\|=1} \|Tx\|.$$

The *n-th Mitiagin number* $v_n(T)$ is defined as

$$v_n(T) = \sup_{F \hookrightarrow Y, \mathrm{codim}\, F \geqslant n} \rho(q_F T),$$

where $q_F : Y \to Y/F$ is the quotient map, and, for an operator $S : Z \mapsto W$,

$$\rho(S) = \sup\{c \geqslant 0 : c\mathbf{B}_W \subset S(\mathbf{B}_Z)\}$$

($\mathbf{B}_W$ and $\mathbf{B}_Z$ refer to the unit balls of W and Z, respectively).

Lemma 4.1. *In the above notation, suppose $\mathbb{S}$ is acting on an unconditional matrix space $\mathfrak{X}$. Then, for every $m \in \mathbb{N}$, $\lambda_m \leqslant \min\{u_m(\mathbb{S}), v_m(\mathbb{S})\}$.*

Proof. To prove $\lambda_m \leqslant u_m(\mathbb{S})$, let $\mathbf{S} = \{(i,j) : a_i b_j < \lambda_m\}$, and $E = \mathrm{span}[E_{ij} : (i,j) \notin \mathbf{S}]$. Then $\dim E \geqslant m$, and, by Corollary 2.2, $\|\mathbb{S}x\| \geqslant \lambda_m \|x\|$ for any $x \in E$.

The inequality $\lambda_m \leqslant v_m(\mathbb{S})$ follows by duality. Indeed, denote by $\mathcal{P}_N$ the operator of truncation to the top left $N \times N$ corner, and let $\mathfrak{X}_N$ be the range of $\mathcal{P}_N$ in $\mathfrak{X}$. Both $\mathfrak{X}_N$ and its dual $\mathfrak{X}_N^*$ are unconditional matrix spaces. The previous paragraph shows that, for N sufficiently large, $u_m(\mathbb{S}_{a \cdot b}^{\mathfrak{X}} \mathcal{P}_N) \geqslant \lambda_m$, where $\mathbb{S}_{a \cdot b}^{\mathfrak{X}} \mathcal{P}_N$ is viewed as an operator on $\mathfrak{X}_N$. By [20] (Lemma 6.1, Theorem 6.4), $v_m(T) = u_m(T^*)$ for any linear operator T. Therefore,

$$v_m(\mathbb{S}) \geqslant v_m(\mathbb{S}\mathcal{P}_N) = u_m((\mathbb{S}\mathcal{P}_N)^*) = u_m(\mathbb{S}^*\mathcal{P}_N) \geqslant \lambda_m$$

(here, we identify $\mathbb{S}^*\mathcal{P}_N$ with the operator $\mathbb{S}\mathcal{P}_N$, acting on $\mathfrak{X}_N^*$). ∎

Lemma 4.2. *In the above notation, $\lambda_m \geqslant a_m(\mathbb{S})$. If, in addition, $\mathbb{S}$ acts on $B(\ell_2)$ or $\mathcal{S}^p$ $(1 \leqslant p \leqslant \infty)$, then $\lambda_m \geqslant a_m^{(cb)}(\mathbb{S})$.*

Proof. Let $\mathbf{S} = \{(i,j) : a_i b_j \leqslant \lambda_m\}$. Then $|\mathbf{S}^c| < m$, and $\|\mathbb{S}|_\mathbf{S}\| \leqslant \lambda_m$. By scaling, we can assume that $\lambda_n = 1$. As in the proof of Theorem 1.1, we can assume that there exist $n \in \mathbb{N}$, $0 = i_0 < i_1 < \ldots < i_n < i_{n+1} = \infty$, and $0 = j_0 < j_1 < \ldots < j_n < j_{n+1} = \infty$, such that $a_k = a_{i_s+1} = \alpha_s$ when $i_s < k \leqslant i_{s+1}$, $b_k = b_{j_s+1} = \beta_s$ when $j_s < k \leqslant j_{s+1}$. Furthermore, we can assume that $\alpha_s \beta_{n-s} = 1$ for $0 \leqslant s \leqslant n$, and $\alpha_0 > \alpha_1 > \ldots > \alpha_n$.

