

OPERATOR SPACES WITH PRESCRIBED SETS OF COMPLETELY BOUNDED MAPS

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ABSTRACT. Suppose \mathcal{A} is a dual Banach algebra, and a representation $\pi : \mathcal{A} \rightarrow B(\ell_2)$ is unital, weak* continuous, and contractive. We use a “Hilbert-Schmidt version” of Arveson Distance Formula to construct an operator space X , isometric to $\ell_2 \otimes \ell_2$, such that the space of completely bounded maps on X consists of Hilbert-Schmidt perturbations of $\pi(\mathcal{A}) \otimes I_{\ell_2}$. This allows us to establish the existence of operator spaces with various interesting properties. For instance, we construct an operator space X for which the group $K_1(CB(X))$ contains \mathbb{Z}_2 as a subgroup, and a completely indecomposable operator space containing an infinite dimensional homogeneous Hilbertian subspace.

1. INTRODUCTION AND THE MAIN RESULT

In this paper, we show that, for certain representations $\pi : \mathcal{A} \rightarrow B(\ell_2)$ (\mathcal{A} is a Banach algebra), there exist an operator space X , isomorphic to ℓ_2 , such that $T \in CB(X)$ if and only if $T = \pi(a) + S$, with $a \in \mathcal{A}$ and $S \in S_2$ (here and below, S_2 denotes the space of Hilbert-Schmidt operators, while $\|\cdot\|_2$ is the corresponding norm). Similar results in the Banach space case were obtained in [17], where the authors constructed a Banach space X such that $T \in B(X)$ if and only if T is a strictly singular perturbation of a member of an algebra of “spreads.” Another predecessor of this work is [22], where it was proved that, for any unital injective von Neumann subalgebra \mathcal{N} of $B(\ell_2)$, there exists an operator space structure X on ℓ_2 s.t. $CB(X) \sim \mathcal{N} + S_2$. In this paper, we explore a wider class of subalgebras of $B(\ell_2)$.

Throughout the paper, we freely use standard operator space results and terminology. The reader is referred to [13, 24, 25] for more information. We work mainly with 1-Hilbertian operator spaces – that is, operator spaces isometric to Hilbert spaces.

Our most interesting result is

Theorem 1.1. *Suppose A_1, A_2, \dots are weak* closed subspaces of $B(\ell_2)$ such that $\mathbb{C}I \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \dots$, and $A_n A_m \subset A_{n+m}$ for any $n, m \in \mathbb{N}$. Suppose, furthermore, that $\gamma_1, \gamma_2, \dots \in (0, 1]$ are such that $\lim_n \gamma_n = 0$, and $\gamma_n \gamma_m \leq \gamma_{n+m}$ for any $n, m \in \mathbb{N}$. Then there exists an operator space X , isometric to $\ell_2 \otimes \ell_2$, such that:*

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- (1) Every completely bounded map on X is of the form $T = a \otimes I_{\ell_2} + S$, with $a \in \mathcal{A}$ (\mathcal{A} is the norm closure of $\cup_n A_n$), and $S \in S_2$. Moreover, $\|S\|_2 \leq 16\|T\|_{cb}$, and

$$\inf_{b \in A_k} \|a - b\| \leq 4\gamma_{k+1} \min\{\|T\|_{cb}, \|a \otimes I_{\ell_2}\|_{cb}\}$$

for any k .

- (2) For any Hilbert-Schmidt operator S on X , $\|S\|_{cb} \leq \|S\|_2$. For any $a \in A_k$, $\|a \otimes I_{\ell_2}\| \leq \|a\|/\gamma_k$.
- (3) The map $\rho : T = a \otimes I_{\ell_2} + S \mapsto a \otimes I_{\ell_2}$ defines a bounded homomorphism on $CB(X)$.

Moreover, suppose $\gamma_n > 2^{-n}$ for any n , and K is a subset of \mathcal{A} such that

$$\sup_{x \in K} \inf_{a \in A_n} \|x - a\| < 4^{-n}$$

for every $n \in \mathbb{N}$. Then X can be constructed in such a way that $\|(a_1 - a_2) \otimes I_{\ell_2}\|_{cb} \leq \max\{44\|a_1 - a_2\|, 18\|a_1 - a_2\|^{1/2}\}$ for any $a_1, a_2 \in K$.

By Local Reflexivity Principle, finite dimensional subspaces of a dual Banach space are weak* closed. In fact, the spaces A_n are finite dimensional in most applications considered below (in Section 3).

Note that Theorem 1.1 does not claim that any operator $a \otimes I_{\ell_2}$ (with $a \in \mathcal{A}$) is completely bounded.

A better result is available for weak* closed subalgebras of $B(\ell_2)$, or, more generally, for “nice” images of Banach algebras which are dual spaces. Here, a *representation* of a Banach algebra \mathcal{A} on $B(\ell_2)$ is a continuous linear map which is also an algebraic homomorphism. A representation is called *faithful* if its kernel is trivial.

Theorem 1.2. *Suppose \mathcal{A} is a unital Banach algebra which is a dual Banach space, and $\pi : \mathcal{A} \rightarrow B(\ell_2)$ is a weak* continuous faithful unital contractive representation. Then there exists an operator space X , isometric to $\ell_2 \otimes \ell_2$, such that $T \in B(X)$ is completely bounded if and only if $T = \pi(a) \otimes I_{\ell_2} + S$, with $a \in \mathcal{A}$ and $S \in S_2$. Moreover,*

$$\max\left\{\|\pi(a)\|, \frac{\|a\|}{4}, \frac{\|S\|_2}{16}\right\} \leq \|a \otimes I_{\ell_2} + S\|_{cb} \leq \|a\| + \|S\|_2.$$

The map $\rho : \pi(a) \otimes I_{\ell_2} + S \mapsto \pi(a) \otimes I_{\ell_2}$ defines a bounded homomorphism on $CB(X)$.

It is known that multiplication of elements of $B(\ell_2)$ is separately weak* continuous. Therefore, if \mathcal{A} is a Banach algebra, and $\pi : \mathcal{A} \rightarrow B(\ell_2)$ is a weak* continuous faithful representation, then multiplication in \mathcal{A} is also separately weak* continuous. In the terminology of Section 4.4 of [27], \mathcal{A} is a *dual Banach algebra*.

Remark 1.3. Theorem 1.2 can be applied to the natural embedding π of a weak* closed unital subalgebra \mathcal{A} into $B(\ell_2)$. In this setting, we obtain the existence of an

operator space structure X on $\ell_2 \otimes \ell_2$ s.t. $T \in CB(X)$ iff $T = a \otimes I_{\ell_2} + S$, with $a \in \mathcal{A}$ and $S \in S_2$, and

$$\max \left\{ \|a\|, \frac{\|S\|_2}{16} \right\} \leq \|a \otimes I_{\ell_2} + S\|_{cb} \leq \|a\| + \|S\|_2.$$

Theorems 1.1 and 1.2 are deduced from a more general Theorem 2.1 in Section 2. In Sections 3 and 4 we use these theorems to construct examples of operator spaces with “unusual” properties. In particular, we prove the existence of the following separable 1-Hilbertian operator spaces:

- A space X for which the group $K_1(CB(X))$ contains \mathbb{Z}_2 (no such examples are known on the Banach space level).
- A space X_ε ($\varepsilon > 0$) which cannot be represented as a direct sum of two infinite dimensional operator spaces, but contains an infinite dimensional subspace Y s.t. $\|T\|_{cb} \leq (1 + \varepsilon)\|T\|$ for any $T \in B(Y)$.
- A space X , completely isomorphic to $X \oplus X$, and such that there is a non-trivial trace on $CB(X)$ (we do not know of any Banach spaces possessing this property).

2. PROOF OF THE MAIN RESULTS

In this section, we state and prove Theorem 2.1, and show that it implies Theorems 1.1 and 1.2. Following the convention, we denote by B_X the closed unit ball of a Banach space X .

Theorem 2.1. *Suppose \mathcal{B} is a subset of $B_{B(\ell_2)}$, which contains the identity, and such that $ab \in \mathcal{B}$ and $\lambda a + \mu b \in \mathcal{B}$ whenever $a, b \in \mathcal{B}$, and the complex numbers λ and μ satisfy $|\lambda| + |\mu| \leq 1$. Denote by $\overline{\mathcal{B}}$ the weak* closure of \mathcal{B} . Let \mathcal{A} be the linear span of $\overline{\mathcal{B}}$ (not necessarily closed). For $a \in \mathcal{A}$ define $\|a\|$ as the infimum of the numbers $c > 0$ for which $a/c \in \overline{\mathcal{B}}$. Then there exists an operator space X , isometric to $\ell_2 \otimes \ell_2$, such that:*

- (1) *Every $T \in CB(X)$ can be written as $T = a \otimes I_{\ell_2} + S$ in a unique way, with $a \in \mathcal{A}$, $S \in S_2$, $\|S\|_2 \leq 16\|T\|_{cb}$, and $\|a\| \leq 4\|T\|_{cb}$.*
- (2) *If S is a Hilbert-Schmidt operator on X , then $\|S\|_{cb} \leq \|S\|_2$. If $a \in \mathcal{A}$, then $\|a \otimes I_{\ell_2}\|_{cb} \leq \|a\|$.*
- (3) *The map $\rho : T = a \otimes I_{\ell_2} + S \mapsto a \otimes I_{\ell_2}$ defines a bounded homomorphism on $CB(X)$.*

