

REPRESENTATIONS OF BANACH ALGEBRAS AS ALGEBRAS OF COMPLETELY BOUNDED MAPS

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ABSTRACT. We use the notion of Π_2 -hyperreflexivity to construct, for a wide variety of Banach algebras \mathcal{A} , an operator space X and a representation $\pi : \mathcal{A} \rightarrow CB(X)$, such that $CB(X)$ consists of 2-summing perturbations of $\pi(\mathcal{A})$. This gives rise to some examples of operator spaces with interesting properties.

1. INTRODUCTION

In this paper, we try to answer two related questions.

- (1) Suppose a unital Banach algebra \mathcal{A} , and a unital contractive faithful representation $\pi : \mathcal{A} \rightarrow B(E)$ (E is a Banach space), are given. Can we equip E with an operator space structure X , such that $CB(X) = \pi(\mathcal{A}) + \Pi_2(X)$, where $\Pi_2(X)$ denotes the set of 2-summing operators on X ?
- (2) Suppose a unital Banach algebra \mathcal{A} is given. Can we construct an operator space X , and a unital contractive faithful representation $\pi : \mathcal{A} \rightarrow CB(X)$, such that $CB(X) = \pi(\mathcal{A}) + \Pi_2(X)$?

Here, Π_2 is the ideal of 2-summing operators (see Section 2 for its definition and basic properties), and $CB(X)$ denotes the set of completely bounded maps on an operator space X (see e.g. [13, 27, 31] for more information). For certain Banach algebras \mathcal{A} (and certain representations π), we give a positive answer to the above questions in Section 3 (Theorem 3.1, Corollary 3.3). In Section 4, we construct further examples of Banach algebras \mathcal{A} , representations π , and operator spaces X as above. This gives rise to operator spaces with unusual and interesting properties (Theorems 4.3, 4.6). The proofs require an investigation into hyperreflexivity of spaces and sets with respect to the ideal Π_2 . In Section 2, we review hyperreflexivity with respect to operator ideals, and prove some preliminary results.

In a slightly different context, the problems of representing Banach algebras in a “nice” way has recently been considered in [8, 9, 20, 36]. We briefly describe these results in Section 3 (Remark 3.7).

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2. PRELIMINARY RESULTS ON HYPERREFLEXIVITY

In this paper, we use the notion of hyperreflexivity with respect to an operator ideal, introduced in [23]. Suppose \mathfrak{B} is a maximal Banach operator ideal. That is, for any pair X, Y of Banach spaces, $\mathfrak{B}(X, Y)$ is a subset of $B(X, Y)$, equipped with the norm $\beta(\cdot)$ in such a way that $(\mathfrak{B}(X, Y), \beta)$ is a Banach space. By convention, we set $\beta(T) = \infty$ if $T \in B(X, Y) \setminus \mathfrak{B}(X, Y)$. The ideal property means that, for any Banach spaces X_0, X, Y, Y_0 , and every $T \in \mathfrak{B}(X, Y)$, $T_X \in B(X_0, X)$, and $T_Y \in B(Y, Y_0)$, we have $\beta(T_Y T T_X) \leq \|T_Y\| \beta(T) \|T_X\|$. The ideal \mathfrak{B} is maximal if, for every $T \in B(X, Y)$, $\beta(T) = \sup \beta(qTi)$, where $i : E \rightarrow X$ is an injection, $q : Y \rightarrow F$ is a quotient, and the spaces E and F are finite dimensional. For further information about operator ideals, see [10, 11, 29, 35].

In this paper, we concentrate on the ideal Π_2 of 2-summing maps. Recall that the 2-summing norm of $T \in B(X, Y)$ (denoted by $\pi_2(T)$) is defined as the smallest positive constant c such that, for every $x_1, \dots, x_n \in X$,

$$\left(\sum_i \|Tx_i\|^2 \right)^{1/2} \leq c \sup_{x^* \in X^*, \|x^*\| \leq 1} \left(\sum_i |x^*(x_i)|^2 \right)^{1/2}.$$

The ideal of 2-summing operators is denoted by Π_2 . If X and Y are Hilbert spaces, then it is well-known that $\Pi_2(X, Y) = \mathcal{S}_2(X, Y)$, and $\pi_2(T) = \|T\|_2$. Here, \mathcal{S}_2 and $\|\cdot\|_2$ denote the space of Hilbert-Schmidt maps, and the Hilbert-Schmidt norm, respectively.

Suppose \mathcal{A} is a non-empty convex balanced subset of $B(X, Y)$, and \mathfrak{B} is an operator ideal. For $T \in B(X, Y)$ denote by $d_{\mathcal{A}, \mathfrak{B}}(T)$ the infimum of all $\lambda > 0$ with the property that, for any $\gamma \geq 1$, $\beta(uTv) \leq \lambda\gamma$ whenever the contractions $v : E \rightarrow X$ and $u : Y \rightarrow F$ (E and F are finite dimensional) satisfy $\beta(uav) \leq \gamma$ for every $a \in \mathcal{A}$. Equivalently, $d_{\mathcal{A}, \mathfrak{B}}(T) > \lambda$ iff there exists $\gamma \geq 1$ and contractions u, v s.t. $\beta(uTv) > \lambda\gamma$, yet $\beta(uav) \leq \gamma \forall a \in \mathcal{A}$.

Observe that, for any $\lambda \in \mathbb{C}$, $a \in \mathcal{A}$, and $S \in \mathfrak{B}(E, F)$, we have $\beta(u(\lambda a + S)v) \leq |\lambda|\gamma + \beta(S)$ for any pair of contractions u, v , with the property that $\beta(uav) \leq \gamma$ for every $a \in \mathcal{A}$. Thus,

$$(2.1) \quad d_{\mathcal{A}, \mathfrak{B}}(\lambda a + S) \leq |\lambda| + \beta(S).$$

The definition of Azoff-Shehada hyperreflexivity given below arose from an attempt to reverse this implication: namely, to conclude that any $T \in B(E, F)$ with finite $d_{\mathcal{A}, \mathfrak{B}}(T)$ belongs to $\mathbb{C}\mathcal{A} + \mathfrak{B}(E, F)$.

For $C_1, C_2 > 0$, we say that \mathcal{A} is $(C_1, C_2) - \mathfrak{B}$ -Azoff-Shehada hyperreflexive ($(C_1, C_2) - \mathfrak{B}$ -ASHR, for short) if, for any $T \in B(X, Y)$ with $d_{\mathcal{A}, \mathfrak{B}}(T)$ finite, and any $\varepsilon > 0$, we can write $T = \lambda a + S$, with $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$, $S \in \mathfrak{B}(X, Y)$, $|\lambda| < C_1 d_{\mathcal{A}, \mathfrak{B}}(T) + \varepsilon$, and $\beta(S) < C_2 d_{\mathcal{A}, \mathfrak{B}}(T) + \varepsilon$. \mathcal{A} is said to be \mathfrak{B} -ASHR if it $(C_1, C_2) - \mathfrak{B}$ -ASHR for some C_1 and C_2 .

Throughout the paper, we work with maximal Banach ideals. In this case, the condition that E and F are finite dimensional is redundant.

Now suppose $\mathcal{A} \hookrightarrow B(X, Y)$ (throughout the paper, we use the notation “ $Z_1 \hookrightarrow Z_2$ ” to mean “ Z_1 is a closed linear subspace of a Banach space Z_2 ”). It is easy to see that, if \mathcal{A} is $(C', C) - \mathfrak{B}$ -ASHR, then it is $(c, C) - \mathfrak{B}$ -ASHR for any $c > 0$ (indeed, in the definition above, $|\lambda|$ can be made arbitrarily small). In this case, we say that \mathcal{A} is $C - \mathfrak{B}$ -hyperreflexive. \mathcal{A} is called \mathfrak{B} -hyperreflexive if it is $C - \mathfrak{B}$ -hyperreflexive for some C . It is observed in [23] that \mathcal{A} is $C - \mathfrak{B}$ -hyperreflexive iff $\inf_{a \in \mathcal{A}} \beta(T - a) \leq Cd_{\mathcal{A}, \mathfrak{B}}(T)$, and $d_{\mathcal{A}, \mathfrak{B}}(T) = \sup \beta(uTv)$, with the sup taken over all finite rank contractions u and v with $u\mathcal{A}v = 0$. Note that, in this case, $d_{\mathcal{A}, \mathfrak{B}}(T) = \sup \beta(qTi)$, where the supremum runs over all injections $i : E \rightarrow X$ and quotient maps $q : Y \rightarrow F$, for which E and F are finite dimensional, and $q\mathcal{A}i = 0$.

The definitions presented above were introduced in [23] (although the definition of the Azoff-Shehada hyperreflexivity given there is slightly different). Numerous examples of sets which are, and are not, ASHR, can be found there. The same paper explains the connection with the classical notion of hyperreflexivity (see e.g. [6, 7, 19] for more information).

The following proposition (essentially proved in [23]) illustrates the connection between the connection between the hyperreflexivity of a space, and the ASHR of its unit ball.