For $1 \leqslant s \leqslant n$, let $\gamma_s = \alpha_s / \alpha_{s-1}$, and consider the sequences

$$c^{(s)} = (-1)^{\chi_{\{1,\ldots,i_s\}}} \ \text{ and } \ d^{(s)} = (-1)^{\chi_{\{j_{n-s+1}+1, j_{n-s+1}+2, \ldots\}}}.$$

In other words,

$$c_k^{(s)} = \begin{cases} -1 & k \leqslant i_s \\ 1 & k > i_s \end{cases} \ \text{ and } \ d_\ell^{(s)} = \begin{cases} 1 & \ell \leqslant j_{n-s+1} \\ -1 & \ell > j_{n-s+1} \end{cases} .$$

Define the Schur multipliers

$$\mathbb{S}_{\phi^{(s)}} = \frac{1+\gamma_s}{2} + \frac{1-\gamma_s}{2} \mathbb{S}_{c^{(s)} \bullet d^{(s)}}. \tag{4.1}$$

Note that the symbol $\phi^{(s)}$ of this multiplier satisfies

$$\phi_{k\ell}^{(s)} = \begin{cases} 1 & (k,\ell) \in [1,i_s] \times (j_{n-s+1}, \infty) \cup (i_s, \infty) \times [1, j_{n-s+1}] \\ \gamma_s & (k,\ell) \in [1,i_s] \times [1, j_{n-s+1}] \cup (i_s, \infty) \times (j_{n-s+1}, \infty) \end{cases} . \tag{4.2}$$

Let $\phi = \phi^{(1)} \diamond \ldots \diamond \phi^{(n)}$. By (4.1) $\mathbb{S}_\phi = \mathbb{S}_{\phi^{(1)}} \ldots \mathbb{S}_{\phi^{(n)}}$ is a contraction.

Next we show that, for $(k,\ell) \in \mathbf{S}$,

$$\phi_{k\ell} = a_k b_\ell \tag{4.3}$$

To this end, fix (k,ℓ), and find $p, q \in \{0, \ldots, n\}$ s.t. $p + q \geqslant n$, $i_p < k \leqslant i_{p+1}$, and $j_q < j \leqslant j_{q+1}$. Then $a_k b_\ell = \alpha_p \beta_q$.

First suppose $p + q = n$ ($q = n - p$). Then, for every s,

$$(k,\ell) \in (i_p, i_{p+1}] \times (j_{n-p}, j_{n-p+1}] \subset [1, i_s] \times (j_{n-s+1}, \infty) \cup (i_s, \infty) \times [1, j_{n-s+1}]$$

(to see this, consider the cases of $s \leqslant p+1$ and $s \geqslant p$ separately). In this case, $a_k b_\ell = 1$, and, by (4.2), $\phi_{k\ell}^{(s)} = 1$ for every s, hence $\phi_{k\ell} = 1$.

Now suppose $q > n - p$. As $\gamma_s = \alpha_{s-1}/\alpha_s$, we have

$$a_k b_\ell = \alpha_p \beta_q = \frac{\alpha_p}{\alpha_{n-q}} = \prod_{s=n-q+1}^{p} \gamma_s.$$

On the other hand, by (4.2), $\phi_{k\ell}^{(s)} = \gamma_s$ iff $n - q + 1 \leqslant s \leqslant p$, hence $\phi_{k\ell} = \gamma_{n-q+1} \ldots \gamma_p$, thus establishing (4.3).