Remark 2.2. Our goal is to construct an operator space for which every operator of the form $a \otimes I_{\ell_2}$ (with $a \in \mathcal{B}$) is completely contractive. Then, $a \otimes I_{\ell_2}$ is completely contractive whenever $a \in \overline{\mathcal{B}}$. Indeed, the unit ball of $CB(X, Y)$ (here, X and Y are operator spaces) is closed in the weak operator topology (in this topology, the net

(T_α) converges to T if $y^*(T_\alpha x) \rightarrow y^*(Tx)$ for any $y^* \in Y^*$ and $x \in X$. To see this, note that

$$\|T\|_{cb} = \sup_{x \in M_n(X)} \|I_{M_n} \otimes T(x)\| = \sup\{\|Tv\|_{cb} \mid v \in CB(M_n^*, X), \|v\|_{cb} \leq 1\}$$

(the operator v corresponds to x). On the other hand, embedding Y into $B(K)$ for a suitable Hilbert space K and truncating, we see that

$$\|T\|_{cb} = \sup\{\|uT\|_{cb} \mid u \in CB(Y, M_n), \|u\|_{cb} \leq 1\}.$$

Putting the two together, we obtain:

$$\|T\|_{cb} = \sup\{\|uTv\|_{cb} \mid u \in CB(Y, M_n), \|u\|_{cb} \leq 1, v \in CB(M_n^*, X), \|v\|_{cb} \leq 1\}.$$

Thus, if $T_\alpha \rightarrow T$ in the weak operator topology, and $\|T_\alpha\|_{cb} \leq 1$ for any α , then $\|T\|_{cb} \leq 1$.

It is easy to see that a bounded net $(a_\alpha \otimes I_{\ell_2})$ converges to $a \otimes I_{\ell_2}$ in the weak operator topology if and only if $a_\alpha \rightarrow a$ in the weak* topology.

Finally, we observe that $\overline{\mathcal{B}}$ is bounded, convex, and closed under multiplication. Indeed, the first two properties are easy to verify. Moreover, \mathcal{B} is closed under multiplication, hence the same is true for its strong operator closure. The latter coincides with $\overline{\mathcal{B}}$, by Theorem 5.1.2 of [18].

To define X as in Theorem 2.1, recall that, by [22], there exists a family $(E_i)_{i=1}^\infty$ of finite dimensional operator spaces such that:

- (1) E_i is isometric to $\ell_2^{n_i}$ for some $n_i \in \mathbb{N}$, and $\{i \mid n_i = j\}$ is infinite for any $j \in \mathbb{N}$.
- (2) For any operator $u : E_i^* \rightarrow E_j$, we have $\|u\|_1 / (4 + 2^{-i}) \leq \|u\|_{cb} \leq \|u\|_1$ if $i = j$, $\|u\|_{cb} = \|u\|_2$ if $i \neq j$ (here, $\|v\|_p$ denotes the norm of v in the Schatten class S_p).

Denote by \mathcal{K} the space of compact operators on ℓ_2 . H stands for the Hilbert space $\ell_2 \otimes \ell_2$. Find a sequence of operators $u_i : H \rightarrow \ell_2^{n_i}$ such that $\|u_i\|_2 = 1$ and, for any $\varepsilon > 0$, $n \in \mathbb{N}$, and $u : H \rightarrow \ell_2^n$, there exists $i \in \mathbb{N}$ for which $n_i = n$ and $\|u_i - u\|_1 < \varepsilon$. On the Banach space level, we identify the range of u_i with E_i described above.

We define the operator space X as follows: for $x \in H \otimes \mathcal{K}$, let

$$(2.1) \quad \|x\|_{X \otimes \mathcal{K}} = \sup \{ \|(u_i(a \otimes I_{\ell_2}) \otimes I_{\mathcal{K}})x\|_{E_i \otimes \mathcal{K}} \mid i \in \mathbb{N}, a \in \mathcal{B} \}.$$

Clearly, X is an operator space (Ruan's axioms are satisfied), X is isometric to H as a Banach space, and $a \otimes I_{\ell_2}$ is completely contractive whenever $a \in \mathcal{B}$. Moreover, Hilbert-Schmidt operators into X are completely bounded:

Lemma 2.3. *If Y is an operator space isometric to ℓ_2 and $T : Y \rightarrow X$ is a Hilbert-Schmidt operator, then $\|T\|_{cb} \leq \|T\|_2$.*

Proof. By (2.1),

$$\|T\|_{cb} = \sup\{\|u_i(a \otimes I_{\ell_2})T\|_{cb} \mid i \in \mathbb{N}, a \in \mathcal{B}\} \leq \sup\{\|u_i(a \otimes I_{\ell_2})T\|_1 \mid i \in \mathbb{N}, a \in \mathcal{B}\}.$$

However, $\|u_i(a \otimes I_{\ell_2})T\|_1 \leq \|u_i\|_2 \|a \otimes I_{\ell_2}\| \|T\|_2 = \|T\|_2$ for such i and a . \blacksquare

Below, we obtain lower estimates for c.b. norms of operators on X . The following lemma is one of our main tools.

Lemma 2.4. *Suppose Y is a subspace of X . Consider the operators $T : Y \rightarrow X$, $u : X \rightarrow \ell_2^n$, and $v : \ell_2^n \rightarrow Y$, such that $\|u\|_2 = \|v\| = 1$. Let $C = \sup\{\|u(a \otimes I_{\ell_2})v\|_1 \mid a \in \mathcal{B}\}$. Then $\|T\|_{cb} \geq \|uTv\|_1 / (4 \max\{C, 1\})$.*

Proof. By a small perturbation argument, we can assume that $n = n_i$, and $u = u_i$ (we identify ℓ_2^n with E_i). We view v as a map from E_i^* to X . By (2.1),

$$\|v\|_{cb} = \sup\{\|u_j(a \otimes I_{\ell_2})v\|_{cb} \mid j \in \mathbb{N}, a \in \mathcal{B}\}$$

If $i = j$, then $\|u_j(a \otimes I_{\ell_2})v\|_{cb} \leq \|u_j(a \otimes I_{\ell_2})v\|_1 = C$. If $j \neq i$,

$$\|u_j(a \otimes I_{\ell_2})v\|_{cb} \leq \|u_j(a \otimes I_{\ell_2})v\|_2 \leq \|u_j\|_2 \|a \otimes I_{\ell_2}\| \|v\| = 1.$$

Therefore, $\|v\|_{cb} \leq \max\{C, 1\}$.

By (2.1), $\|u_i\|_{cb} = 1$, and therefore,

$$\|T\|_{cb} \geq \frac{\|u_iTv\|_{cb}}{\|u_i\|_{cb}\|v\|_{cb}} \geq \frac{\|uTv\|_1}{(4 + 2^{-i}) \max\{C, 1\}}.$$

However, i can be chosen to be arbitrarily large. \blacksquare

Corollary 2.5. *Suppose Y is a subspace of X , P and Q are orthogonal projections on subspaces of X and Y , respectively, and $P(a \otimes I_{\ell_2})Q = 0$ for any $a \in \mathcal{B}$. Then $\|PTQ\|_2 \leq 4\|T\|_{cb}$ for any $T \in B(Y, X)$.*

Proof. It suffices to consider P and Q as above having finite (and equal) rank. Let $u = (PTQ)^* / \|PTQ\|_2$. Then $u(a \otimes I_{\ell_2})Q = 0$ for any $a \in \mathcal{B}$, hence the constant C from Lemma 2.4 equals 0. Therefore, by Lemma 2.4, $\|uTQ\|_1 \leq 4\|T\|_{cb}$. However, $\|uTQ\|_1 = \|PTQ\|_2$, which implies the desired inequality. \blacksquare

Proof of Theorem 2.1. By (2.1) and stability of \mathcal{B} under product, $a \otimes I_{\ell_2}$ is completely contractive whenever $a \in \mathcal{B}$. Hence, by Remark 2.2 and Lemma 2.3, $\|a \otimes I_{\ell_2} + S\|_{cb} \leq \|a\| + \|S\|_2$ for $a \in \mathcal{A}$ and $S \in S_2$.

Now fix $T \in CB(X)$. By Corollary 2.5, $\|PT(I - P)\|_2 \leq 4\|T\|_{cb}$ whenever the projection P commutes with $\mathcal{B} \otimes I_{\ell_2}$. In particular, this inequality must hold for all projections P commuting with the hyperfinite von Neumann algebra $B(\ell_2) \otimes I_{\ell_2}$. By the averaging argument from the proof of Theorem 5.2 in [22], there exists $a \in B(\ell_2)$ for which $\|T - a \otimes I_{\ell_2}\|_2 \leq 16\|T\|_{cb}$. Such an a is unique, since $\|a \otimes I_{\ell_2}\|_2 = \infty$ for any $a \in B(\ell_2) \setminus \{0\}$. Moreover, $\rho : a \otimes I_{\ell_2} + S \mapsto a \otimes I_{\ell_2}$ is a bounded algebraic

homomorphism on $CB(X)$. It remains to describe $a \in B(\ell_2)$ for which $a \otimes I_{\ell_2} \in CB(X)$.