Proposition 2.1. *Suppose (\mathfrak{B}, β) is a maximal ideal, and let \mathcal{A} be a $C - \mathfrak{B}$ -hyperreflexive subspace of $B(X, Y)$. Then $\text{Ba}(\mathcal{A})$ is $(C + 1, C) - \mathfrak{B}$ -Azoff-Shehada hyperreflexive.*

As in the classical theory of hyperreflexivity, we can use ampliations to create hyperreflexive spaces (or ASHR sets). To this end, consider a symmetric sequence space \mathcal{E} (that is, a space with a 1-symmetric basis). Denote by \mathcal{E}_n ($n \in \mathbb{N}$) the span of n elements of the canonical basis of \mathcal{E} . For $T \in B(E, F)$ and $n \in \mathbb{N} \cup \{\infty\}$, the n -th ampliation of T (on \mathcal{E}) is defined as $T^{(n, \mathcal{E})} = I_{\mathcal{E}} \otimes T \in B(\mathcal{E}(E), \mathcal{E}(F))$. For $\mathcal{A} \subset B(E, F)$, set $\mathcal{A}^{(n, \mathcal{E})} = \{T^{(n, \mathcal{E})} \mid T \in \mathcal{A}\}$. When there is no possibility of confusion, we will omit \mathcal{E} , and simply write $T^{(n)}$ and $\mathcal{A}^{(n)}$.

Theorem 2.2. *Suppose \mathcal{A} is a convex balanced subset subset of the unit ball of $B(E, F)$ (E and F are Banach spaces), closed in the topology induced by $E \widehat{\otimes} F^*$. Suppose, furthermore, that (\mathfrak{B}, β) is a maximal Banach operator ideal, and $\lim_n \beta(I_{\mathcal{E}_n}) = \infty$, where \mathcal{E} is ℓ_p ($1 \leq p < \infty$) or c_0 . Then $\mathcal{A}^{(\infty, \mathcal{E})}$ is a $(5, 25) - \mathfrak{B}$ -ASHR subset of $B(\mathcal{E}(E), \mathcal{E}(F))$.*

The lemmas below lead to the proof of Theorem 2.2. For the sake of brevity, introduce some notation. For \mathcal{E} as above, $\tilde{E} = \mathcal{E}(E)$ and $\tilde{F} = \mathcal{E}(F)$. For $\mathcal{F} \subset \mathbb{N}$,

define the contractive projection $P_{\mathcal{F}} \in B(\mathcal{E})$ by setting

$$P_{\mathcal{F}}\delta_i = \begin{cases} \delta_i & i \in \mathcal{F} \\ 0 & i \notin \mathcal{F} \end{cases},$$

where (δ_i) is the canonical basis for \mathcal{E} . Abusing the notation somewhat, we use $P_{\mathcal{F}}$ for $P_{\mathcal{F}} \otimes I_E \in B(\tilde{E})$ and $P_{\mathcal{F}} \otimes I_F \in B(\tilde{F})$. We denote by Q_n the projection $P_{\{1, \dots, n\}}$ on the first n elements of the basis of \mathcal{E} (with the conventions described above). An operator $T \in B(\tilde{E}, \tilde{F})$ is called *diagonal* if $T = \sum_i P_{\{i\}} T P_{\{i\}}$ (convergence in the strong operator topology). We also write $T = (\text{diag}(P_{\{i\}} T P_{\{i\}}))$.

Lemma 2.3. *Suppose \mathcal{E} is an unconditional sequence space, \mathfrak{B} is a maximal operator ideal, E and F are Banach spaces, and $\mathcal{A} \subset B(E, F)$ is closed in the topology induced by $E \hat{\otimes} F^*$. Suppose, furthermore, that $T' \in B(\tilde{E}, \tilde{F})$ satisfies $d_{\mathcal{A}(\infty), \mathfrak{B}}(T') < 1$. Then there exists a diagonal operator T s.t. $\beta(T - T') \leq 4$. Consequently, $d_{\mathcal{A}(\infty), \mathfrak{B}}(T) < 5$.*

Proof. Denote by T the diagonal part of T' , that is $T = \sum_n P_{\{n\}} T' P_{\{n\}}$ (convergence in the strong operator topology). Let $T'' = T' - T$. For any $n \in \mathbb{N}$,

$$Q_n T'' Q_n = 2^{2-n} \sum_{\mathcal{F} \subset \{1, \dots, n\}} (Q_n - P_{\mathcal{F}}) T' P_{\mathcal{F}}.$$

However, $(Q_n - P_{\mathcal{F}})B(E, F)^{(\infty)}P_{\mathcal{F}} = 0$, hence $\beta((Q_n - P_{\mathcal{F}})T'P_{\mathcal{F}}) < 1$. Therefore, $\beta(Q_n T'' Q_n) < 4$ for every n .

By the maximality of \mathfrak{B} , $\beta(T'') \leq 4$. Indeed, otherwise there exists a finite rank contraction u s.t. $\beta(T''u) > 4$. However, $\lim_n (Q_n u - u) = 0$ (this follows from the fact that, for any finite rank map S , $\beta(S) \leq \nu_1(S)$, where $\nu_1(\cdot)$ denotes the nuclear norm). Therefore, we can assume that $Q_m u = u$ (m is sufficiently large). In a similar fashion, we find $n \geq m$ s.t. $\beta(Q_n T'' Q_m u) > 4$. Therefore, $\beta(Q_n T'' Q_n) > 4$, which contradicts the results of the previous paragraph.

By (2.1), $d_{\mathcal{A}(\infty), \mathfrak{B}}(T) < 5$. ■

Lemma 2.4. *Suppose \mathcal{E} is a symmetric sequence space, \mathfrak{B} , E , F , and \mathcal{A} are as in the previous lemma. Suppose, furthermore, $T = \text{diag}(T_i) \in B(\tilde{E}, \tilde{F})$ is a diagonal operator, and $d_{\mathcal{A}(\infty), \mathfrak{B}}(T)$ is finite. Then the sequence (T_i) converges in norm.*

Proof. Without loss of generality, $d_{\mathcal{A}(\infty), \mathfrak{B}}(T) < 1$. We show that there exists $T_0 \in B(E, F)$ s.t. $\lim_i \|T_i - T_0\| = 0$. Indeed, suppose otherwise. Then there exist $c > 0$ a sequence $n_1 < m_1 < n_2 < m_2 < \dots$, and a sequence of norm-1 vectors $e_i \in E$, s.t. $\|(T_{n_i} - T_{m_i})e_i\| > c$ for every i . Consequently, we can find a sequence of norm-1 vectors $f_i^* \in F^*$, s.t. $\langle (T_{n_i} - T_{m_i})e_i, f_i^* \rangle > c$.

Fix $N \in \mathbb{N}$ s.t. $\beta(I_{\mathcal{E}_N}) > 4/c$. Denote by $(\delta_i)_{i=1}^{\infty}$ the canonical basis of \mathcal{E} . Consider the contractions $u : \mathcal{E}_N = \text{span}[\delta_1, \dots, \delta_N] \rightarrow \tilde{E}$ and $v : \tilde{F} \rightarrow \mathcal{E}_N$, defined by

$$u\delta_i = e_i \otimes (\delta_{n_i} + \delta_{m_i})/2, \quad v\left(\sum_j f_j \otimes \delta_j\right) = \sum_{i=1}^N \frac{1}{2} \langle f_{n_i} - f_{m_i}, f_i^* \rangle \delta_i.$$

Then $v\mathcal{A}^{(\infty)}u = 0$, and $vTu\delta_i = c_i\delta_i$, where, for each i , $c_i \geq c/4$. Therefore,

$$1 > \beta(vTu) \geq \frac{c}{4}\beta(I_{\mathcal{E}_N}) > 1,$$

a contradiction. ■

Consider $T = \text{diag}(T_i) \in B(\tilde{E}, \tilde{F})$, for which $d_{\mathcal{A}^{(\infty)}, \mathfrak{B}}(T)$ is finite. By the previous result, $T_0 = \lim T_i$ exists. For brevity, write $\tilde{T} = T_0^{(\infty)}$.

Lemma 2.5. *Suppose $E, F, \mathcal{A}, \mathcal{E}$, and \mathfrak{B} are as above, and the diagonal operator $T \in B(\tilde{E}, \tilde{F})$ satisfies $d_{\mathcal{A}^{(\infty)}, \mathfrak{B}}(T) < 1$. Then $\beta(T - \tilde{T}) \leq 4$, and $d_{\mathcal{A}^{(\infty)}, \mathfrak{B}}(\tilde{T}) < 1$.*

Proof. Suppose, for the sake of contradiction, $\beta(T - \tilde{T}) > 4$. Then $\beta(Q_n(T - \tilde{T})Q_n) > 4$ for some n . Find a finite rank contraction $u_0 : E_0 \rightarrow \text{ran } Q_n$ s.t. $\beta(Q_n(T - \tilde{T})Q_n u_0) > 4$. Fix $\varepsilon > 0$, and find $N > n$ so large that $\|T_i - T_0\| < \varepsilon/(n \text{rank } u_0)$ for $i > N$. Let S_E be the right shift operator on \tilde{E} , defined by $S_E((e_1, e_2, \dots)) = (0, e_1, e_2, \dots)$. Then $\beta((T - \tilde{T})S_E^N Q_n u_0) < \varepsilon$.

Let $u = (u_0 + S_E^N u_0)/2$. Define $v \in B(\tilde{F})$ by setting

$$v((f_1, f_2, \dots)) = \frac{1}{2}(f_1 - f_{N+1}, \dots, f_n - f_{N+n}, 0, 0, \dots)$$

Then u and v are contractions, satisfying $vB(E, F)^{(\infty)}u = 0$. Furthermore,

$$vTu = \frac{1}{2}v(T - \tilde{T})(u_0 + S_E^N u_0) = \frac{1}{4}((T - \tilde{T})u_0 + (S_E^N)^{-1}(T - \tilde{T})S_E^N u_0),$$

where S_F is defined similarly to S_E . However, $\beta((T - \tilde{T})S_E^N u_0) < \varepsilon$, hence $\beta(vTu) > 1 - \varepsilon/4$. As ε is an arbitrary positive number, we arrive at a contradiction.