Now let $\psi = a \bullet b - \phi$. Then ψ has fewer than m non-zero entries, and $\mathbb{S}_{a \bullet b} - \mathbb{S}_\psi = \mathbb{S}_\phi$. Therefore,

$$a_m(\mathbb{S}_{a \bullet b}) \leqslant \|\mathbb{S}_\psi\| = 1 = \lambda_m,$$

as desired.

It remains to estimate $\|\mathbb{S}_\phi\|_{cb}$ when $\mathfrak{X}$ is $B(\ell_2)$ or $\mathcal{S}^p$, for $1 \leqslant p \leqslant \infty$. It is well known that the norm and c.b. norm of any Schur multiplier on $B(\ell_2)$ coincide. By [15], $\mathbb{S}_\phi$ maps $\mathcal{S}^\infty$ into itself, hence, by "parallel duality," it maps $\mathcal{S}^1$ into itself, with

$$\|\mathbb{S}_\phi\|_{CB(\mathcal{S}^1)} = \|\mathbb{S}_\phi\|_{CB(\mathcal{S}^\infty)} = \|\mathbb{S}_\phi\|_{B(B(\ell_2))} = 1.$$

Complex interpolation yields $\|\mathbb{S}_\phi\|_{CB(\mathcal{S}^p)} \leqslant 1$ for $1 < p < \infty$. ■

Remark 4.3. If $\mathbb{S}_{a \cdot b}$ acts on $\mathfrak{X} = B(\ell_2)$, one can prove the above lemma by using the version of Stinespring's Extension Theorem for restricted Schur multipliers, developed in [19]. However, the above proof has the advantage of being constructive, and of working for arbitrary unconditional matrix spaces.

Proof of Theorem 1.3. Recall that, for any operator T, $a_n(T) \geqslant \max\{c_n(T), d_n(T)\}$, and similar inequality holds for the c.b. versions of these s-numbers. On the other hand, by [20], $u_n(T) \leqslant c_n(T)$ and $v_n(T) \leqslant d_n(T)$. Thus, Lemmas 4.2 and 4.1 provides the upper, respectively lower, estimates, for the s-numbers in question. ■

We have not been able to obtain sharp individual estimates for other s-sequences (see the next section for "collective" estimates). We did, however, obtain some results on the sequence of *isomorphism numbers* i_n. Recall that, for $T \in B(X, Y)$, we set $i_n(T) = 0$ if $\operatorname{rank} T < n$. Otherwise, we set $i_n(T) = \sup\{\|A\|^{-1}\|B\|^{-1}\}$, where the supremum is taken over all the spaces G of dimension at least n, and all operators $A : G \to X$ and $B : Y \to G$ satisfying $BTA = I_G$.

Proposition 4.4. *Suppose $\mathfrak{X}$ is an unconditional matrix space. Consider an operator $\mathbb{S} = \mathbb{S}_{a \cdot b}$ (with $a = \operatorname{diag}(a_i)$, $b = \operatorname{diag}(b_j)$, $a_1 \geqslant a_2 \geqslant \ldots > 0$, $b_1 \geqslant b_2 \geqslant \ldots > 0$) as acting on $\mathfrak{X}$. Let $\lambda_m = \widetilde{[ab]}_m$ $(m \in \mathbb{N})$.*

1. *There exists universal constant c_1 and c_2 such that, for any $n \in \mathbb{N}$, $i_n(\mathbb{S}) \geqslant c_1 \max\{\lambda_{\lceil c_2 n(\log n+1)\rceil}, \lambda_n/(\log n + 1)\}$.*
2. *Moreover, if $\mathfrak{X} = \mathcal{S}^{\mathcal{E}}$, where the symmetric sequence space $\mathcal{E}$ has non-trivial Boyd indices, then $i_n(\mathbb{S}) \geqslant c\lambda_n$ (c is a constant, depending on $\mathcal{E}$).*