More precisely, we show that $\|T\|_{cb} \geq |f(a)|/4$, with $T = a \otimes I_{\ell_2} + S$, whenever $a \in B(\ell_2)$, $S \in S_2$, and $f \in B(\ell_2)_*$ satisfies

$$(2.2) \quad |f(b)| < 1 \text{ whenever } b \in \mathcal{B}.$$

Identifying $B(\ell_2)_*$ with the trace class, write $f = \sum_{i=1}^{\infty} \xi_i \otimes \eta_i$, where (ξ_i) and (η_i) are orthogonal systems, and $\|\xi_i\| = 1$ for every i . In other words, $f(b) = \sum_{i=1}^{\infty} \langle b\xi_i, \eta_i \rangle$ for $b \in B(\ell_2)$. Moreover, $\sum_i \|\xi_i\| \|\eta_i\| < \infty$, and \mathcal{B} is bounded, hence we may assume that $f = \sum_{i=1}^n \xi_i \otimes \eta_i$.

We identify $H = \ell_2 \otimes \ell_2$ with $\ell_2 \otimes (\ell_2 \otimes \ell_2)$, and $a \otimes I_{\ell_2}$ with $a \otimes I_{\ell_2} \otimes I_{\ell_2}$. Let $(\delta_i)_{i \in \mathbb{N}}$ be the canonical basis in ℓ_2 . Select an integer $m \geq n^2 c^2$, where $c = (\sum_{i=1}^n \|\eta_i\|^2)^{1/2}$. Fix $\varepsilon > 0$. By compactness of S , there exists $N \in \mathbb{N}$ such that $\|S(\xi \otimes \delta_j)\| < \varepsilon \|\xi\| / \sqrt{m}$ for any $j \geq N$ and $\xi \in \ell_2 \otimes \ell_2$. Define the operators $u : X \rightarrow \ell_2^m$ and $v : \ell_2^m \rightarrow X$ by setting, for $1 \leq i \leq m$,

$$u^* e_i = \frac{1}{c\sqrt{mn}} \sum_{k=1}^n \eta_k \otimes \delta_k \otimes \delta_{i+N} \quad \text{and} \quad v e_i = \frac{1}{\sqrt{n}} \sum_{k=1}^n \xi_k \otimes \delta_k \otimes \delta_{i+N},$$

where $(e_i)_{i=1}^m$ is the canonical basis of ℓ_2^m . Clearly, $\|u\|_2 = \|v\| = 1$. Moreover,

$$\langle e_j, u(b \otimes I_{\ell_2} \otimes I_{\ell_2}) v e_i \rangle = \langle u^* e_j, (b \otimes I_{\ell_2} \otimes I_{\ell_2}) v e_i \rangle = \begin{cases} f(b)/(cn\sqrt{m}) & i = j \\ 0 & i \neq j \end{cases}$$

for any $b \in B(\ell_2)$. In particular, $\|u(b \otimes I_{\ell_2} \otimes I_{\ell_2}) v\|_1 = c'|f(b)|$, where $c' = \sqrt{m}/(cn) \geq 1$. Moreover, $\|S v e_i\| \leq \varepsilon / \sqrt{m}$ for $1 \leq i \leq m$, hence

$$\|u S v\|_1 \leq \|u\|_2 \|S v\|_2 = \left(\sum_{i=1}^m \|S v e_i\|^2 \right)^{1/2} \leq \varepsilon.$$

Therefore,

$$\|u T v\|_1 \geq \|u(a \otimes I_{\ell_2} \otimes I_{\ell_2}) v\|_1 - \|u S v\|_1 > c'|f(a)| - \varepsilon.$$

However, $\|u(b \otimes I_{\ell_2} \otimes I_{\ell_2}) v\|_1 \leq c'$ for $b \in \mathcal{B}$. Lemma 2.4 shows that

$$\|T\|_{cb} \geq \frac{\|u T v\|_1}{4 \max\{c', 1\}} > \frac{c'|f(a)| - \varepsilon}{4c'} = \frac{|f(a)|}{4} - \frac{\varepsilon}{4c'}.$$

Moreover, $\varepsilon > 0$ is arbitrary, hence $\|T\|_{cb} \geq |f(a)|/4$. This inequality holds for any $f \in B(\ell_2)_*$ satisfying (2.2). Hence, by the bipolar theorem, $\|T\|_{cb} \geq \|a\|/4$. \blacksquare

Remark 2.6. We say that a subspace A of $B(K)$ (K is a Hilbert space) is *Hilbert-Schmidt hyperreflexive* if there exists a constant $c > 0$ such that, for any $b \in B(\ell_2)$,

$$\inf\{\|b - a\|_2 \mid a \in A\} \leq c \sup \|ubv\|_1,$$

where the supremum runs over all operators $u \in B(K, \ell_2^n)$ and $v \in B(\ell_2^n, K)$, for which $\|u\|_2 = \|v\| = 1$, and $uAv = 0$. Examining the proof of Corollary 2.5, one can see that the above inequality holds if

$$\inf\{\|b - a\|_2 \mid a \in A\} \leq c \sup \|Pb(I - P)\|_2,$$

where the supremum runs over all orthogonal projections P acting on K and satisfying $PA(I - P) = 0$. The proof of Theorem 2.1 shows that $B \otimes I_{\ell_2}$ is Hilbert-Schmidt hyperreflexive if B is a weak* closed subspace of $B(\ell_2)$, and $K = \ell_2 \otimes \ell_2$. It was shown in [22] that any injective unital von Neumann subalgebra of $B(\ell_2)$ is Hilbert-Schmidt hyperreflexive.

The notion of Hilbert-Schmidt hyperreflexivity is a variation on the theme of hyperreflexivity. Recall that a subspace A of $B(K)$ is *hyperreflexive* if, for some $c > 0$ and any $b \in B(K)$,

$$\inf\{\|b - a\| \mid a \in A\} \leq c \sup \|Pb(I - P)\|$$

(as before, we take the supremum over all orthogonal projections P acting on K and satisfying $PA(I - P) = 0$). W. Arveson [3] showed that nest algebras are hyperreflexive (see also Chapter 9 of [11], or [12]).

Proof of Theorem 1.1. For the sake of simplicity, let $A_0 = \mathbb{C}I$ and $\gamma_0 = 1$. Denote by \mathcal{B} the convex hull of $\cup_{k \geq 0} \{a \in A_k \mid \|a\| \leq \gamma_k\}$. It is easy to check that \mathcal{B} satisfies the assumption of Theorem 2.1. Namely, \mathcal{B} is a subset of the unit ball of $B(\ell_2)$, containing I , and $\lambda a + \mu b$ and ab belong to \mathcal{B} whenever $a, b \in \mathcal{B}$ and $|\lambda| + |\mu| \leq 1$. For instance, to check the stability of \mathcal{B} under multiplication, select two elements of \mathcal{B} : $a = \sum_k \gamma_k a_k$ and $b = \sum_k \gamma_k b_k$, where a_k, b_k are elements of the space A_k such that $\sum_k \|a_k\|, \sum_k \|b_k\| \leq 1$ (both sums are finite). Then $ab = \sum_k \gamma_k c_k$, where

$$c_k = \sum_{i+j=k} \frac{\gamma_i \gamma_j}{\gamma_k} a_i b_j \in A_k.$$

However, $\gamma_i \gamma_j \leq \gamma_{i+j}$, thus

$$\sum_k \|c_k\| \leq \sum_k \sum_{i+j=k} \|a_i\| \|b_j\| = \sum_i \|a_i\| \cdot \sum_j \|b_j\| \leq 1,$$

which implies $ab \in \mathcal{B}$.

We define X as in Theorem 2.1, with \mathcal{B} described above. The statements (2) and (3) easily follow from Theorem 2.1. Furthermore, any completely bounded operator on X is of the form $a \otimes I_{\ell_2} + S$, with $a \in B(\ell_2)$ and $\|S\|_2 \leq 16\|T\|_{cb}$. To show that $a \in \overline{\cup_k A_k}$ and to prove the minoration result from part (1) of our theorem, let F_k be the set of functionals $f \in B(\ell_2)_*$ s.t. $\|f\| \leq 1/\gamma_{k+1}$ and $f|_{A_k} = 0$. By construction, $|f(b)| \leq 1$ if $f \in F_k$ and $b \in \overline{\mathcal{B}}$. Then, by the estimate preceding (2.2),

$$\|a \otimes I_{\ell_2} + S\|_{cb} \geq \frac{1}{4} \sup_{f \in F_k} |f(a)| = \frac{1}{4\gamma_{k+1}} \inf_{b \in A_k} \|b - a\|$$

for any $k \in \mathbb{N}$ (the last equality follows from Hahn-Banach theorem). In particular, for every k there exists $b_k \in A_k$ satisfying $\|b_k - a\| \leq 8\gamma_{k+1}\|a \otimes I_{\ell_2} + S\|_{cb}$. Then $a = \lim_k b_k$, which implies $a \in \overline{\bigcup_k A_k} = \mathcal{A}$. This yields (1).

