PRODUCTS OF PROJECTIONS IN VON NEUMANN ALGEBRAS

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Abstract. We describe the elements of von Neumann algebras which can be represented as products of orthogonal projections and idempotents, and estimate the minimal number of terms in the product.

1. Introduction and the main results

We investigate whether an element a in a von Neumann algebra $\mathcal{N}$ can be represented as a product of projections in $\mathcal{N}$, and if yes, what is the minimal number of projections required (this number is denoted by $\mathbf{M}(a)$).

Throughout this paper, all von Neumann algebras and C^*-algebras are assumed to be acting on a fixed separable Hilbert space. An element $a \in \mathcal{N}$ is called an *an idempotent* if $a = a^2$. A self-adjoint idempotent is called *a projection*. For a von Neumann algebra $\mathcal{N}$, we denote the set of its projections by $\mathcal{P}(\mathcal{N})$. If E is a subspace of a Hilbert space H, $\mathbf{pr}(E)$ denotes the projection onto the closure of E. E is said to be *affiliated* with a von Neumann algebra $\mathcal{N}$ if $\mathbf{pr}(E) \in \mathcal{N} \hookrightarrow B(H)$ (equivalently, by [10], E is the range of an idempotent in $\mathcal{N}$).

Throughout, **ran** and **ker** denote the range and kernel of an operator, respectively. The usual Murray-von Neumann relations on $\mathcal{P}(\mathcal{N})$ are denoted by $\prec$, $\succ$, and $\sim$. For $p, q \in \mathcal{P}(\mathcal{N})$, we say that p *n-majorizes* q ($p \gg_n q$) if there exist n mutually orthogonal projections $q_1, \ldots, q_n \in \mathcal{N}$ such that $q = q_1 + \ldots + q_n$, and q_j is equivalent to a subprojection of p for $1 \leqslant j \leqslant n$. We say that a projection $p \in \mathcal{N}$ is *n-majorant* if $p \gg_n \mathbf{1}_{\mathcal{N}} - p = p^\perp$. Similar notation is used for subspaces affiliated with $\mathcal{N}$, which are identified with the projections onto them. For instance, we say that subspaces E and F are equivalent, and write $E \sim F$, if $\mathbf{pr}(E) \sim \mathbf{pr}(F)$.

Theorem 1.1. *Suppose $\mathcal{N}$ is a von Neumann algebra, acting on a separable Hilbert space H.*

1. *Suppose $p \in \mathcal{P}(\mathcal{N})$, $a \in \mathcal{N}$ satisfy $a = p^\perp + pap$, $\|pap\| < 1$, $p \ll_n \mathbf{ker}\, a$, and $(\mathbf{ran}\, a)^\perp \sim \mathbf{ker}\, a$. Then a can be represented as a product of at most $\gamma n/(1 - \|pap\|)$ projections (γ is an absolute constant).*
2. *Suppose $a = p_n \ldots p_1$, with $p_1, \ldots, p_n \in \mathcal{P}(\mathcal{N})$. Then $\mathbf{ker}\, a \sim (\mathbf{ran}\, a)^\perp$, and there exists $p \in \mathcal{P}(\mathcal{N})$ such that $a = p^\perp + pap$, with $\|a\xi\| < \|\xi\|$ for any $\xi \in \mathbf{ran}\, p \backslash \{0\}$, and $p - \mathbf{pr}(\mathbf{ran}\,(pap)) \ll_{n-1} \mathbf{pr}(\mathbf{ker}\, a)$.*

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The estimate on the minimal number of projections in part (1) is optimal:

Proposition 1.2. *(1) Suppose p is a projection in a von Neumann algebra $\mathcal{N}$, such that $\mathbf{ker}\, p$ is k-majorant, but not $(k-1)$-majorant. Then $\mathbf{M}(-\lambda p) \geqslant \max\{k, (1+\lambda)/(1-\lambda)\}$.*

(2) Moreover, suppose τ is a faithful normal finite trace on a von Neumann algebra $\mathcal{N}$, $p \in \mathcal{P}(\mathcal{N})$, and $\lambda \in (0,1)$. Then

$$\mathbf{M}(-\lambda p) \geqslant \frac{\tau(p)}{1-\tau(p)} \frac{1+\lambda}{1-\lambda}.$$

More can be said about self-adjoint elements.

Theorem 1.3. *Suppose $\mathcal{N}$ is a von Neumann factor, and $a \in \mathcal{N}$ is self-adjoint. Then a is a product of projections if and only if $\sigma(a) \subset (-1,1]$, and $\mathbf{ker}\, a$ is n-majorant, for some n. If $\mathbf{ker}\, a$ is n-majorant, and $\sigma(a) \subset [-\lambda, 1]$ for some $\lambda \in [0,1)$, then $\mathbf{M}(a) \leqslant \gamma' n/(1-\lambda)$, where γ' is a constant.*

Furthermore, we can describe precisely which positive operators can be represented as symmetric products of projections.