Now suppose u and v are finite rank contractions s.t. $\beta(va^{(\infty)}u) \leq \gamma$ ($\gamma \geq 1$) for any $a \in \mathcal{A}$. We shall show that $\beta(v\tilde{T}u) \leq \gamma$. Let $v_N = S_F^{-N}$ and $u_N = S_E^N$. Then, for any N , $\beta(vb^{(\infty)}u) = \beta(v_N b^{(\infty)}u_N)$ for any $b \in B(E, F)$, hence, in particular, $\beta(v_N a^{(\infty)}u_N) \leq \gamma$. On the other hand, $\lim_N \beta((T - \tilde{T})u_N) = 0$, and therefore,

$$\beta(v\tilde{T}u) = \beta(v_N \tilde{T}u_N) = \lim_N \beta(v_N T u_N) \leq \gamma,$$

as desired. ■

Lemma 2.6. *Suppose, $E, F, \mathcal{E}, \mathfrak{B}$, and \mathcal{A} as above, and $T \in B(E, F)$ is such that $d_{\mathcal{A}^{(\infty)}, \mathfrak{B}}(T^{(\infty)}) < 1$. Then $T \in \mathcal{A}$.*

Proof. As \mathcal{A} is convex, balanced, and $(B(E, F), E \widehat{\otimes} F^*)$ -closed, there exist $c > 0$ and a norm-1 $s \in E \widehat{\otimes} F^*$, s.t. $|\langle a, s \rangle| < c$ for any $a \in \mathcal{A}$, yet $\langle T, s \rangle > c$. Moreover, we can always assume that s belongs to the algebraic tensor product $E \otimes F^*$. Write $s = \sum_{i=1}^N \gamma_i e_i \otimes f_i^*$, where $e_1, \dots, e_N, f_1^*, \dots, f_N^*$ are norm-1 elements of E and F^* , respectively, the numbers (γ_i) are positive, adding up to 1. We can write $\gamma_i = \alpha_i \beta_i$, where the positive scalars (α_i) and (β_i) satisfy $\|(\alpha_i)_{i=1}^N\|_{\mathcal{E}} = 1 = \|(\beta_i)_{i=1}^N\|_{\mathcal{E}^*}$.

Fix $M > 2/c$, and define the linear maps $u : \mathcal{E}_M \rightarrow \tilde{E}$ and $v : \tilde{F} \rightarrow \mathcal{E}_M$ by

$$u\delta_i = \sum_{j=1}^N \alpha_j \delta_{Ni+j} \otimes e_i, \quad v\left(\sum_j \delta_j \otimes f_j\right) = \sum_{i=1}^M \sum_{j=1}^N \beta_j \langle f_{Ni+j}, f_j^* \rangle \delta_i$$

(here, as before, (δ_i) stands for the canonical basis of \mathcal{E}). Clearly, u and v are contractions. For any $a \in B(E, F)$, $va^{(\infty)}u = \text{diag}(\langle a, s \rangle)$ (repeated M times). Thus, for $a \in \mathcal{A}$, $\beta(va^{(\infty)}u) < cM\beta(I_{\mathcal{E}_N})$, and $\beta(vT^{(\infty)}u) > cM\beta(I_{\mathcal{E}_N})$. This contradicts our assumption that $d_{\mathcal{A}, \mathfrak{B}}(T^{(\infty)}) < 1$. \blacksquare

Proof of Theorem 2.2. Suppose $T' \in B(\mathcal{E}(E), \mathcal{E}(F))$ is such that $d_{\mathcal{A}, \mathfrak{B}}(T') < 1/5$. By Lemma 2.3, there exists a diagonal $T \in B(\mathcal{E}(E), \mathcal{E}(F))$ s.t. $\beta(T - T') < 4/5$, and $d_{\mathcal{A}, \mathfrak{B}}(T) < 1$. By Lemmas 2.5 and 2.6, there exists $a \in \mathcal{A}$ s.t. $\beta(T - a^{(\infty)}) \leq 4$. Therefore, $\beta(T' - a^{(\infty)}) \leq 24/5$. \blacksquare

3. REPRESENTATIONS OF BANACH ALGEBRAS

In this section, we apply Π_2 -ASHR to constructing operator spaces with prescribed families of c.b. maps (see e.g. [21, 22, 24, 25] for other work in this direction). More precisely, for a given subset \mathcal{A} of $B(E)$, we seek to equip E with an operator space structure X s.t. $CB(X)$ consists of 2-summing perturbations of $\mathbb{C}\mathcal{A}$. Furthermore, for a given unital Banach algebra \mathcal{A} , we look for a representation $\pi : \mathcal{A} \rightarrow CB(X)$, s.t. $CB(X) = \pi(\mathcal{A}) + \Pi_2(X)$. 2-summing operators are strictly singular, hence $CB(X)$ is comprised of “small perturbations” of $\pi(\mathcal{A})$.

Theorem 3.1. *Suppose E is a Banach space with separable dual, and \mathcal{A} is a balanced convex subset of $\text{Ba}(B(E))$, closed in the topology induced by $E \widehat{\otimes} E^*$, containing the identity, and such that $ab \in \mathcal{A}$ whenever $a, b \in \mathcal{A}$. Suppose, furthermore, that \mathcal{A} is (C_1, C_2) - Π_2 -ASHR. Then there exists an operator space X , isometric to E , such that $CB(X) = \mathbb{C}\mathcal{A} + \Pi_2(E)$. More precisely:*

- (1) *If $a \in \mathcal{A}$, then $\|a\|_{cb} \leq 1$. If $S \in \Pi_2(X)$, then $\|S\|_{cb} \leq \pi_2(S)$.*
- (2) *If $T \in CB(X)$, then there exists $S \in \Pi_2(X)$ and $a \in \mathcal{A}$, so that $T = \lambda a + S$, $a \in \mathcal{A}$, $|\lambda| \leq 4C_1\|T\|_{cb}$, and $\pi_2(S) \leq 4C_2\|T\|_{cb}$.*

Remark 3.2. Proposition 2.1 shows that the above theorem can be applied when \mathcal{A} is a Π_2 -hyperreflexive subspace of $B(E)$.

An application of Theorem 2.2 immediately yields:

Corollary 3.3. *Suppose E is a Banach spaces with separable dual, and \mathcal{A} is a convex balanced subset of $\text{Ba}(B(E))$, closed in the topology induced by $E \widehat{\otimes} E^*$, containing the identity, and such that $ab \in \mathcal{A}$ whenever $a, b \in \mathcal{A}$. Denote by \mathcal{E} the sequence space ℓ_p ($1 < p < \infty$) or c_0 . Then there exists an operator space X , isometric to $\mathcal{E}(E)$, such that $CB(X) = \mathcal{A}^{(\infty)} + \Pi_2(X)$. More precisely:*

- (1) If $a \in \mathcal{A}$, then $\|a^{(\infty)}\|_{cb} \leq 1$. If $S \in \Pi_2(X)$, then $\|S\|_{cb} \leq \pi_2(S)$.
- (2) If $T \in CB(X)$, then there exists $S \in \Pi_2(X)$ and $a \in \mathcal{A}$, so that $T = \lambda a^{(\infty)} + S$, $a \in \mathcal{A}$, $|\lambda| \leq 25\|T\|_{cb}$, and $\pi_2(S) \leq 120\|T\|_{cb}$.

Note that the decomposition $T = a^{(\infty)} + S$ is unique, since any 2-summing operator is strictly singular.

In our previous work (see [23] and references therein), we dealt with representations of Banach algebras on Hilbertian operator spaces. However, for a given algebra, non-trivial representations on a Hilbert space need not exist (see [3]). Thus, the results of this paper cover a wider class of Banach algebras.

Recall that a *dual Banach algebra* is a Banach algebra which is dual as a Banach space, such that multiplication is separately weak* continuous.

Corollary 3.4. *Suppose \mathcal{A} is a dual Banach algebra with a separable predual. Then there exists a separable reflexive Banach space E , a unital isometric weak*-to-weak* continuous representation $\pi : \mathcal{A} \rightarrow B(E)$, and an operator space X , isometric to $\ell_2(E)$, such that $CB(X) = \pi(\mathcal{A}) \otimes I_{\ell_2} + \Pi_2(X)$. More precisely, $T \in B(X)$ is completely bounded if and only if $T = \pi(a) \otimes I_{\ell_2} + S$, with $a \in \mathcal{A}$ and $S \in \Pi_2(X)$.*

- (1) If $S \in \Pi_2(X)$, then $\|S\|_{cb} \leq \pi_2(S)$. If $a \in \mathcal{A}$, then $\|\pi(a)\|_{cb} = \|a\|$.
- (2) Every $T \in CB(X)$ can be written as $T = \pi(a) + b$, with $a \in \mathcal{A}$, $S \in \Pi_2(X)$, $\pi_2(S) \leq 120\|T\|_{cb}$, and $\|a\| \leq \|T\|$.

Remark 3.5. Reasoning as in [25], we can show that the space X we construct in the proof of this corollary fails the Operator Approximation Property (OAP). Examples of other “interesting” spacing without the OAP can be found, for instance, in [23, 24].