To prove the last part of the theorem, fix $a_1, a_2 \in K$. Then there exist sequences $(a_{in})_{n \in \mathbb{N}}$ ($i = 1, 2$) such that $a_{in} \in A_n$ for any n , and $\|a_i - a_{in}\| < 4^{-n}$. Denote by N the smallest positive integer satisfying $\|a_1 - a_2\| \geq 2 \cdot 4^{-N}$. Then $\|a_{1N} - a_{2N}\| \leq 2\|a_1 - a_2\|$, and $\|a_{i,n+1} - a_{in}\| \leq 5 \cdot 4^{-(n+1)}$ for $n \geq N$ and $i = 1, 2$. Moreover,

$$\|a_{i,n+1} \otimes I_{\ell_2} - a_{in} \otimes I_{\ell_2}\|_{cb} \leq \frac{1}{\gamma_{n+1}} \|a_{i,n+1} - a_{in}\| < \frac{5}{2^{n+1}},$$

and similarly,

$$\|a_{1N} \otimes I_{\ell_2} - a_{2N} \otimes I_{\ell_2}\|_{cb} \leq 2^N \|a_{1N} - a_{2N}\|.$$

By the triangle inequality,

$$\begin{aligned} & \|a_1 \otimes I_{\ell_2} - a_2 \otimes I_{\ell_2}\|_{cb} \\ & \leq \|a_{1N} \otimes I_{\ell_2} - a_{2N} \otimes I_{\ell_2}\|_{cb} + \sum_{i=1,2} \sum_{n=N}^{\infty} \|a_{i,n+1} \otimes I_{\ell_2} - a_{in} \otimes I_{\ell_2}\|_{cb} \\ & \leq 2^N \|a_{1N} - a_{2N}\| + \frac{10}{2^N} \leq 2^{N+1} \|a_1 - a_2\| + \frac{10}{2^N}. \end{aligned}$$

If $N = 1$, then

$$\|a_1 \otimes I_{\ell_2} - a_2 \otimes I_{\ell_2}\|_{cb} \leq 4\|a_1 - a_2\| + 5 \leq 44\|a_1 - a_2\|.$$

If $N > 1$, then $2 \cdot 4^{-N} \leq \|a_1 - a_2\| < 8 \cdot 4^{-N}$, hence $2^{-N} \leq \|a_1 - a_2\|^{1/2}$, and $2^N \leq 4\|a_1 - a_2\|^{-1/2}$. Thus,

$$\|a_1 \otimes I_{\ell_2} - a_2 \otimes I_{\ell_2}\|_{cb} \leq 18\|a_1 - a_2\|^{1/2}.$$

Combining these two inequalities, we complete the proof. ■

Proof of Theorem 1.2. Apply Theorem 2.1 with $\mathcal{B} = \pi(B_{\mathcal{A}})$. ■

3. APPLICATIONS OF THEOREM 1.1

In this section we use Theorem 1.1 to construct several Hilbertian operator spaces with unusual properties.

First recall that the ℓ_{∞} direct sum (or, simply, the direct sum) $X \oplus Y$ of operator spaces X and Y coincides with $X \oplus_{\infty} Y$ as a Banach space, and has the operator space structure defined by setting, for $x = \sum_i a_i \otimes x_i + \sum_j b_j \otimes y_j \in M_n(X \oplus Y)$ (here $a_i, b_j \in M_n$, $x_i \in X$, and $y_j \in Y$), $\|x\| = \max\{\|\sum_i a_i \otimes x_i\|_{M_n(X)}, \|\sum_j b_j \otimes y_j\|_{M_n(Y)}\}$. An operator space X is called *completely indecomposable* if it is not completely isomorphic to a direct sum of two infinite dimensional operator spaces (in other words, if $P \in CB(X)$ is a projection, then either the kernel or the range of P is finite dimensional).

The definition of complete indecomposability is inspired by Banach space theory. A Banach space is called *indecomposable* if it is not isomorphic to a direct sum of two infinite dimensional Banach spaces. For a long time, no examples of indecomposable Banach spaces were known. In 1993, T. Gowers and B. Maurey [16] constructed a Banach space X all of whose infinite dimensional subspaces are indecomposable. Later, examples of indecomposable spaces containing unconditional basic sequences were given (see e.g. [1], [2]).

Below, we consider a non-commutative analogue of the same problem. More precisely, we construct a completely indecomposable operator space, containing a subspace with a prescribed algebra of c.b. maps.

Proposition 3.1. *Suppose \mathcal{N} is a Hilbert-Schmidt hyperreflexive weak* closed unital subalgebra of $B(\ell_2)$ (see Remark 2.6 for the definition). For every $\varepsilon > 0$ there exists a completely indecomposable operator space X , isometric to ℓ_2 , and containing uncountably many infinite dimensional subspaces Y , not completely isomorphic to each other, such that $T \in CB(Y, X)$ if and only if $T = J_Y T_0 + S$, with $T_0 \in \mathcal{N}$ and $S \in S_2$ (here, we identify \mathcal{N} with a subalgebra of $B(Y)$, and J_Y is the canonical embedding of Y into X). For such T_0 and S , $\|J_Y T_0 + S\|_{CB(Y, X)} \leq (1 + \varepsilon)\|T_0\| + \|S\|_2$. Moreover, for any $T \in CB(Y, X)$ there exists $\tilde{T} \in CB(X)$ such that $\tilde{T}|_Y = T$.*

Perhaps the most interesting application of this result occurs if we take $\mathcal{N} = B(\ell_2)$. Then we obtain a completely indecomposable operator space X , containing uncountably many infinite dimensional subspaces Y , not completely isomorphic to each other, such that every bounded operator T on Y is completely bounded, with $\|T\|_{CB(Y)} \leq (1 + \varepsilon)\|T\|$. Moreover, an inspection of the proof shows that any such T extends to an operator \tilde{T} on X , with $\|\tilde{T}\|_{CB(X)} \leq (1 + \varepsilon)\|T\|$. Finally, although there are no non-trivial projections on X (that is, if $P \in CB(X)$ satisfies $P^2 = P$, then either the domain or the range of P is finite dimensional), the space $CB(X)$ is still rich with operators. In particular, both $CB(X)$ and $CB(X)/S_2$ contain a $(1 + \varepsilon)$ -isomorphic copy of $B(\ell_2)$.

Proof. Denote by L the separable Hilbert space $(\sum_{i=-1}^{\infty} L_i)_2$ (direct sum), where the spaces L_i are copies of ℓ_2 . Let z_0, z_1, z_2, \dots be a sequence of distinct points, dense in the unit circle \mathbb{T} , with $z_0 = 1$. For $k \in \mathbb{N}$ denote by B_k the set of block-diagonal operators $\phi_k(a)$ (with $a \in \mathcal{N}$), defined by setting $\phi_k(a)|_{L_i} = z_i^k a$ for $i \geq 0$, $\phi_k(a)|_{L_{-1}} = 0$. Let $A_0 = B_0 = \mathbb{C}I$, $A_k = \text{span}[\mathbb{C}I, B_1, \dots, B_k]$, and $\gamma_k = (1 + \varepsilon)^{-k}$.

First show that the spaces A_k are weak* closed. To this end, suppose a net $(u_\alpha) \subset A_k$ converges weak*. Write $u_\alpha = \sum_{j=0}^k \phi_j(a_{j\alpha})$. Passing to blocks, one sees that, for every i , $\sum_{j=0}^k z_i^j a_{j\alpha}$ converges weak*. The matrix $(z_i^j)_{i,j=0}^n$ has non-zero

determinant, hence there exists a matrix $(t_{ij})_{i,j=0}^n$ s.t.

$$a_{j\alpha} = \sum_{i=1}^k t_{ij} \left(\sum_{j=0}^k z_i^j a_{j\alpha} \right).$$

Therefore, the net $(a_{j\alpha})$ converges weak* to some $a_j \in \mathcal{N}$ for any j , and the net (u_α) converges weak* to $\sum_{j=0}^k \phi_j(a_j) \in A_k$.

Applying Theorem 1.1, we obtain an operator space X , isometric to $L \otimes \ell_2$, such that every c.b. map on X is a Hilbert-Schmidt perturbation of an element of $\mathcal{A} \otimes I_{\ell_2}$, where $\mathcal{A} = \overline{\cup_k A_k}$. Fix a non-zero $\xi \in \ell_2$, and consider $Y = Y(\xi) = L_0 \otimes \xi$ as a subspace of X . If $T_0 \in \mathcal{N}$, we can view $J_Y T_0$ as a restriction of $\phi_1(T_0) \in A_1$ to Y , hence

$$\|J_Y T_0\|_{CB(Y,X)} \leq \|\phi_1(T_0)\|_{cb} \leq (1 + \varepsilon) \|T_0\|.$$

Therefore, $\|J_Y T_0 + S\|_{cb} \leq (1 + \varepsilon) \|T_0\| + \|S\|_2$ for any $T_0 \in \mathcal{N}$ and $S \in S_2$. In this case, $\tilde{T} = \phi_1(T_0) + S$ is a completely bounded extension of T to the whole space X .