Proof. We first present (somewhat sketchily) the proof of part (1). As before, let $\mathbf{S} = \{(i, j) : a_i b_j \geqslant \lambda_n\}$, and $E = \operatorname{span}[E_{ij} : (i, j) \notin \mathbf{S}]$. If the set $\mathbf{S}$ is infinite, then there exist $N, M \in \mathbb{N}$ s.t. NM is arbitrarily large, and $[1, N] \times [1, M] \subset \mathbf{S}$. Since the truncation projection onto the top left $N \times M$ corner is contractive, Corollary 2.2, we conclude that $i_n(\mathbb{S}) \geqslant \lambda_n$ (this is witnessed by $G = \operatorname{span}[E_{k\ell} : 1 \leqslant k \leqslant M, 1 \leqslant \ell \leqslant N]$).

Now suppose $\mathbf{S}$ is finite. For $i \in \mathbb{N}$, denote by $\tilde{j}_i$ the largest value of j for which $(i, j) \in \mathbf{S}$. If there are no such j's, set $\tilde{j}_i = 0$. Then $\sum_{i=1}^\infty \tilde{j}_i \geqslant n$. We shall show that there exists i s.t. $i\tilde{j}_i \geqslant \alpha n/(\log n + 1)$, where α is a constant. Indeed, otherwise we would have

$$n \leqslant \sum_i \tilde{j}_i \leqslant \frac{\alpha n}{\log n + 1} \sum_{i=1}^{\lfloor \alpha n/(\log n+1) \rfloor} \frac{1}{i},$$

and the right hand side is less than n for sufficiently small values of α. Estimating $i_{\lfloor \alpha n/(\log n+1) \rfloor}(\mathbb{S})$ using the space $G = \mathrm{span}[E_{k\ell} : 1 \leqslant k \leqslant i,\ 1 \leqslant \ell \leqslant \tilde{j}_i]$, we show that

$$i_{\lfloor \alpha n/(\log n+1) \rfloor}(\mathbb{S}) \geqslant a_i b_{\tilde{j}_i} \geqslant \lambda_n.$$

Therefore, $i_n(\mathbb{S}) \geqslant c\lambda_{\lceil c_2 n(\log n+1) \rceil}$, for some c.

Next let $\mathbf{S}' = \{(i,j) : i \leqslant n,\ a_i b_j \geqslant \lambda_n\}$. This set has at least n elements, and can be written as $\cup_{k=1}^{\ell} [1, i_k] \times [1, j_k]$, with $\ell \leqslant n$. Then the "natural" truncation operator onto $\mathbf{S}'$ has the norm not exceeding $C(\log n + 1)$, where C is an absolute constant [16]. Factoring the identity on $G = \mathrm{span}[E_{k\ell} : (k,\ell) \in \mathbf{S}']$, we see that $i_n(\mathbb{S}) \geqslant \lambda_n/(C(\log n + 1))$.

To deal with part (2), suppose $\mathcal{E}$ has non-trivial Boyd indices. Then, by Section 4 of [3] (or Theorem 3.3 of [9]), the "natural" truncation onto $\mathbf{S}'$ (viewed as an operator on $\mathcal{S}^{\mathcal{E}}$) has the norm not exceeding some constant, depending only on $\mathcal{E}$. ■

5. Membership in operator ideals

In this section, we consider the membership of elementary operators in operator ideals. We denote by $M_{A,B}$ the multiplication operator: if X_0, X, Y, Y_0 are Banach spaces, $A \in B(X_0, X)$, and $B \in B(Y, Y_0)$, we define $M_{A,B} : B(X,Y) \to B(X_0, Y_0) : T \mapsto BTA$. Throughout this section, we assume that A and B are non-zero.

If $X_0 = X = Y = Y_0 = \ell_2$, and $A = \mathrm{diag}\,(a)$, and $B = \mathrm{diag}\,(b)$, then $M_{A,B} = \mathbb{S}_{a\cdot b}$. If $\mathcal{E}$ is a symmetric sequence space, and $A, B \in B(\ell_2)$, then $M_{A,B}^{\mathcal{E}}$ denotes the corresponding map from $\mathcal{S}^{\mathcal{E}}$ into itself.