Remark 3.6. The term $\Pi_2(X)$ in the statement of Corollary 3.4 cannot be avoided. Indeed, $\ell_1(\mathbb{Z})$, equipped with its convolution product, is a dual Banach algebra (see Section 4.4 of [33] for further results in this direction). However, the space $CB(X)$ cannot be isomorphic (in the Banach algebra sense) to $\ell_1(\mathbb{Z})$. Indeed, if this were the case, X would need to be infinite dimensional. As any finite dimensional subspace of X is completely complemented, $CB(X)$ contains infinitely many idempotents. However, $\ell_1(\mathbb{Z})$ contains only two idempotents: 0 and 1.

Remark 3.7. Gelfand-Naimark-Segal theorem shows that every C^* -algebra can be represented as an algebra of operators on a Hilbert space. For a general Banach algebra, however, non-trivial representations on a Hilbert space need not exist [3]. Thus, we have to consider representations on larger classes of spaces. Recently, several results reminiscent of Corollary 3.4 appeared. M. Daws [8] showed that any dual Banach algebra has an isometric representation on E , for some Banach space E . M. Daws [9] and, independently, F. Uygul [36] showed that every completely contractive dual Banach algebra is completely isometric to a weak*-closed subalgebra

of $CB(X)$, for some reflexive operator space X . However, in these constructions, a generic element of $B(E)$ or $CB(X)$ need not be a “small perturbation” of the image of an element of the algebra.

Finally, [20] gives examples of “natural” representations of certain algebras \mathcal{A} , arising from a non-abelian group G , of the form $\pi : \mathcal{A} \rightarrow CB(B(L_2(G)))$. Here, once again, the small perturbations of $\pi(\mathcal{A})$ do not “fill” $CB(B(L_2(G)))$.

The proof of Corollary 3.4 follows from the fact that any 2-summing operator is strictly singular, and the following lemma:

Lemma 3.8. *Suppose \mathcal{A} is a dual unital Banach algebra with a separable predual. Then there exists a separable reflexive Banach space E , and a unital contractive weak*-to-weak* continuous representation $\pi : \mathcal{A} \rightarrow B(E)$.*

Proof. A recent result of M. Daws [8] states that, for any dual Banach algebra, \mathcal{A} , there exists a weak*-to weak* continuous isometric representation $\psi : \mathcal{A} \rightarrow B(F)$, where F is a reflexive Banach space. We first show that F may be selected to be separable if \mathcal{A} is separable in the weak* topology.

The existence of a countable fundamental, total, and bounded biorthogonal system in $(\mathcal{A}_*, \mathcal{A})$ (see [26, 28]) implies the existence of a weak*-dense sequence $(a_n)_{n \in \mathbb{N}}$ in $\psi(\mathcal{A})$. For each $n, m \in \mathbb{N}$, find a norm one $f_{n,m} \in F$ s.t. $\|a_n f_{n,m}\| \geq \|a_n\|/(1+2^{-m})$. Let $F_1 = \text{span}f_{1,1}$, and, for $n > 1$,

$$F_n = \text{span}[a_k f_{i,j}, a_k F_{n-1}, F_{n-1} \mid 1 \leq i, j, k, \leq n].$$

Clearly, F_n is finite dimensional, hence $G = \overline{\cup_n F_n}$ is separable. Moreover, $a_k F_n \subset F_{n+1}$ for $1 \leq k \leq n$, hence $a_k G \subset G$ for each k . We shall show that $aG \subset G$ for every $a \in \psi(\mathcal{A})$. Indeed, suppose $af \notin G$ for some $f \in G$. Pick $f^* \in F^*$ s.t. $f^*|_G = 0$, and $\langle f^*, af \rangle \neq 0$. Pick a sequence $(n_k)_{k \in \mathbb{N}}$ s.t. $a_{n_k} \rightarrow a$ in the weak* topology. In particular,

$$0 = \langle f^*, a_{n_k} f \rangle \rightarrow \langle f^*, af \rangle \neq 0,$$

a contradiction.

Define $\phi : \mathcal{A} \rightarrow B(G)$ by setting, for $a \in \mathcal{A}$, $\phi(a) = \psi(a)|_G$. Clearly, ϕ is an isometric representation. To show that ϕ is weak*-to-weak* continuous, suppose $a_i \rightarrow 0$ in the weak* topology of \mathcal{A} , and show that $\lim_i \sum_{k=1}^{\infty} \langle g_k^*, \phi(a)g_k \rangle = 0$ whenever $(g_k) \subset G$ and $(g_k^*) \subset G^*$ satisfy $\sum_k \|g_k\| \|g_k^*\| < \infty$. Find $(f_k^*) \subset F^*$ s.t. $\|f_k^*\| = \|g_k^*\|$, and $f_k^*|_G = g_k^*$. Recall that $\psi(a_i) \rightarrow \psi(a)$ weak* in $B(F)$, hence

$$\sum_{k=1}^{\infty} \langle g_k^*, \phi(a)g_k \rangle = \sum_{k=1}^{\infty} \langle f_k^*, \psi(a)g_k \rangle \rightarrow 0.$$

Now denote by e the identity in \mathcal{A} . Then $p = \phi(e)$ is a norm 1 idempotent in $B(G)$. Let $E = p(G)$, and define $\pi : \mathcal{A} \rightarrow B(E)$ by setting $\pi(a) = \phi(a)|_E$. It is easy to show that π is isometric, unital, and weak*-to-weak* continuous. \blacksquare

The proof of Theorem 3.1 uses some ideas of [23]. To define the space X , pick a sequence $(n_i)_{i=1}^\infty \subset \mathbb{N}$, in which every positive integer occurs infinitely many times. By [25], there exists a family $(E_i)_{i=1}^\infty$ of finite dimensional operator spaces such that: (i) E_i is isometric to $\ell_2^{n_i}$, and (ii) for any operator $u : E_i^* \rightarrow E_j$, we have $\|u\|_1/(4+2^{-i}) \leq \|u\|_{cb} \leq \|u\|_1$ if $i = j$, $\|u\|_{cb} = \|u\|_2$ if $i \neq j$. Here, $\|\cdot\|_1$ and $\|\cdot\|_2$ are, respectively, the trace class norm and the Hilbert-Schmidt norm on operators between Hilbert spaces (they coincide with $\nu_1(\cdot)$ and $\pi_2(\cdot)$).

Find a sequence of operators $u_i : E \rightarrow \ell_2^{n_i}$ such that $\pi_2(u_i) = 1$ and, for any $\varepsilon > 0$, $n \in \mathbb{N}$, and $u : E \rightarrow \ell_2^n$ with $\pi_2(u) = 1$, there exists $i \in \mathbb{N}$ for which $n_i = n$ and $\nu_1(u_i - u) < \varepsilon$. We identify the range of u_i with the space E_i described above. Denote by \mathcal{K}_0 the space of compact operators on ℓ_2 with finitely many non-zero entries. We define the operator space X as follows: for $x \in E \otimes \mathcal{K}_0$, let

$$(3.1) \quad \|x\|_{X \otimes \mathcal{K}_0} = \sup \{ \|(u_i a \otimes I_{\mathcal{K}_0})x\|_{E_i \otimes \mathcal{K}_0} \mid i \in \mathbb{N}, a \in \mathcal{A} \}.$$

In other words, X is Banach isometric to the space E , embedded into $(\sum_i E_i)_\infty$ via the map $e \rightarrow (u_i e)_i$. Therefore, X is an operator space (Ruan's axioms hold). It is easy to see that X is isometric to E (as a Banach space). \mathcal{A} is a unital semigroup, hence $\|a\|_{cb} \leq 1$ for any $a \in \mathcal{A}$. Moreover, all 2-summing operators on X are completely bounded:

Lemma 3.9. *If Y is an operator space, and $T : Y \rightarrow X$ is a 2-summing operator, then $\|T\|_{cb} \leq \pi_2(T)$.*

Proof. By (3.1),

$$\|T\|_{cb} = \sup \{ \|u_i a T\|_{cb} \mid i \in \mathbb{N}, a \in \mathcal{A} \} \leq \sup \{ \nu_1(u_i a T) \mid i \in \mathbb{N}, a \in \mathcal{A} \}.$$

We have to show that, for any Banach spaces X and Y , $u \in B(Y, X)$, and $v \in B(X, \ell_2^n)$, we have $\nu_1(vu) \leq \pi_2(v)\pi_2(u)$. By duality,

$$\nu_1(vu) = \sup \{ \text{tr}(vuw) \mid w \in B(\ell_2^n, Y), \|w\| \leq 1 \}.$$

By Lemma 1.16 of [30], $\text{tr}(vuw) \leq \pi_2(v)\pi_2(uw) \leq \pi_2(v)\pi_2(u)$. ■

To estimate the c.b. norms of operators from below, we need:

Lemma 3.10. *Suppose Y is a subspace of X . Consider the operators $T : Y \rightarrow X$, $u : X \rightarrow \ell_2^n$, and $v : \ell_2^n \rightarrow Y$, such that $\pi_2(u) = 1 = \|v\|$. Let $C = \sup \{ \|uav\|_1 \mid a \in \mathcal{A} \}$. Then $\|T\|_{cb} \geq \|uTv\|_1 / (4 \max\{C, 1\})$.*

Proof. Fix $\varepsilon > 0$, and find $i \in \mathbb{N}$ s.t. $n = n_i$, and $\nu_1(u - u_i) < \varepsilon$, and $4^{-i} < \varepsilon$ (we identify ℓ_2^n with E_i). We view u and v as maps from X to E_i and from E_i^* to Y , respectively. By (3.1), $\|v\|_{cb} = \sup \{ \|u_j a v\|_{cb} \mid j \in \mathbb{N}, a \in \mathcal{A} \}$. If $i = j$, then, for any $a \in \mathcal{A}$,

$$\|u_i a v\|_{cb} \leq \|u_i a v\|_1 \leq \|u a v\|_1 + \nu_1(u - u_i) = C + \varepsilon.$$

If $j \neq i$,

$$\|u_j av\|_{cb} \leq \|u_j av\|_2 = \pi_2(u_j av) \leq \pi_2(u_j) \|a\| \|v\| \leq 1.$$

Therefore, $\|v\|_{cb} \leq \max\{C + \varepsilon, 1\}$.