Next we show that any operator $T \in B(Y, X)$ which cannot be represented as $J_Y T_0 + S$ (with $T_0 \in \mathcal{N}$ and $S \in S_2$) is not completely bounded. To this end, we show that for every $C > 0$ there exist operators $u \in B(X, \ell_2^n)$ and $v \in B(\ell_2^n, Y)$, such that

$$(3.1) \quad \|u\|_2 = \|v\| = 1, \quad \|uTv\|_1 \geq C, \quad \text{and} \quad uJ_Y T_0 v = 0 \quad \text{for any } T_0 \in \mathcal{N}$$

(hence $u(a \otimes I_{\ell_2})v = 0$ for any $a \in \mathcal{B}$, with \mathcal{B} defined as in the proof of Theorem 1.1). Indeed, for any $T_0 \in \mathcal{N}$,

$$T - J_Y T_0 = P_Y(T - J_Y T_0) + P_{Y^\perp}(T - J_Y T_0) = (P_Y T - T_0) + P_{Y^\perp} T,$$

where P_Y and P_{Y^\perp} are the orthogonal projection from X onto Y and Y^\perp , respectively. Moreover, the operators $P_Y T - T_0$ and $P_{Y^\perp} T$ have mutually orthogonal ranges, hence

$$\|T - J_Y T_0\|_2^2 = \|P_Y T - T_0\|_2^2 + \|P_{Y^\perp} T\|_2^2.$$

By our assumption, $\|T - J_Y T_0\|_2 = \infty$ whenever $T_0 \in \mathcal{N}$, hence either $\|P_{Y^\perp} T\|_2 = \infty$, or $\|P_Y T - T_0\|_2 = \infty$ for any such T_0 . In the first case, we have $\infty = \|P_{Y^\perp} T\|_2 = \sup \|u P_{Y^\perp} T v\|_1$, where the supremum runs over all $v : \ell_2^n \rightarrow Y$ and $u : X \rightarrow \ell_2^n$ such that $u|_Y = 0$, and $\|u\|_2 = \|v\| = 1$. Thus, for any $C > 0$ there exist u and v satisfying (3.1). In the second case, the Hilbert-Schmidt hyperreflexivity of \mathcal{N} guarantees that for every $C > 0$ there exist $u : Y \rightarrow \ell_2^n$ and $v : \ell_2^n \rightarrow Y$ for which (3.1) holds.

If (3.1) holds, then, by Lemma 2.4, $\|T\|_{cb} \geq C/4$. However, C in (3.1) can be arbitrarily large, hence T is not completely bounded. Therefore, $T \in B(Y, X)$ is completely bounded if and only if $T = J_Y T_0 + S$ for some $T_0 \in \mathcal{N}$ and $S \in S_2$.

Clearly, the spaces $Y(\xi_1)$ and $Y(\xi_2)$ coincide if the vectors ξ_1 and ξ_2 are colinear. Next we prove that $Y(\xi_1)$ and $Y(\xi_2)$ are not completely isomorphic to each other, provided ξ_1 and ξ_2 are not colinear. Indeed, suppose $\|\xi_1\| = \|\xi_2\| = 1$, and $|\langle \xi_1, \xi_2 \rangle| <$

1. Denote by P the orthogonal projection onto $Y(\xi_1)^\perp$. Any element of $Y(\xi_2)$ is of the form $\eta \otimes \xi_2$ ($\eta \in L_0$), and $\|(I - P)\eta \otimes \xi_2\| = \|\eta\| |\langle \xi_1, \xi_2 \rangle|$. Therefore, by the Pythagorean theorem,

$$\|P\eta \otimes \xi_2\|^2 = \|\eta \otimes \xi_2\|^2 - \|(I - P)\eta \otimes \xi_2\|^2 = (1 - |\langle \xi_2, \xi_2 \rangle|^2) \|\eta\|^2,$$

which means that the restriction of P to $Y(\xi_2)$ is an isomorphism. Consequently, if $T : Y(\xi_1) \rightarrow Y(\xi_2)$ is a c.b. map with a bounded inverse, then PT also has a bounded inverse. However, $T = J_{Y(\xi_1)}T_0 + S$ (with $T_0 \in \mathcal{N}$ and $S \in S_2$), hence $PT = PS$ is a Hilbert-Schmidt operator. This contradiction shows that the spaces $Y(\xi_1)$ and $Y(\xi_2)$ are not completely isomorphic to each other.

Finally, we show that the space X is completely indecomposable. Suppose, for the sake of contradiction, that a projection $P \in CB(X)$ has infinite dimensional kernel and range. Then $\rho(P)$ is also a projection (ρ is an algebraic homomorphism, hence $\rho(P) = \rho(P^2) = \rho(P)^2$), and $P - \rho(P)$ is Hilbert-Schmidt. By the results of Section 2.c of [20], $\rho(P) = P + (P - \rho(P))$ has infinite dimensional kernel and range. Thus, we can assume without loss of generality that $P = Q \otimes I_{\ell_2}$, where Q is a projection on L .

Observe that, for any $k \geq 0$, the space A_k is isometric to a subspace of $C(\mathbb{T}, \mathcal{N})$ (the space of \mathcal{N} -valued continuous functions on the unit circle \mathbb{T}). Indeed, let $A = \cup_k A_k$ (this space need not be closed), and define $\psi : A \rightarrow C(\mathbb{T}, \mathcal{N})$ by setting $\psi(I) = \mathbf{1}$, and $\psi(\phi_j(a))(z) = z^j a$ for $j \geq 1$ and $a \in \mathcal{N}$. Note that ψ is an algebraic homomorphism, and an isometry. Clearly, $\psi(B_k)$ is contractively complemented in $C(\mathbb{T}, \mathcal{N})$. Hence, for any $k \geq 0$ there exists a contractive projection P_k from A onto B_k . Moreover, $P_k P_j = 0$ if $k \neq j$, and $\sum_{k=0}^{\infty} P_k = I$.

Let $C = 4\|Q \otimes I_{\ell_2}\|_{cb}$. By Theorem 1.1, for any $k \in \mathbb{N}$ there exists $a_k \in A_k$ s.t. $\|a_k - Q\| < C(1 + \varepsilon)^{-(k+1)}$. For $j \geq 0$ let $a_{jk} = P_j a_k$ (then $a_{jk} = 0$ for $j > k$). By the triangle inequality, $\|a_{k+1} - a_k\| < 2C(1 + \varepsilon)^{-(k+1)}$, hence $\|a_{j,k+1} - a_{jk}\| < 2C(1 + \varepsilon)^{-(k+1)}$ for any j . Therefore, for any j there exists $b_j = \lim_{k \rightarrow \infty} a_{jk} \in B_j$, and

$$\|a_{jk} - b_j\| \leq \sum_{n=k}^{\infty} \|a_{j,n+1} - a_{jn}\| < 2C\varepsilon^{-1}(1 + \varepsilon)^{-k}$$

whenever $k \geq 0$. In particular, $\|b_j\| \leq \|a_{j,j-1} - b_j\| < 2C\varepsilon^{-1}(1 + \varepsilon)^{1-j}$, hence we can define $b = \sum_{j=0}^{\infty} b_j$ (the series converges absolutely). We have

$$\|a_k - b\| \leq \sum_{j=0}^k \|a_{jk} - b_j\| + \sum_{j=k+1}^{\infty} \|b_j\| < 2C\varepsilon^{-1}(1 + \varepsilon)^{-k}((k+1) + \varepsilon^{-1}(1 + \varepsilon)^{-1}),$$

and therefore, $\lim_k \|a_k - b\| = 0$. Thus, $b = Q$.

Note that $a_{0k} = c_k I$ (with $c_k \in \mathbb{C}$), hence the restriction of a_k to $L_{-1} \otimes \ell_2$ equals $c_k I$. Moreover, $Q^2 = Q$, hence $\lim_{k \rightarrow \infty} \|a_k - a_k^2\| = 0$. In particular, $\lim_{k \rightarrow \infty} |c_k - c_k^2| = 0$,

which means that b_0 equals either I or 0 . Passing from Q to $I - Q$ if necessary, we can assume that $b_0 = 0$.

We complete the proof by showing that $b_n = 0$ for every $n \in \mathbb{N}$. Indeed, find the smallest n for which $b_n \neq 0$. Let $T = \sum_{j=0}^N b_j$, where $N \geq n$ is so large that $\|T^2 - T\| < \|b_n\|$. However, $P_n(T) = b_n$, and $P_n(T^2) = 0$ (to see this, consider $\psi(T^2)$). Hence, $\|b_n\| = \|P_n(T^2 - T)\| < \|b_n\|$, a contradiction. \blacksquare

In order to construct two more examples, we introduce some notation. For an operator space X , we denote by $CI(X)$ the set of *completely invertible* operators on X – that is, operators T such that both T and T^{-1} are completely bounded. We use X^n for $X \oplus \dots \oplus X$ (n times).

Proposition 3.2. *There exists an operator space X , isometric to ℓ_2 , and $T \in CI(X)$, such that:*

- (1) *There exists a continuous path in $CI(X)$, connecting T^2 to I_X*
- (2) *There is no $n \in \mathbb{N}$ for which there exists a continuous path in $CI(X^n)$, connecting $T \oplus I_{X^{n-1}}$ with I_{X^n} .*

Before proving this result, recall some facts concerning K -theory of Banach algebras. For more information, the reader is referred to [6].

Suppose \mathcal{A} is a Banach algebra with the identity I . We denote by $M_n(\mathcal{A})$ the algebra of $n \times n$ matrices with entries from \mathcal{A} , with obvious multiplication. For $n \in \mathbb{N}$, $IP_n(\mathcal{A})$ denotes the set of idempotents (or projections) in $M_n(\mathcal{A})$. We let $IP_\infty(\mathcal{A}) = \cup_{n \in \mathbb{N}} IP_n(\mathcal{A})$. We say that $p \in IP_m(\mathcal{A})$ and $q \in IP_n(\mathcal{A})$ are *0-equivalent* ($p \sim_0 q$) if there exist $u \in M_{m+k, n+k}(\mathcal{A})$ and $v \in M_{n+k, m+k}(\mathcal{A})$ satisfying

$$uv = \begin{pmatrix} p & 0 \\ 0 & I^{(k)} \end{pmatrix}, \quad vu = \begin{pmatrix} q & 0 \\ 0 & I^{(k)} \end{pmatrix}$$

(here $I^{(k)}$ is the element of $M_k(\mathcal{A})$, consisting of k copies of I on the diagonal, and zeroes elsewhere). The 0-equivalence class of p is denoted by $[p]_0$. Addition is defined as follows:

$$[p]_0 + [q]_0 = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right]_0.$$

Then the (Abelian) group $K_0(\mathcal{A})$ can be represented as the set of all differences $[p]_0 - [q]_0$, with $p, q \in IP_\infty(\mathcal{A})$.