In this section, we determine whether $M_{A,B}^{\mathcal{E}}$ belongs to a given operator ideal (see e.g. [22] for the terminology and the main results).

Proposition 5.1. *Suppose A and B are operators between Hilbert spaces, and $2 < p < \infty$. Suppose, furthermore, that $\mathcal{E}$ is a symmetric sequence space, such that the formal identity $\ell_2 \to \mathcal{E}$ is contractive. Then $\|A\|_p \|B\|_p \leqslant \pi_{p,2}(M_{A,B}^{\mathcal{E}}) \leqslant C_p \|A\|_p \|B\|_p$ (C_p is an absolute constant).*

Denote by $x_n(T)$ the sequence of Weyl numbers of an operator $T \in B(X,Y)$, defined by

$$x_n(T) = \sup\{a_n(Tu) : u \in B(\ell_2, X),\ \|u\| \leqslant 1\}.$$

The quasi-Banach operator ideal $\mathfrak{L}_p^{(x)}$ consists of all operators T, for which $\sum_n x_n(T)^p$ is finite, equipped with then norm $\|T\|_{\mathfrak{L}_p^{(x)}} = (\sum_n x_n(T)^p)^{1/p}$. See Section 2.4 of [23] for more information.

Suppose A and B are operators between Hilbert spaces, and consider the operator $M_{A,B}$ and its Weyl numbers. Note that, if either A or B is non-compact (and both are non-zero), $M_{A,B}$ preserves a copy of ℓ_2, hence $\limsup x_n(M_{A,B}) > 0$.

Proposition 5.2. *Suppose A and B are compact operators on a Hilbert space, $\mathcal{E}$ is a symmetric sequence space, and $1 \leqslant p < \infty$. Then*

$$c\|A\|_p\|B\|_p \leqslant \Big(\sum_n x_n(M^{\mathcal{E}}_{A,B})^p\Big)^{1/p} \leqslant \|A\|_p\|B\|_p$$

(c is an absolute constant).

Proof. Suppose $T : X \to X$ is a compact operator with eigenvalues $(\lambda_i(T))$, enumerated in such a way that $|\lambda_1(T)| \geqslant |\lambda_2(T)| \geqslant \ldots$. A Banach space version of the classical Weyl inequality (see e.g. [21]) shows that there exists an absolute constant κ such that, for any $n \in \mathbb{N}$,

$$\prod_{k=1}^n |\lambda_k(T)| \leqslant \kappa^n \prod_{k=1}^n \dot{x}_k(T), \text{ where } \dot{x}_k(T) = x_{\lfloor (k+1)/2 \rfloor}(T).$$

A simple domination argument (cf. Chapter II of [5]) shows that $\|(|\lambda_k(T)|)\|_{\mathcal{F}} \leqslant \kappa\|(\dot{x}_k(T))\|_{\mathcal{F}} \leqslant 2\kappa\|(x_k(T))\|_{\mathcal{F}}$ for any symmetric sequence space $\mathcal{F}$.

In our situation, we may assume that $A = \mathrm{diag}\,(a_i)$ and $B = \mathrm{diag}\,(b_j)$. Then the eigenvalues of $M^{\mathcal{E}}_{A,B}$ are precisely the numbers $a_i b_j$. As $\sum_{i,j} |a_i b_j|^p = \sum_i |a_i|^p \sum_j |b_j|^p$, we are done. This proves the left hand side of the lemma. To deal with the right hand side, observe that

$$\sum_n x_n(M^{\mathcal{E}}_{A,B})^p \leqslant \sum_n a_n(M^{\mathcal{E}}_{A,B})^p = \sum_{i,j} a_i^p b_j^p = \|A\|_p\|B\|_p,$$

since, by Theorem 1.3, the sequence $(a_n(M^{\mathcal{E}}_{A,B}))$ is a rearrangement of the double sequence $(a_i b_j)$. ∎