By (3.1), $\|u_i\|_{cb} = 1$, hence $\|u\|_{cb} \leq \|u_i\|_{cb} + \nu_1(u - u_i) < 1 + \varepsilon$. Therefore,

$$\|T\|_{cb} \geq \frac{\|uTv\|_{cb}}{\|u\|_{cb}\|v\|_{cb}} \geq \frac{\|uTv\|_1}{(1 + \varepsilon)(4 + \varepsilon) \max\{C + \varepsilon, 1\}}.$$

However, ε can be chosen to be arbitrarily small. ■

Proof of Theorem 3.1. By (3.1) and the remark following it, a is completely contractive whenever $a \in \mathcal{A}$. By Lemma 3.9, $\|S\|_{cb} \leq \pi_2(S)$ for any $S \in \Pi_2(S)$. This proves part (1) of the theorem.

To prove part (2), pick $T \in CB(X)$ with $\|T\|_{cb} < 1/4$. Suppose, for the sake of contradiction, that we cannot represent $T = \lambda a + S$, with $a \in \mathcal{A}$, $|\lambda| \leq C_1$, and $\pi_2(S) \leq C_2$. Then there exist $\gamma > 1$, finite dimensional spaces E_0 and F_0 , and contractions $u_0 \in B(E_0, E)$, $v \in B(E, F_0)$, s.t. $\pi_2(v_0 a u_0) \leq \gamma$ for any $a \in \mathcal{A}$, yet $\pi_2(v_0 T u_0) > \gamma$. By the remark on page 10 of [30], we can assume that $E_0 = \ell_2^n$. By trace duality (p. 19 of [30], or [35]), there exists $w \in B(F_0, \ell_2^n)$, s.t. $\pi_2(w) = 1$, and $\text{tr}(w v_0 T u_0) > \gamma$. Moreover, $\|w v_0 a u_0\|_1 \leq \pi_2(w) \pi_2(v_0 a u_0) \leq \gamma$ for any $a \in \mathcal{A}$. Lemma 3.10 then yields $\|T\|_{cb} \geq 1/4$, a contradiction. ■

4. EXAMPLES AND APPLICATIONS

By Corollary 3.3 (see also Corollary 3.4), an infinite ampliation of any contractive unital weak*-to-weak* continuous representation $\pi : \mathcal{A} \rightarrow B(E)$ gives rise to an operator space X , isometric to $\ell_2(E)$, s.t. $CB(X) = \pi(\mathcal{A}) \otimes I_{\ell_2} + \Pi_2(X)$. In some cases, however, we can prove that $\pi(\mathcal{A})$ is Π_2 -ASHR without resorting to ampliation. Some examples can be found in [22, 24]. These examples are used to construct “pathological” operator spaces. Below, we present further representations π with this property, and operator spaces arising from them (Theorems 4.2, 4.3, 4.6).

4.1. The long James algebra. We show that the unit ball of the “long James algebra”, viewed as a set of diagonal operators on ℓ_2 , is Π_2 -ASHR. As a consequence, we construct a separable Hilbertian operator space with a complete transfinite basis, but without the Complete Bounded Approximation Property (and consequently, without a complete basis). This answers the non-commutative version of a question asked in [32]. For the construction, we need the notion of the long James space, introduced in [12]. The construction we present here is slightly different, but it is easy to check that it yields an equivalent norm on the same Banach space.

Suppose η is an infinite ordinal. Denote by \mathcal{I}_η , or simply \mathcal{I} , the set of all ordinals $\alpha < \eta$ which are not limit ordinals. For $f : \mathcal{I} \rightarrow \mathbb{C}$, define

$$\|f\|_J = \sup \left\{ \left(\sum_{k=1}^n |f(\alpha_{2k-1}) - f(\alpha_{2k})|^2 \right)^{1/2} \mid (\alpha_i)_{i=1}^{2n} \in \mathcal{F}_n, n \in \mathbb{N} \right\}.$$

Here, the set $\mathcal{F}_n = \mathcal{F}_n(\eta)$ consists of all $2n$ -tuples $(\alpha_i)_{i=1}^{2n}$ of members of \mathcal{I} , for which either $\alpha_1 < \alpha_2 < \dots < \alpha_{2n-1} < \alpha_{2n}$, or $\alpha_{2n} < \alpha_1 < \alpha_2 < \dots < \alpha_{2n-1}$. Then $\|\cdot\|_J$ is a seminorm on $c_{00}(\mathcal{I}) + \mathbb{C}\mathbf{1}$, with the kernel $\mathbb{C}\mathbf{1}$. We define by $J(\eta)$ the completion of $c_{00}(\mathcal{I}) + \mathbb{C}\mathbf{1}$ with respect to $\|\cdot\|_J$.

For the sake of convenience, we introduce a new norm on J . Let $\mathcal{I}' = \mathcal{I} \cup \{\eta, \eta+\}$. Extend $f : \mathcal{I} \rightarrow \mathbb{C}$ to $\tilde{f} : \mathcal{I}' \rightarrow \mathbb{C}$ by setting $\tilde{f}|_{\mathcal{I}} = f$, $\tilde{f}(\eta) = \tilde{f}(\eta+) = 0$. Set

$$\|f\| = \sup \left\{ \left(\sum_{k=1}^n |\tilde{f}(\alpha_{2k-1}) - \tilde{f}(\alpha_{2k})|^2 \right)^{1/2} \mid (\alpha_i)_{i=1}^{2n} \in \mathcal{F}_n(\eta++), n \in \mathbb{N} \right\}.$$

Then $\|\cdot\|$ is a norm on $J(\eta)$. Moreover, any $f \in J(\eta)$ has a unique representation $f_0 + \lambda\mathbf{1}$, with $\lambda \in \mathbb{C}$ and $f_0 \in \overline{c_{00}}^{\|\cdot\|_J}$. For future reference, we need the following easy inequality: for any $f \in J(\eta)$,

$$(4.1) \quad \max\{\|f\|_J, \|f\|_\infty\} \leq \|f\| \leq (\|f\|_J^2 + 2\|f\|_\infty^2)^{1/2}.$$

Next we use the techniques of [2] to make $J(\eta)$ into a commutative unital Banach algebra. For any $f \in \ell_\infty(\mathcal{I})$, set $\|f\|_{\mathcal{A}} = \sup\{\|fg\| \mid \|g\| \leq 1\}$. We call $\mathcal{A}(\eta) = \{f \in \ell_\infty(\mathcal{I}) \mid \|f\|_{\mathcal{A}} < \infty\}$ the long James algebra. Clearly, it is a unital Banach algebra. The next lemma shows that $\mathcal{A}(\eta)$ and $J(\eta)$ coincide as sets, and describes useful properties of $\mathcal{A}(\eta)$.

Lemma 4.1. *Suppose η is an infinite ideal.*

- (1) For any $f \in \ell_\infty(\mathcal{I})$, $\|f\|_{\mathcal{A}} \geq \|f\|_\infty$.
- (2) Suppose $\gamma_1 < \gamma_2 \leq \eta$ are ordinals. Define

$$f(\beta) = \begin{cases} 1 & \gamma_1 \leq \beta < \gamma_2 \\ 0 & \text{otherwise} \end{cases}.$$

Then either $f = 0$, or $\|f\|_{\mathcal{A}} = 1$.

- (3) $\mathcal{A}(\eta)$ and $J(\eta)$ coincide as sets. For any $f \in J(\eta)$,

$$\|f\|/\sqrt{2} \leq \|f\|_{\mathcal{A}} \leq 2\|f\|.$$

Proof. (1) Fix $\alpha \in \mathcal{I}$, and define $g : \mathcal{I} \rightarrow \mathbb{C}$ by setting

$$f(\beta) = \begin{cases} 1 & \beta = \alpha \\ 0 & \beta \neq \alpha \end{cases}.$$

Then $fg = f(\alpha)g$, hence $\|f\|_{\mathcal{A}} \geq |f(\alpha)|$.

(2) It suffices to show that, for any $g \in J(\eta)$ and $(\alpha_i)_{i=1}^{2n} \in \mathcal{F}_n(\eta++)$, we have

$$(4.2) \quad \sum_{k=1}^n |\tilde{f}(\alpha_{2k-1})\tilde{g}(\alpha_{2k-1}) - \tilde{f}(\alpha_{2k})\tilde{g}(\alpha_{2k})|^2 \leq \|g\|^2.$$

We consider the case of $\alpha_1 < \alpha_2 < \dots < \alpha_{2n}$; the other possibility is handled similarly. By erasing the pairs $(\alpha_{2i-1}, \alpha_{2i})$ for which $f(\alpha_{2i-1}) = f(\alpha_{2i}) = 0$, we can restrict our attention to one of the following three cases:

- (i) $\gamma_1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_{2n} < \gamma_2$;
- (ii) $\alpha_1 < \gamma_1 \leq \alpha_2 < \dots < \alpha_{2n} < \gamma_2$;
- (iii) $\gamma_1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_{2n-1} < \gamma_2 \leq \alpha_{2n}$;
- (iv) $\alpha_1 < \gamma_1 \leq \alpha_2 < \dots < \alpha_{2n-1} < \gamma_2 \leq \alpha_{2n}$.