To define the group $K_1(\mathcal{A})$, consider the set $GL_n(\mathcal{A})$ of all invertible elements of $M_n(\mathcal{A})$, and let $GL_\infty(\mathcal{A}) = \cup_n GL_n(\mathcal{A})$. We say that $U \in GL_m(\mathcal{A})$ and $V \in GL_n(\mathcal{A})$ are *1-equivalent* ($U \sim_1 V$) if, for some $k \geq \max\{m, n\}$, there exists a continuous path $h : [0, 1] \rightarrow GL_k(\mathcal{A})$ such that

$$h(0) = \begin{pmatrix} U & 0 \\ 0 & I^{(k-m)} \end{pmatrix}, \quad vu = \begin{pmatrix} V & 0 \\ 0 & I^{(k-n)} \end{pmatrix}.$$

$[U]_1$ denotes the equivalence class of U . The group $K_1(\mathcal{A})$ is the set of equivalence classes $\text{GL}_\infty(\mathcal{A})/\sim_1$, equipped with addition

$$[U]_1 + [V]_1 = \left[\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \right]_1.$$

Now we can re-phrase Proposition 3.2 in terms of K -theory: we establish that $[T^2]_1 = [T]_1^2 = [I_X]_1$, yet $[T]_1 \neq [I_X]_1$. Therefore, $K_1(CB(X))$ contains a copy of \mathbb{Z}_2 . In particular, this group has torsion. Note that no examples of Banach spaces X for which $K_1(B(X))$ has torsion are known (see [19], where several examples of Banach spaces with exotic K_0 and K_1 groups are given).

For our construction, we use the Cuntz algebra \mathcal{O}_n ($n \geq 2$) – the C^* -algebra of operators on ℓ_2 generated by the identity I and isometries s_1, s_2, \dots, s_n s.t. $s_i^* s_i = I$, the projections $s_i s_i^*$ are mutually orthogonal, and $I = \sum_{i=1}^n s_i s_i^*$.

Proof of Proposition 3.2. It is known (see e.g. [8], or Exercise 10.8.11 of [6]) that $K_0(\mathcal{O}_3) = \mathbb{Z}_2$. By suspension and unitization (combining Theorem 8.2.2 and Corollary 8.3.7 of [6]), we construct a separable unital C^* algebra \mathcal{A} for which $K_1(\mathcal{A}) = K_0(\mathcal{O}_3) = \mathbb{Z}_2$. Passing from \mathcal{A} to $M_n(\mathcal{A})$ if necessary, we obtain an element $x \in \text{GL}_1(\mathcal{A})$ such that it is not 1-equivalent to I , yet there exists a continuous function $h : [0, 1] \rightarrow \text{GL}_1(\mathcal{A})$ s.t. $h(1) = I$ and $h(0) = x^2$. By the GNS construction, we may assume that \mathcal{A} is a unital subalgebra of $B(\ell_2)$.

Let $\gamma_n = (2/3)^n$, and find finite dimensional subspaces $A_1 \hookrightarrow A_2 \hookrightarrow \dots \hookrightarrow \mathcal{A}$ such that $x, x^{-1} \in A_1$, $A_n A_m \subset A_{n+m}$ for $n, m \in \mathbb{N}$, and for any $t \in [0, 1]$, there exist $a_{n1}, a_{n2} \in A_n$ s.t. $\|h(t) - a_{n1}\|, \|h(t)^{-1} - a_{n2}\| < 4^{-n}$ (this is possible, since both $h([0, 1])$ and the set of the inverses of its elements are compact). Applying Theorem 1.1 with

$$K = \{h(t) \mid t \in [0, 1]\} \cup \{h(t)^{-1} \mid t \in [0, 1]\} \cup \{0\},$$

we obtain an operator space X , isometric to $\ell_2 \otimes \ell_2$, such that every $T \in CB(X)$ is of the form $a \otimes I_{\ell_2} + S$, with $a \in \mathcal{A}$ and $S \in S_2$. Moreover, the map $a \mapsto a \otimes I_{\ell_2}$ is uniformly continuous on K (by the last part of Theorem 1.1), hence the function $H : [0, 1] \rightarrow CI(X) : t \mapsto h(t) \otimes I_{\ell_2}$ is continuous. Therefore, $x^2 \otimes I_{\ell_2} \sim_1 I_X$.

Now suppose $x \otimes I_{\ell_2} \sim_1 I_X$, that is, there exist $n \in \mathbb{N}$ and a continuous function $H : [0, 1] \rightarrow CI(X)$ such that $H(0) = x \otimes I_{\ell_2} \oplus I_{X^{n-1}}$, and $H(1) = I_{X^n}$. We know that the map $\phi : a \otimes I_{\ell_2} + S \mapsto a$ (here, $a \in \mathcal{A}$ and $S \in S_2$) defines a bounded algebraic homomorphism from $CB(X)$ to \mathcal{A} . It is easy to see that $\phi_n = I_{M_n} \otimes \phi$ is a bounded algebraic homomorphism from $M_n(CB(X))$ to $M_n(\mathcal{A})$. Then $h = \phi_n \circ H$ is a continuous map from $[0, 1]$ into $\text{GL}_1(\mathcal{A})$ s.t.

$$h(0) = \begin{pmatrix} x & 0 \\ 0 & I^{(n-1)} \end{pmatrix} \text{ and } h(1) = I^{(n)}.$$

This is, however, impossible by construction of x . ■

Remark 3.3. In a similar way, we can use the fact that $K_0(\mathcal{O}_{n+1}) = \mathbb{Z}_n$ ($n \geq 2$) to construct an operator space X for which $K_1(CB(X))$ contains \mathbb{Z}_n as a subgroup.

Proposition 3.4. *Suppose $n \geq 3$. Then there exists a 1-Hilbertian operator space X , such that X^k is completely isomorphic to X^m if and only if $k - m = 0 \pmod{n-1}$.*

Proof. Suppose s_1, \dots, s_n are the isometries from the definition of Cuntz algebra. Let $\gamma_n = (2/3)^n$, and define finite dimensional subspaces $A_1 \hookrightarrow A_2 \hookrightarrow \dots \hookrightarrow \mathcal{O}_n$ as follows: $A_1 = \text{span}[I, s_1, \dots, s_n, s_1^*, \dots, s_n^*]$, $A_n = \text{span}[A_1^n]$ for $n > 1$. Construct the operator space X as in Theorem 1.1. For $1 \leq k \leq n$ let $X_k = (s_k \otimes I_{\ell_2})X$. Then X_k is completely isomorphic to X , since $\|s_k \otimes I_{\ell_2}\|_{cb}, \|s_k^* \otimes I_{\ell_2}\|_{cb} \leq 3/2$. Moreover, X_k is the range of the orthogonal projection $P_k = s_k^* s_k \otimes I_{\ell_2}$, with c.b. norm not exceeding $(3/2)^2$. Hence, X is completely isomorphic to X^n .

Suppose X^k is completely isomorphic to X^m . In other words, there exist $U \in M_{km}(CB(X))$ and $V \in M_{mk}(CB(X))$ such that $UV = I_{X^k}$ and $VU = I_{X^m}$. As in the proof of Proposition 3.2, consider the homomorphism $\phi : CB(X) \rightarrow \mathcal{O}_n$, sending $a \otimes I_{\ell_2} + S$ into a . Denote the maps $I_{km} \otimes \phi$ and $I_{mk} \otimes \phi$ (acting on $M_{km}(CB(X))$ and $M_{mk}(CB(X))$, respectively) by ϕ_{km} and ϕ_{mk} . Let $u = \phi_{km}(U) \in M_{km}(\mathcal{O}_n)$ and $v = \phi_{mk}(V) \in M_{mk}(\mathcal{O}_n)$. Then $uv = I^{(k)}$ and $vu = I^{(m)}$, which implies $k[I]_0 = m[I]_0$. By [8], the last inequality holds if and only if $n - 1$ divides $k - m$. \blacksquare

4. APPLICATIONS OF THEOREM 1.2

In this section we deal with examples of operator spaces with interesting properties, arising from Theorem 1.2.

Proposition 4.1. *Suppose E is the dual of a separable Banach space. Then there exists an operator space X , isometric to ℓ_2 , such that $CB(X)$ is isomorphic to $E \oplus \ell_2$.*

Proof. Note that E is isomorphic to $E' = \mathbb{C} \oplus_1 F$, where F is a dual Banach space (the kernel of any element of E_* can serve as F). We can view E' as a unital Banach algebra, with the multiplication $(\lambda, f) \cdot (\lambda', f') = (\lambda\lambda', \lambda'f + \lambda f')$.