Remark 5.3. In the same way, we can prove that, when $\mathbb{S}_{a\cdot b}$ acts on an unconditional matrix space $\mathfrak{X}$, we have

$$c\|a\|_p\|b\|_p \leqslant \Big(\sum_n x_n(M^{\mathcal{E}}_{A,B})^p\Big)^{1/p} \leqslant \|a\|_p\|b\|_p$$

(c is an absolute constant). Here, $\|\cdot\|_p$ refers to the ℓ_p norm of a sequence.

Proof of Proposition 5.1. Note that, if either A or B is not compact, then $M^{\mathcal{E}}_{A,B}$ preserves a copy of ℓ_2, hence it doesn't belong to $\Pi_{p,2}$. Thus, we can assume that $A = \mathrm{diag}\,(a)$ and $B = \mathrm{diag}\,(b)$, with $a = (a_i), b = (b_j) \in c_0$ have non-negative entries.

First show that $\|A\|_p\|B\|_p \leqslant \pi_{p,2}(M^{\mathcal{E}}_{A,B})$. By definition,

$$\Big(\sum_{i,j} \|M^{\mathcal{E}}_{A,B}E_{ij}\|^p\Big)^{1/p} \leqslant \pi_{p,2}(M^{\mathcal{E}}_{A,B}) \sup_{T \in \mathcal{S}^{\mathcal{E}^*}, \|T\|_{\mathcal{E}^*} \leqslant 1} \Big(\sum_{i,j} |\langle T, E_{ij}\rangle|^2\Big)^{1/2}. \tag{5.1}$$

But

$$\sum_{i,j} \|M^{\mathcal{E}}_{A,B} E_{ij}\|^p = \sum_{i,j} |a_i|^p |b_j|^p = \|A\|_p^p \|B\|_p^p.$$

On the other hand, $\langle T, E_{ij} \rangle = \mathrm{tr}(T\,{}^tE_{ij}) = T_{ij}$, and

$$\sum_{i,j} |\langle T, E_{ij} \rangle|^2 = \sum_{i,j} |T_{ij}|^2 = \|T\|_2^2 \leqslant \|T\|_{\mathcal{E}^*}^2 \leqslant 1.$$

By (5.1), $\|A\|_p \|B\|_p \leqslant \pi_{p,2}(M^{\mathcal{E}}_{A,B})$.

To prove the opposite inequality, recall that, by Theorem 17 of [21], for $2 < p < \infty$ there exists a constant γ_p s.t. $\pi_{p,2}(T) \leqslant \gamma_p \big(\sum_n x_n(T)^p \big)^{1/p}$ for every operator T. Applying Proposition 5.2, we complete the proof. ■

Remark 5.4. Some results related to those from Sections 4 and 5 need to be mentioned. By Proposition 3.3 of [6], for $1 \leqslant p < \infty$, $A \in B(X_1, X)$, and $B \in B(Y, Y_1)$, we have $\pi_p(M_{A,B}) = \pi_p(A^*)\pi_p(B)$ ($M_{A,B}$ is viewed as a map from $B(X, Y)$ to $B(X_1, Y_1)$). If X, X_1, Y, and Y_1 are Hilbert spaces, then $\pi_p(A^*) \sim \|A\|_2$ and $\pi_p(B) \sim \|B\|_2$, with the equality for $p = 2$ (see e.g. [22]). In this case, $\pi_p(M_{A,B}) \sim \|A\|_2 \|B\|_2$, with equality when $p = 2$. The ideal norms and s-numbers of formal identity maps between non-commutative sequence spaces have been studied by A. Defant and his co-authors, see e.g. [8] and references therein. Finally, some estimates on the s-numbers of multiplication operators (and, more generally, of elementary operators) on C^*-algebras have recently been obtained in [2].

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