In case (i), the inequality (4.2) clearly holds. In case (ii), consider $\alpha'_1 = \eta$, and $\alpha'_i = \alpha_i$ for $2 \leq i \leq 2n$. Then $(\alpha'_i)_{i=1}^{2n} \in \mathcal{F}_n(\eta++)$, and

$$\begin{aligned} & \sum_{k=1}^n |\tilde{f}(\alpha_{2k-1})\tilde{g}(\alpha_{2k-1}) - \tilde{f}(\alpha_{2k})\tilde{g}(\alpha_{2k})|^2 \\ &= \sum_{k=1}^n |\tilde{g}(\alpha'_{2k-1}) - \tilde{g}(\alpha'_{2k})|^2 \leq \|g\|^2. \end{aligned}$$

The case (iii) is handled in the similar fashion. Finally, if (iv) occurs, we pass to $\alpha'_1 = \eta+$, $\alpha'_{2n} = \eta$, and $\alpha'_i = \alpha_i$ for $2 \leq i \leq 2n-1$.

(3) To show the inequality $\|f\|_{\mathcal{A}} \geq \|f\|/\sqrt{2}$, note that, by part (1),

$$\|f\|_{\mathcal{A}} \geq \frac{\|f\mathbf{1}\|}{\|\mathbf{1}\|} = \frac{\|f\|}{\sqrt{2}}.$$

To obtain the upper estimate on $\|f\|_{\mathcal{A}}$, observe that $\tilde{f}g = \tilde{f}\tilde{g}$. Suppose $(\alpha_i)_{i=1}^{2n} \in \mathcal{F}_n(\eta++)$. Then

$$\begin{aligned} & \sum_{k=1}^n |\tilde{f}(\alpha_{2k-1})\tilde{g}(\alpha_{2k-1}) - \tilde{f}(\alpha_{2k})\tilde{g}(\alpha_{2k})|^2 = \\ & \sum_{k=1}^n |\tilde{f}(\alpha_{2k-1})(\tilde{g}(\alpha_{2k-1}) - \tilde{g}(\alpha_{2k})) + (\tilde{f}(\alpha_{2k-1}) - \tilde{f}(\alpha_{2k}))\tilde{g}(\alpha_{2k})|^2 \leq \\ & 2 \sum_{k=1}^n (|\tilde{f}(\alpha_{2k-1})|^2 |\tilde{g}(\alpha_{2k-1}) - \tilde{g}(\alpha_{2k})|^2 + |\tilde{f}(\alpha_{2k-1}) - \tilde{f}(\alpha_{2k})|^2 |\tilde{g}(\alpha_{2k})|^2) \\ & \leq 2(\|f\|_{\infty}^2 \|g\|^2 + \|f\|^2 \|g\|_{\infty}^2) \leq 4\|f\|^2 \|g\|^2 \end{aligned}$$

(here, we use the left hand side of (4.1)). Taking the supremum over all $(\alpha_i)_{i=1}^{2n} \in \mathcal{F}_n(\eta++)$, we obtain the inequality $\|fg\| \leq 2\|f\|\|g\|$, as desired. \blacksquare

Denote by $(e_{\alpha})_{\alpha \in \mathcal{I}}$ the canonical orthonormal basis in $\ell_2(\mathcal{I})$. Consider the ‘‘natural’’ representation $\pi : \mathcal{A}(\eta) \rightarrow B(\ell_2(\eta))$ by setting $\pi(f)e_{\alpha} = f(\alpha)e_{\alpha}$. It follows from

the discussion above that π is a unital contractive representation. A particular case of the following lemma is implicit in [22].

Theorem 4.2. *Suppose η is an infinite ordinal. Then $\pi(\text{Ba}(\mathcal{A}(\eta)))$ is $(10\sqrt{6}, 4)$ - Π_2 -ASHR.*

Proof. For the sake of brevity, let $H = \ell_2(\mathcal{I})$. Suppose $T \in B(H)$ is such that $\|uTv\|_2 < 1/5$ whenever the finite rank contractions u and v satisfy $\|u\pi(a)v\|_2 \leq 1$ for any $a \in \text{Ba}(\mathcal{A}(\eta))$. We shall show that $T = T_1 + \pi(f)$, where $\|T_1\|_2 \leq 4/5$, and $\|f\|_{\mathcal{A}} \leq 2\sqrt{6}$.

For $\alpha \in \mathcal{I}$, set $f(\alpha) = \langle Te_\alpha, e_\alpha \rangle$. Let T_0 be the diagonal part of T (that is, $T_0e_\alpha = f(\alpha)e_\alpha$ for any $\alpha \in \mathcal{I}$), and let $T_1 = T - T_0$ be the off-diagonal part. Suppose S_1 and S_2 are disjoint finite subsets of \mathcal{I} . Let P_{S_1} and P_{S_2} be the corresponding orthogonal projections. Then $P_{S_1}\pi(a)P_{S_2} = 0$ for any $a \in \text{Ba}(\mathcal{A}(\eta))$, hence $\|P_{S_1}TP_{S_2}\|_2 < 1/5$. An averaging (cf. the proof of Lemma 2.3) argument yields $\|T_1\|_2 \leq 4/5$. Moreover, if the finite rank contractions u and v satisfy $\|u\pi(a)v\|_2 \leq 1$ for any $a \in \text{Ba}(\mathcal{A}(\eta))$, then $\|uT_0v\|_2 \leq \|uTv\|_2 + \|T_1\|_2 \leq 1$.

Recall that $T_0e_\alpha = f(\alpha)e_\alpha$. It suffices to show that $\|f\|_{\mathcal{A}} \leq 2\sqrt{6}$. To this end, note first that $\|f\|_\infty \leq 1$. Indeed, otherwise there exists $\alpha \in \mathcal{I}$ s.t. $|f(\alpha)| > 1$. Let u and v be equal to the orthogonal projection onto $\mathbb{C}e_\alpha$. Then $\|u\pi(a)v\|_2 \leq 1$ for any $a \in \text{Ba}(\mathcal{A}(\eta))$, yet $\|uT_0v\|_2 > 1$, a contradiction.

Next consider $(\alpha_i)_{i=1}^{2n} \in \mathcal{F}_n$, and show that $\sum_{k=1}^n |f(\alpha_{2k-1}) - f(\alpha_{2k})|^2 \leq 4$ (and therefore, $\|f\|_J \leq 2$). Indeed, denote by $(\delta_k)_{k=1}^n$ an orthonormal basis in ℓ_2^n , and consider

$$v : \ell_2^n \rightarrow H : \delta_k \mapsto (e_{\alpha_{2k-1}} + e_{\alpha_{2k}})/\sqrt{2},$$

and

$$u : H \rightarrow \ell_2^n : \xi \mapsto \sum_{k=1}^n \frac{1}{\sqrt{2}} \langle \xi, e_{\alpha_{2k-1}} - e_{\alpha_{2k}} \rangle \delta_k.$$

If S is a diagonal operator with $Se_\alpha = g(\alpha)e_\alpha$ for $\alpha \in \mathcal{I}$, then

$$uSv\delta_k = \frac{1}{2}(g(\alpha_{2k-1}) - g(\alpha_{2k}))\delta_k,$$

hence

$$(4.3) \quad \|uSv\|_2^2 = \sum_{k=1}^n |g(\alpha_{2k-1}) - g(\alpha_{2k})|^2/4$$

Then, for any $a \in \mathcal{A}(\eta)$, $\|u\pi(a)v\|_2 \leq \|a\|_J \leq \|a\| < 1$, by (4.1) and Lemma 4.1. Thus, $\|uT_0v\|_2 \leq 1$. By (4.3), $\|f\|_J \leq 2$. By (4.1), $\|f\| \leq \sqrt{6}$. By Lemma 4.1, $\|f\|_{\mathcal{A}} \leq 2\sqrt{6}$. ■

4.2. Transfinite bases. The notion of a transfinite basis in a Banach space was introduced by C. Bessaga [4]. We refer the reader to that paper, and to Section 2 of [32], for more information. Here we present a non-commutative version of the same definition. Suppose X is an operator space, η is an ordinal, and $(P_\gamma)_{0 \leq \gamma \leq \eta}$ is a family of projections on X such that $\sup_\gamma \|P_\gamma\|_{cb} < \infty$, $P_\alpha P_\beta = P_{\min\{\alpha, \beta\}}$, $P_0 = 0$, $P_\eta = I_X$, and for each $x \in X$, the map $\gamma \mapsto P_\gamma x$ is continuous on $[0, \eta]$ (here, $[0, \eta]$ and X are equipped with the order, resp. norm, topology). Then the family $(P_\gamma)_{0 \leq \gamma \leq \eta}$ is called a *complete transfinite Schauder decomposition (of length η)*. If, in addition, $\dim(P_{\gamma+} - P_\gamma) = 1$ for any γ , then $(P_\gamma)_{0 \leq \gamma \leq \eta}$ is called a *complete transfinite (Schauder) basis (of length η)*. Note that (complete) transfinite bases of length ω (the first infinite ordinal) in Banach (operator) spaces are just the usual (complete) bases.