Suppose P and Q are mutually orthogonal projections of infinite rank on ℓ_2 . Define a homomorphism $\pi : E' \rightarrow B(\ell_2)$ by setting $\pi(\lambda, f) = \lambda I + \phi(f)$, where $\phi : F \rightarrow PB(\ell_2)Q$ is a weak* continuous isometry ($PB(\ell_2)Q$ contains ℓ_∞ as a weak* closed subspace, hence such a ϕ exists). π is contractive, unital, faithful, and weak* continuous. An application of Theorem 1.2 completes the proof. \blacksquare

In [22], we gave examples of operator spaces X for which there are precisely n ($n \in \mathbb{N} \cup \{\infty\}$) multiplicative functionals on $CB(X)$. It is easy to see that, if X is completely isomorphic to $X \oplus \dots \oplus X$ (n times), then there are no non-zero multiplicative functionals on $CB(X)$. Indeed, if X is completely isomorphic to $X \oplus \dots \oplus X$, then there exist n pairs of operators $U_i, V_i \in CB(X)$ s.t. $U_i V_i = I_X$ for

any i , and $\sum_{i=1}^n V_i U_i = I_X$. This precludes the existence of non-zero multiplicative functionals, but not the existence of (non-unital) traces on $CB(X)$. Recall that a functional τ , acting on a Banach algebra \mathcal{A} , is called a *trace* if $\tau(ab) = \tau(ba)$ for any $a, b \in \mathcal{A}$.

Proposition 4.2. *There exists an operator space X , isometric to ℓ_2 , such that X is completely isomorphic to $X \oplus X$, and there exists a non-zero trace on $CB(X)$.*

Proof. Below, we work with the *second Cuntz semigroup* \mathcal{C}_2 , consisting of the “zero element” θ , neutral element e , and generators s_1, s_2, s'_1 , and s'_2 , subject to the relations

$$s\theta = \theta s = \theta, \quad se = es = s \text{ for any } s \in \mathcal{C}_2, \quad s'_i s_j = \begin{cases} e & \text{if } i = j \\ \theta & \text{otherwise} \end{cases}.$$

This semigroup was introduced and described in Section III.2 of [26] (see also Chapters 1-2 of [23]). One can see that every element of $\mathcal{C}_2^\square = \mathcal{C}_2 \setminus \{\theta\}$ can be written in a unique way as $s_{\mathbf{i}} s'_{\mathbf{j}}$, where $\mathbf{i} = (i_1, \dots, i_n)$, $\mathbf{j} = (j_1, \dots, j_m)$, $s_{\mathbf{i}} = s_{i_1} \dots s_{i_n}$, and $s'_{\mathbf{j}} = s'_{j_m} \dots s'_{j_1}$, with $i_1, \dots, i_n, j_1, \dots, j_m$ equal to 1 or 2. If $\mathbf{i} = \emptyset$, we let $s_{\mathbf{i}} = e$, with a similar convention governing $s'_{\mathbf{j}}$. Multiplication is subject to cancellation rules:

$$s_{\mathbf{i}} s'_{\mathbf{j}} s_{\mathbf{k}} s'_{\mathbf{l}} = \begin{cases} s_{\mathbf{i}\mathbf{h}} s'_{\mathbf{l}} & \mathbf{k} = \mathbf{j}\mathbf{h} \\ s_{\mathbf{i}} s'_{\mathbf{h}\mathbf{k}} & \mathbf{j} = \mathbf{k}\mathbf{h} \\ \theta & \text{otherwise} \end{cases}.$$

Recall that a semigroup Γ is *inverse* if for any $s \in \Gamma$ there exists a unique element s^* such that $ss^*s = s$ and $s^*s^*s^* = s^*$ (see Chapter 1 of [7] for more information). By the above, \mathcal{C}_2 is an inverse semigroup, with $(s_{\mathbf{i}} s'_{\mathbf{j}})^* = s_{\mathbf{j}} s'_{\mathbf{i}}$.

Let $\mathcal{C}_2^\square = \mathcal{C}_2 \setminus \{\theta\}$, and observe that $\ell_1(\mathcal{C}_2^\square)$, equipped with convolution $\widehat{\star}$, is a unital Banach algebra. Here, $\widehat{\star}$ is defined by setting, for $s, t \in \mathcal{C}_2^\square$,

$$\delta_s \widehat{\star} \delta_t = \begin{cases} 0 & \text{if } st = \theta \\ \delta_{st} & \text{otherwise} \end{cases}$$

(as usual, δ_s is the characteristic function of $s \in \mathcal{C}_2^\square$).

Following [4], define the *left regular $*$ -representation* $\lambda : \ell_1(\mathcal{C}_2^\square) \rightarrow B(\ell_2(\mathcal{C}_2^\square))$: for $s \in \mathcal{C}_2^\square$ let

$$\lambda(\delta_s) \delta_t = \begin{cases} \delta_{st} & \text{if } s^* st = t \\ 0 & \text{otherwise} \end{cases}.$$

In order to apply Theorem 1.2, we need to show that λ is unital, contractive, faithful, and weak* continuous.

It is easy to see that λ is unital. Indeed, $e^* = e$, hence $\lambda(\delta_e) \delta_t = \delta_t$ for any $t \in \mathcal{C}_2^\square$.

To show that λ is contractive, it suffices to prove that $t_1 = t_2$ whenever $t_1, t_2 \in \mathcal{C}_2^\square$ satisfy $\lambda(\delta_s) \delta_{t_1} = \lambda(\delta_s) \delta_{t_2} \neq 0$. Indeed, in this situation $s^* st_1 = t_1$, $s^* st_2 = t_2$, and $st_1 = st_2$. Multiplying the last equality by s^* from the left, we conclude that $t_1 = t_2$.

The fact that λ is faithful can be deduced from [28]. For the sake of completeness, we present a direct proof. Consider a non-zero

$$f = \sum \alpha_{\mathbf{ij}} \delta_{s_{\mathbf{i}} s'_{\mathbf{j}}} \in \ell_1(\mathcal{C}_2^\square),$$

where the sum is taken over all pairs of (possibly empty) finite strings \mathbf{i} and \mathbf{j} , consisting of 1's and 2's. Pick a pair (\mathbf{i}, \mathbf{j}) such that $\alpha_{\mathbf{ij}} \neq 0$, and $\alpha_{\mathbf{kl}} = 0$ whenever $\mathbf{j} = \mathbf{lh}$ with $\mathbf{h} \neq \emptyset$ (this can be done by finding the “shortest” \mathbf{j} with $\alpha_{\mathbf{ij}} \neq 0$). We shall show that

$$(4.1) \quad \langle \delta_{s_{\mathbf{i}}}, \lambda(f) \delta_{s_{\mathbf{j}}} \rangle = \alpha_{\mathbf{ij}}.$$

Indeed, if $\alpha_{\mathbf{kl}} \neq 0$, then either $\mathbf{j} \neq \mathbf{l}$ and $\lambda(\delta_{s_{\mathbf{k}} s'_{\mathbf{l}}}) \delta_{s_{\mathbf{j}}} = 0$, or $\mathbf{j} = \mathbf{l}$, and $\lambda(\delta_{s_{\mathbf{k}} s'_{\mathbf{l}}}) \delta_{s_{\mathbf{j}}} = \delta_{s_{\mathbf{k}}}$. Therefore,

$$\langle \delta_{s_{\mathbf{i}}}, \alpha_{\mathbf{kl}} \lambda(\delta_{s_{\mathbf{k}} s'_{\mathbf{l}}}) \delta_{s_{\mathbf{j}}} \rangle = \begin{cases} \alpha_{\mathbf{ij}} & (\mathbf{k}, \mathbf{l}) = (\mathbf{i}, \mathbf{j}) \\ 0 & \text{otherwise} \end{cases},$$

which implies (4.1).

To prove the weak* continuity of λ , it suffices to show that, for any $u, v \in \mathcal{C}_2^\square$, $\langle \lambda(\delta_s) \delta_u, \delta_v \rangle = 0$ for all but finitely many $s \in \mathcal{C}_2^\square$. To achieve this, write $s = s_{\mathbf{i}} s'_{\mathbf{j}}$, and $u = s_{\mathbf{k}} s'_{\mathbf{l}}$. By definition of λ , $\lambda(\delta_s) \delta_u = 0$ unless $\mathbf{k} = \mathbf{jh}$ (this is possible only for finitely many strings \mathbf{j}). In the latter case, $\lambda(\delta_s) \delta_u = \delta_{s_{\mathbf{ih}} s'_{\mathbf{l}}}$, which is orthogonal to δ_v , except for at most finitely many strings \mathbf{i} .

Denote by \mathcal{A} the Banach algebra $\sigma \widehat{\star} \ell_1(\mathcal{C}_2^\square) \widehat{\star} \sigma$, where $\sigma = \delta_e - \delta_{s_1 s'_1}$ is an idempotent (that is, $\sigma \widehat{\star} \sigma = \sigma$). Multiplication by δ_s ($s \in \mathcal{C}_2^\square$) is weak* continuous on $\ell_1(\mathcal{C}_2^\square)$, hence \mathcal{A} is a weak* closed subalgebra of $\ell_1(\mathcal{C}_2^\square)$, with σ playing the role of identity. Let $P = \lambda(\sigma)$, $H = P(\ell_2(\mathcal{C}_2^\square))$, and define $\pi : \mathcal{A} \rightarrow B(H)$ by setting, for $a \in \mathcal{A}$, $\pi(a) = P \lambda(a) P$. By the above, π is a contractive unital weak* continuous algebraic homomorphism from \mathcal{A} into $B(H)$. Moreover,

$$\lambda(a) = \lambda(\sigma \widehat{\star} a \widehat{\star} \sigma) = \lambda(\sigma) \lambda(a) \lambda(\sigma) = P \lambda(a) P$$

for any $a \in \mathcal{A}$, hence π is faithful.

By Theorem 1.2, there exists an operator space X , isometric to $H \otimes \ell_2$, and such that every c.b. map on X is of the form $\pi(a) \otimes I_{\ell_2} + S$, with $a \in \mathcal{A}$ and $S \in S_2$. Moreover, $\rho : \pi(a) \otimes I_{\ell_2} + S \mapsto \pi(a) \otimes I_{\ell_2}$ is an algebraic homomorphism.

It was shown in [10] that there exists $a_1, b_1, a_2, b_2 \in \mathcal{A}$ s.t. $a_1 \widehat{\star} b_1 = a_2 \widehat{\star} b_2 = \sigma$, and $b_1 \widehat{\star} a_1$ and $b_2 \widehat{\star} a_2$ are idempotents whose product is 0. Let $U_i = \pi(a_i) \otimes I_{\ell_2}$ and $V_i = \pi(b_i) \otimes I_{\ell_2}$, ($i = 1, 2$). Then $U_i V_i = I_X$, and $P_i = V_i U_i$ ($i = 1, 2$) are projections on X satisfying $P_1 P_2 = P_2 P_1 = 0$. Thus, the ranges of P_1 and P_2 are completely isomorphic to X , and X is completely isomorphic to the direct sum of these ranges.

By [10], there exists a non-trivial trace τ_0 on \mathcal{A} . For $T = \pi(a) \otimes I_{\ell_2} + S \in CB(X)$ ($a \in \mathcal{A}$, $S \in S_2$) define the linear functional τ by setting $\tau(T) = \tau_0(a)$. It is easy to check that τ is well-defined, and is a trace. \blacksquare

Finally, we examine the scope of applicability of Theorem 1.2. As noted in the introduction, we can apply this theorem for any unital weak* closed subalgebra of $B(\ell_2)$ (just by considering the identity representation). Below, we give some examples of Banach algebras which are not algebraically isomorphic to algebras of operators, yet which have representations π as in the statement of Theorem 1.2.

Suppose G is a locally compact group. It is well known that there exists a left Haar measure μ on this group (see e.g. Theorem 3.3.2 of [9]). The *full C^* -algebra* $C^*(G)$ is the completion of the space $L_1(G)$ of all complex-valued integrable functions of G , equipped with the norm

$$\|f\|_{C^*(G)} = \sup \left\| \int \pi(x)f(x) d\mu(x) \right\|,$$

where the supremum is taken over all continuous unital representations $\pi : G \rightarrow B(H)$ (see e.g. [14] for more information). The *Fourier-Stieltjes algebra* $b(G)$ is the space of complex-valued functions g on the groups G for which there exists a continuous unital representation $\pi : G \rightarrow B(H)$ and vectors $\xi, \eta \in H$ such that $g(x) = \langle \pi(x)\xi, \eta \rangle$ for any $x \in G$. We set $\|g\|_{b(G)} = \inf \|\xi\| \|\eta\|$, where the infimum is taken over the set of all π, η , and ξ as above. The pointwise product turns $b(G)$ into a Banach algebra (once again, see [14]). Moreover, $b(G)$ is the dual space of $C^*(G)$, with the duality given by

$$\langle f, g \rangle = \int f(x)g(x) d\mu(x).$$

Clearly, the algebra $b(G)$ has an identity – that is, the function which is identically equal to 1 on G (indeed, such a function arises from the representation π sending every element of G into the identity).

In the above notation, we have:

Proposition 4.3. *Suppose G is a locally compact group, whose left Haar measure μ is σ -finite. Then there exists a weak* continuous faithful unital representation $\pi : b(G) \rightarrow B(\ell_2)$. Consequently, there exists an operator space X , isometric to $\ell_2 \otimes \ell_2$, such that $T \in CB(X)$ if and only if there exists $g \in b(G)$ for which $T - \pi(g) \otimes I_{\ell_2}$ is Hilbert-Schmidt, and moreover,*

$$\frac{1}{16} \max\{\|g\|, \|T - \pi(g) \otimes I_{\ell_2}\|_2\} \leq \|T\|_{cb} \leq \|g\| + \|T - \pi(g) \otimes I_{\ell_2}\|_2.$$

Remark 4.4. This proposition gives us an example of a “nice” representation of a Banach algebra which is not an operator algebra. Indeed, suppose G is an Abelian group with infinitely many elements. By [14], $b(G)$ can be identified with the algebra of measures $M(\hat{G})$, where \hat{G} is the dual group. However, by [29], if $\pi : M(\hat{G}) \rightarrow B(H)$ is a representation, then π^{-1} is unbounded.

It may be interesting to note that, by [15], the algebra $M(G)$ can be isometrically represented as a subalgebra of $B(B(L_2(G)))$ whenever the locally compact group G

is commutative. This result has recently been generalized to the non-commutative case in [21].

Proof of Proposition 4.3. Note that the formal identity map $\pi_* : L_1(G) \rightarrow C^*(G)$ is contractive. Therefore, its dual $\pi : b(G) \rightarrow L_\infty(\mu)$ is a unital faithful contractive weak* closed representation. Representing $L_\infty(\mu)$ as the space of “diagonal” operators on the separable Hilbert space $L_2(\mu)$, we regard π as mapping $b(G)$ into $B(L_2(\mu))$. We complete the proof by applying Theorem 1.2 to π . ■

Remark 4.5. There exist Banach algebras with no non-trivial representations on $B(\ell_2)$ (by [5], $B(\ell_p)$ is such an algebra for $p \neq 2$).

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REFERENCES

- [1] S. Argyros and V. Felouzis. Interpolating hereditarily indecomposable Banach spaces, *J. Amer. Math. Soc.*, 13:243–294, 2000.
- [2] S. Argyros and A. Manoussakis. An indecomposable and unconditionally saturated Banach space, *Studia Math.*, 159:1–32, 2003.
- [3] W. Arveson. Interpolation problems in nest algebras, *J. Funct. Anal.*, 20:208–233, 1975.
- [4] B. Barnes. Representations of the l_1 -algebra of an inverse semigroup, *Trans. Amer. Math. Soc.*, 218:361–396, 1976.
- [5] E. Berkson and H. Porta. Representations of $B(X)$, *J. Funct. Anal.*, 3:1–34, 1969.
- [6] B. Blackadar. *K-theory for operator algebras. Second edition*, Cambridge University Press, Cambridge, 1998.
- [7] A. Clifford and G. Preston. *The algebraic theory of semigroups I*, Amer. Math. Soc., Providence RI, 1961.
- [8] J. Cuntz. K -theory for certain C^* -algebras, *Ann. Math.*, 113:181–197, 1981.
- [9] H. Dales. *Banach algebras and automatic continuity*, Oxford University Press, New York, 2000.
- [10] H. Dales, N. Laustsen, and C. Read. A properly infinite Banach $*$ -algebra with a non-zero, bounded trace, *Studia Math.*, 155:107–129, 2003.
- [11] K. Davidson. *Nest algebras. Triangular forms for operator algebras on Hilbert space*, Longman Scientific & Technical, Harlow, 1988.
- [12] K. Davidson and S. Szarek. Local operator theory, random matrices and Banach spaces, in *Handbook of the geometry of Banach spaces*, Vol. 1, 317–366, North-Holland, Amsterdam, 2001.
- [13] E. Effros and Z.-J. Ruan. *Operator spaces*, Oxford University Press, New York, 2000.
- [14] P. Eymard. L’algèbre de Fourier d’un groupe localement compact, *Bull. Soc. Math. France*, 92:181–236, 1964.
- [15] F. Ghahramani. Isometric representation of $M(G)$ on $B(H)$, *Glasgow Math. J.*, 23:119–122, 1982.
- [16] T. Gowers and B. Maurey. The unconditional basic sequence problem, *J. Amer. Math. Soc.*, 6:851–874, 1993.
- [17] T. Gowers and B. Maurey. Banach spaces with small spaces of operators, *Math. Ann.*, 307:543–568, 1997.

- [18] R. Kadison and J. Ringrose. *Fundamentals of the theory of operator algebras I*, American Mathematical Society, Providence, RI, 1997.
- [19] N. Laustsen. *K*-theory for algebras of operators on Banach spaces *J. London Math. Soc. (2)*, 59:715-728, 1999.
- [20] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces I*, Springer-Verlag, Berlin, 1977.
- [21] M. Neufang, Z.-J. Ruan, and N. Spronk. Completely isometric representations of $M_{cb}A(G)$ and $UCB(\hat{G})^*$, preprint.
- [22] T. Oikhberg and E. Ricard. Operator spaces with few completely bounded maps, *Math. Ann.*, 328:229–259, 2004.
- [23] A. Paterson. *Groupoids, inverse semigroups, and their operator algebras*, Birkhauser, Boston, MA, 1999.
- [24] V. Paulsen. *Completely bounded maps and operator algebras*, Cambridge University Press, 2002.
- [25] G. Pisier. *An introduction to the theory of operator spaces*, Cambridge University Press, 2003.
- [26] J. Renault. *A groupoid approach to C^* -algebras*, Springer, Berlin, 1980.
- [27] V. Runde. *Lectures on amenability*, Springer, Berlin, 2002.
- [28] J. Wordingham. The left regular $*$ -representation of an inverse semigroup, *Proc. Amer. Math. Soc.*, 86:55–58, 1982.
- [29] N. Young. The irregularity of multiplication in group algebras, *Quart J. Math.* 24:59–62, 1973.

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