Recall that an operator space X is said to have the *Completely Bounded Compact Approximation Property (CBCAP for short)* if there exists a constant C s.t. for every finite dimensional $E \hookrightarrow X$ there exists a compact operator $T \in CB(X)$ s.t. $\|T\|_{cb} \leq C$, and $T|_E = I_E$. The definition of the *Completely Bounded Approximation Property (CBAP)* is similar, but more restrictive: the operators T are required to have finite rank. For Banach spaces, the Bounded (Compact) Approximation Property (BAP, CBAP) are defined in a similar fashion.

Clearly, every operator (Banach) space with a (complete) basis has the CBAP (resp. BAP). It was shown in [32] that every Banach space with a transfinite basis has the Approximation Property (which is strictly weaker than the BAP – see e.g. [5] for the connections between the two properties). The problem of whether the existence of the transfinite basis implies the Bounded Approximation Property is open. We give a negative answer to the non-commutative version of this question.

Theorem 4.3. *There exists an operator space with a transfinite basis, failing the Completely Bounded Compact Approximation Property.*

Proof. Consider the ordinal $\eta = \omega^2$. For simplicity, let $\mathcal{A} = \mathcal{A}(\eta)$. In the previous subsection, it is shown that $\pi(\text{Ba}(\mathcal{A}))$ is Π_2 -ASHR. We build an operator space X “around” this set. More precisely, we use the results of Chapter 3 to construct an operator space X , isometric to $\ell_2(\mathcal{I})$ on the Banach space level, such that $T \in B(X)$ is completely bounded iff $T \in \pi(\mathcal{A}) + \mathcal{S}_2$. Moreover, for any $a \in \mathcal{A}$ and $S \in \mathcal{S}_2$, we have $\|\pi(a) + S\|_{cb} \leq \|a\|_{\mathcal{A}} + \|S\|_2$, while there exists a constant c s.t. every $T \in CB(X)$ can be represented as $T = \pi(a) + S$, with $\|a\|_{\mathcal{A}} + \|S\|_2 \geq c\|T\|_{cb}$.

For $\gamma \leq \eta$, define f_γ by setting, for $\alpha \in \mathcal{I}$,

$$f_\gamma(\alpha) = \begin{cases} 1 & \alpha < \gamma \\ 0 & \alpha \geq \gamma \end{cases}.$$

By Lemma 4.1(b), $\|f_\gamma\|_{\mathcal{A}} = 1$ unless $\gamma = 1$, and $\|\mathbf{1} - f_\gamma\|_{\mathcal{A}} = 1$ unless $\gamma = \eta$.

Therefore, for any $\gamma \leq \eta$, the projection $P_\gamma = \pi(f_\gamma)$ is completely contractive. Moreover,

$$P_\gamma e_\alpha = \begin{cases} e_\alpha & \alpha < \gamma \\ 0 & \alpha \geq \gamma \end{cases}$$

for any $\alpha \in \mathcal{I}$. Thus, the family (P_γ) defines a completely transfinite basis on X .

Suppose, for the sake of contradiction, that X has the CBCAP. For each n , consider $E_n = \text{span}[e_\omega, e_{2\omega}, \dots, e_{n\omega}] \hookrightarrow X$. Then there exists a compact operator $T = \pi(f) + S$ s.t. $\|f\|_{\mathcal{A}} + \|S\|_2 < C$ (C is a constant), and $T|_{E_n} = I_{E_n}$. In particular,

$$\sum_{j=1}^n |f(j\omega) - 1|^2 = \sum_{j=1}^n \|Se_j\|^2 \leq \|S\|_2^2 < C^2.$$

Suppose $n > 32C^2$. Then the set

$$\mathcal{G}_n = \{1 \leq k \leq n \mid |f(j\omega) - 1| < 1/4\}$$

is such that $|\mathcal{G}_n^c| < 16C^2$.

On the other hand, $\pi(f)$ is compact, hence for any $j \in [1, n]$, there exists $\alpha_j \in (j\omega, (j+1)\omega)$ s.t. $|f(\alpha_j)| < 1/4$. Then

$$\|f\|^2 \geq \sum_{k \in \mathcal{G}_n} |f(\alpha_j) - f(j\omega)|^2 \geq \frac{|\mathcal{G}_n|}{4} \geq \frac{n}{8}.$$

By Lemma 4.1, $\|f\|_{\mathcal{A}} \geq \sqrt{n}/4$, which yields the desired contradiction. \blacksquare

Remark 4.4. As in Section 2 of [32], we can show that any operator space with a complete transfinite basis has the Strong Operator Approximation Property (SOAP). Thus, Theorem 4.3 yields an example of an operator space with the SOAP, but without the CBCAP. The classical counterpart of this result is the existence of Banach spaces with the Approximation Property, but without the Bounded Compact Approximation Properties (see Section 8 of [5] for the examples).

Remark 4.5. We can generalize the definition of the long James space in the following way. Suppose \mathcal{E} is a symmetric separable sequence space, which is mononormalizing (in the terminology of p. 88 of [14]; see also Section 3 of [23]). That is, we require that two conditions be satisfied: (i) if $\lim_n \|(x_1, \dots, x_n, 0, 0, \dots)\|_{\mathcal{E}} = C < \infty$, then $(x_i)_{i \in \mathbb{N}} \in \mathcal{E}$, and $\|(x_i)_{i \in \mathbb{N}}\|_{\mathcal{E}} = C$; and (ii) if $(x_i)_{i \in \mathbb{N}} \in \mathcal{E}$, then $\lim_n \|(x_n, x_{n+1}, \dots)\|_{\mathcal{E}} = 0$. The space \mathcal{E} is *nice* if, in addition, $\lim_n \|(1, \dots, 1, 0, \dots)\|_{\mathcal{E}} = \infty$.

We denote by $\mathcal{S}_{\mathcal{E}}$ the operator ideal, corresponding to a nice space \mathcal{E} : for a compact operator T , denote by $(\sigma_i(T))$ the sequence of singular numbers of T , and set $\|T\|_{\mathcal{E}} = \|(\sigma_i(T))\|_{\mathcal{E}}$.

For any infinite ordinal η , and for any nice sequence space \mathcal{E} , we define a modified long James space: for $f : \mathcal{I} \rightarrow \mathbb{C}$, define

$$\|f\|_{J_{\mathcal{E}}} = \sup \left\{ \|(f(\alpha_{2k-1}) - f(\alpha_{2k}))_{1 \leq k \leq n}\|_{\mathcal{E}} \mid (\alpha_i)_{i=1}^{2n} \in \mathcal{F}_n, n \in \mathbb{N} \right\}.$$

Here, the set $\mathcal{F}_n = \mathcal{F}_n(\eta)$ consists of all $2n$ -tuples $(\alpha_i)_{i=1}^{2n}$ of members of \mathcal{I} , for which either $\alpha_1 < \alpha_2 < \dots < \alpha_{2n-1} < \alpha_{2n}$, or $\alpha_{2n} < \alpha_1 < \alpha_2 < \dots < \alpha_{2n-1}$. Then $\|\cdot\|_{J_\mathcal{E}}$ is a seminorm on $c_{00}(\mathcal{I}) + \mathbb{C}\mathbf{1}$, with the kernel $\mathbb{C}\mathbf{1}$. We define by $J_\mathcal{E}(\eta)$ the completion of $c_{00}(\mathcal{I}) + \mathbb{C}\mathbf{1}$ with respect to $\|\cdot\|_{J_\mathcal{E}}$.

Similarly, we define on $c_{00} + \mathbb{C}\mathbf{1}$ the norms $\|\cdot\|_\mathcal{E}$ and $\|\cdot\|_{\mathcal{A}_\mathcal{E}(\eta)}$ (analogues of $\|\cdot\|$ and $\|\cdot\|_{\mathcal{A}(\eta)}$). As above, we show that the norms $\|\cdot\|_\mathcal{E}$ and $\|\cdot\|_{\mathcal{A}_\mathcal{E}(\eta)}$ are equivalent to each other, and both of them are equivalent to $\|\cdot\|_{J_\mathcal{E}}$ on c_{00} .

Denote by $\pi_\mathcal{E}$ the identity representation of $\mathcal{A}_\mathcal{E}(\eta)$ on the diagonal of $B(\ell_2(\mathcal{I}))$. As before, we show that $\pi_\mathcal{E}(\text{Ba}(\mathcal{A}_\mathcal{E}(\eta)))$ is $\mathcal{S}_\mathcal{E}$ -ASHR.

Now suppose the ordinal η is countable, and the formal identity map $\ell_2 \rightarrow \mathcal{E}$ is bounded. By [23], we can equip $\ell_2(\mathcal{I})$ with an operator space structure X , s.t. $CB(X) = \pi(\mathcal{A}_\mathcal{E}(\eta)) + \mathcal{S}_\mathcal{E}(\ell_2(\mathcal{I}))$.

4.3. Almost multiplicative functionals. An old problem in Banach algebras asks whether every ‘‘almost multiplicative’’ linear functional on a given Banach algebra is near multiplicative. More precisely: suppose \mathcal{A} is a Banach algebra. We say that $f \in \mathcal{A}^*$ is δ -multiplicative $|f(a)f(b) - f(ab)| < \delta\|a\|\|b\|$ for any $a, b \in \mathcal{A}$. 0-multiplicative functionals are referred to simply as *multiplicative*. A Banach algebra \mathcal{A} is called *AMNM* (*almost multiplicative near multiplicative*) if there exists a function ε on $(0, \infty)$, with $\lim_{\delta \rightarrow 0} \varepsilon(\delta) = 0$, such that for any δ -multiplicative $f \in \mathcal{A}^*$ there exists a multiplicative $g \in \mathcal{A}^*$ s.t. $\|f - g\| < \varepsilon(\delta)$.

Describing the AMNM Banach algebras is a long-standing open problem. There reader is referred to [16, 17], and references therein, for a brief introduction. Here, we mention several results: (1) The algebras $C_0(K)$ (K locally compact Hausdorff) are AMNM. (2) The disk algebra is AMNM. Moreover, any singly generated uniform algebra is AMNM [15] (3) There exists a uniform algebra which is not AMNM [34]. (4) If G is a locally compact abelian group, then $L_1(G)$ is AMNM. (5) It is not known whether H^∞ is AMNM (that is, whether H^∞ has ‘‘almost corona’’).

No examples of Banach spaces E for which $B(E)$ is not AMNM are known. Note that, if E is isomorphic to $\ell_\infty^n(E)$ for some n , then $B(E)$ is AMNM. Indeed, in this case, there exist idempotents $p_1, \dots, p_n \in B(E)$, s.t. $p_i p_j = \delta_{ij} p_i$, and $\sum_{i=1}^n p_i = I_E$, and $u_1, v_1, \dots, u_n, v_n \in B(E)$ s.t. $u_i v_i = p_i$, and $v_i u_i = I_E$. Let $K = \max_i \|u_i\| \|v_i\|$. If $f \in B(E)^*$ is δ -multiplicative, then $|f(I_E)^2 - f(I_E)| \leq \delta$. Therefore, either $|f(I_E)| < \phi(\delta)$, or $|f(I_E) - 1| < \phi(\delta)$, where ϕ is a continuous map on $[0, \infty)$, s.t. $\phi(0) = 0$. In the former case,

$$|f(TI_E) - f(T)f(I_E)| = |f(T)| |1 - f(I_E)| \leq \phi(\delta) \|T\|$$

for any $T \in B(E)$, hence $\|f\| \leq \phi(\delta)/(1 - \phi(\delta))$. The latter case can be ruled out as follows: if $|f(I_E) - 1| < \phi(\delta)$, then

$$|f(u_i)f(v_i) - 1| \leq |f(v_i)f(u_i) - f(I_E)| + |f(I_E) - 1| \leq K\delta + \phi(\delta).$$

However, $|f(u_i)f(v_i) - f(p_i)| \leq K\delta$, hence $|f(p_i) - 1| \leq 2K\delta + \phi(\delta)$ for each i . Then

$$n - 1 - \phi(\delta) \leq |f(I_E) - n| \leq \sum_i |f(p_i) - 1| \leq n(2K\delta + \phi(\delta)),$$

which is clearly impossible for δ small enough.

The spaces E which are isomorphic to $\ell_\infty^n(E)$ include the Orlicz spaces, the Schatten spaces, and more.

For many classes of Banach spaces, however, it is not known whether $B(E)$ is AMNM. For instance, suppose E a hereditarily indecomposable space. Then for every $T \in B(E)$ there exists a (unique) $\lambda_T \in \mathbb{C}$ s.t. $T - \lambda_T I_E$ is strictly singular. It is well-known (see e.g. [18]) that the spectrum of any strictly singular operator is countable, 0 its only possible cluster point, and all isolated points are finite rank eigenvalues. By [1], $f(S) = 0$ whenever $f \in B(E)^*$ is multiplicative, and S is finite rank. Moreover, by [1] again, $f(S) \in \sigma(S)$ whenever $S \in B(E)$, and f is multiplicative.

This shows that the only non-trivial multiplicative functional on $B(E)$, with E as above, is defined by $f(T) = \lambda_T$, with λ_T defined in the previous paragraph. We have no good description of δ -multiplicative members of $B(E)^*$.

The same reasoning could be repeated for the algebras $CB(X)$. In this case, however, we can exhibit an example of X s.t. $CB(X)$ is not AMNM.

Theorem 4.6. *There exists an operator space X such that $CB(X)$ is not AMNM.*

Proof. We recapitulate B. Johnson's construction (Example 9.1 of [16]). There, it is shown that there exists a sequence of unital Banach algebras \mathcal{A}_n s.t., for each $n \in \mathbb{N}$, (1) \mathcal{A}_n is \mathbb{C}^n with pointwise multiplication; (2) there exists $f_n \in \mathcal{A}_n^*$ s.t. $\|f_n\| = f(\mathbf{1}_n)$ ($\mathbf{1}_n$ is the identity of the algebra \mathcal{A}_n); (3) $|f_n(a)f_n(b) - f_n(ab)| \leq \|a\|\|b\|/n$ for any $a, b \in \mathcal{A}_n$; (4) $\|f_n - f\| \geq 1 - 1/n$ whenever $f \in \mathcal{A}_n^*$ is multiplicative.

Consider now the Banach algebra $\mathcal{A} = \mathbb{C}\mathbf{1} \oplus_1 (\sum_i \mathcal{A}_i)_{\ell_1}$. It has a predual $A_* = \mathbb{C} \oplus_\infty (\sum_i \mathcal{A}_i^*)_{c_0}$. We shall show that the multiplication in \mathcal{A} is weak* continuous. By commutativity, it suffices to show the following: suppose $y \in \mathcal{A}$, and a net (x_α) converges weak* to $x \in \mathcal{A}$. Then $x_\alpha y \rightarrow xy$ weak*. By trivality of multiplication by $\mathbf{1}$, we it suffices to consider $y = (y_i)$, with $y_i \in \mathcal{A}_i$. Furthermore, write $x = a\mathbf{1} + (x_i)$, and $x_\alpha = a_\alpha\mathbf{1} + (x_{\alpha i})$. Clearly, $\lim_\alpha a_\alpha = a$. Therefore, we can assume $a = 0$, and $a_\alpha = 0$ for each α . Then $xy = (x_i y_i)$, and $x_\alpha y = (x_{\alpha i} y_i)$. We have to show that, for any $(z_i) \in (\sum \mathcal{A}_i^*)_{c_0}$,

$$\lim_\alpha \sum_i \langle x_{\alpha i} y_i, z_i \rangle = \sum_i \langle x_i y_i, z_i \rangle.$$

We can identify \mathcal{A}_i^* with \mathbb{C}^n as a vector space. Let $z'_i = y_i z_i$ (with pointwise multiplication). A standard Banach algebra calculation yields $\|z'_i\| \leq \|y_i\| \|z_i\|$. Therefore,

$z' = (z'_i) \in (\sum \mathcal{A}_i^*)_{c_0}$. Thus,

$$\lim_{\alpha} \sum_i \langle x_{\alpha i} y_i, z_i \rangle = \lim_{\alpha} \langle x_{\alpha}, z' \rangle = \langle x z' \rangle = \sum_i \langle x_i y_i, z_i \rangle,$$

as desired.

Now extend the functional $f_n \in \mathcal{A}_n^*$ to the whole algebra \mathcal{A} , by setting $g_n(a\mathbf{1} \oplus (x_i)_{i \in \mathbb{N}}) = a + f_n(x_n)$ (here, $x_i \in \mathcal{A}_i$). By definition, $\|g_n\| = 1$. We shall show that g_n is n^{-1} -multiplicative. Indeed, consider norm 1 $x = a\mathbf{1} \oplus (x_i)$ and $y = b\mathbf{1} \oplus (y_i)$. Then $g_n(x) = a + f_n(x_n)$, $g_n(y) = b + f_n(y_n)$, $g_n(xy) = ab + af(y_n) + bf(x_n) + f_n(x_n y_n)$, and

$$|g_n(x)g_n(y) - g_n(xy)| = |f_n(x_n)f_n(y_n) - f_n(x_n y_n)| \leq n^{-1}\|x_n\|\|y_n\| \leq n^{-1}.$$

The last inequality shows that g_n is n^{-1} -multiplicative.

Next we show that $\|g_n - g\| \geq 1 - n^{-1}$ whenever $g \in \mathcal{A}^*$ is multiplicative. Indeed, then $g|_{\mathcal{A}_n}$ is also multiplicative, and $\|g_n - g\| \geq \|f_n - g|_{\mathcal{A}_n}\| \geq 1 - n^{-1}$, per the result of B. Johnson described above.

By Corollary 3.4, there exists a Banach space E , an operator space X , isometric to $\ell_2(E)$, and a unital isometric representation $\pi : \mathcal{A} \rightarrow B(E)$ s.t. $CB(X) = \pi(\mathcal{A}) \otimes I_{\ell_2} + \Pi_2(X)$. For $n \in \mathbb{N}$, define $\tilde{g}_n \in CB(X)^*$ by setting $\tilde{g}_n(\pi(a) \otimes I_{\ell_2} + S) = g_n(a)$ whenever $a \in \mathcal{A}$, and S is 2-summing. Clearly, \tilde{g}_n is well-defined, $\|\tilde{g}_n\| = 1$, and \tilde{g}_n is n^{-1} -multiplicative. Moreover, for any multiplicative $\tilde{g} \in CB(X)^*$, define $g \in \mathcal{A}^*$ by setting $g(a) = \tilde{g}(\pi(a) \otimes I_{\ell_2})$. Then $\|\tilde{g}_n - \tilde{g}\| \geq \|g_n - g\| \geq 1 - n^{-1}$, which shows that $CB(X)$ is not AMNM. \blacksquare

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