Theorem 1.4. *Suppose $0 \leqslant a \leqslant \mathbf{1}$ is an element of a von Neumann algebra $\mathcal{N}$, and n is a natural number. Then the following two statements are equivalent:*

(1) $\overline{\mathbf{ran}\, a} \ominus \mathbf{ker}\,(\mathbf{1}-a) \ll_n \mathbf{ker}\, a$.

(2) $a = p_1 \ldots p_n p_{n+1} p_n \ldots p_1$ *for some* $p_1, \ldots, p_{n+1} \in \mathcal{P}(\mathcal{N})$.

A similar result was obtained in [1], in relation to almost sharp quantum effects.

Observe that our estimate for the minimal number of projections in a "symmetric" product representing a is sharp: if $\overline{\mathbf{ran}\, a} \ominus \mathbf{ker}\,(\mathbf{1}-a) \ll_{n-1} \mathbf{ker}\, a$ is not true, then we cannot write $a = q_1 \ldots q_{n-1} q_n q_{n-1} \ldots q_1$. Indeed, suppose, for the sake of contradiction, that the representation as above exists. By the observations in the beginning of Section 2.1, it suffices to consider the case of $\mathbf{ker}\,(\mathbf{1}-a) = 0$. Let $u = q_n q_{n-1} \ldots q_1$, and observe that $a = u^* u$. By Lemma 2.15, $\overline{\mathbf{ran}\, u} \ll_{n-1} \mathbf{ker}\, u$. However, $\overline{\mathbf{ran}\, u} \sim \overline{\mathbf{ran}\, a}$, yielding a contradiction.

We also describe the closure of the set of products of projections in the strong or weak operator topologies.

Theorem 1.5. *Denote by $\boldsymbol{\tau}$ the weak operator topology, the strong operator topology, or the weak* topology on a separably acting von Neumann algebra $\mathcal{N}$. Then the $\boldsymbol{\tau}$-closure of all products of projections in $\mathcal{N}$ coincides with the unit ball of $\mathcal{N}$ if and only if $\mathcal{N}$ has no M-summands of type I_n $(n \in \mathbb{N})$.*

Finally, we deal with the products of idempotents (not necessarily self-adjoint) in von Neumann algebras. We say that subspaces E and F of a Hilbert space H are *at positive angles* with each other if $\|\mathbf{pr}(E)\mathbf{pr}(F)\| < 1$, or equivalently, $\mathbf{pr}(E)^{\perp}|_F$ is an isomorphism.

Theorem 1.6. *Consider an element u of a von Neumann algebra $\mathcal{N}$, acting on a separable Hilbert space H.*

1. *If u is a product of n idempotents from $\mathcal{N}$, then $\mathbf{ker}\, u \sim (\mathbf{ran}\, u)^\perp$, and there exists a subspace E, affiliated with $\mathcal{N}$, such that (i) $u|_E = I_E$, (ii) E and $\overline{u(F)}$ are at positive angles, where $F = H \ominus (E + \mathbf{ker}\, u)$, and (iii) $F \ll_{n-1} \mathbf{ker}\, u$. If $\mathbf{ker}\, u$ is trivial, then $E = H$.*
2. *Conversely, suppose for for a given u there exist E, F, and n as above. Then u is a product of at most $\gamma'' n$ idempotents, where γ'' is a constant.*

Note that, in part (1), E is at positive angles with $\mathbf{ker}\, u$.

The minimal number of idempotents needed to represent an element of $\mathcal{N}$ is not known to us. It is tempting to conjecture that this number coincides with n from part (1) of the theorem above. Indeed, this is true for finite type I factors [2]. However, the "conjecture" fails for infinite factors, see Propositions 3.6 and 3.4, as well as [5].

For the particular case of the von Neumann algebras $B(H)$, some descriptions of products of projections were obtained by the author in [13]. That paper also contains references to other related articles. An overview of products of other types of operators can be found in [16]. Linear combinations of products of projections of fixed length were described in [3].

2. Proofs of the main results: products of projections

2.1. **Preliminary notes.** (i) Suppose $p_1, \ldots, p_n$ are projections in a von Neumann algebra $\mathcal{N}$ (acting on a Hilbert space H), and $p = p_1 \wedge p_2 \wedge \ldots \wedge p_n$. Then, for each j, $p_j' := p_j p^\perp = p^\perp p_j = p_j - p$ is a projection. Moreover, $p_n \ldots p_1 = p + p_n' \ldots p_1'$. The domain and range of $p_n' \ldots p_1'$ are orthogonal to $\mathbf{ran}\, p$. Finally, $\mathbf{ran}\, p = \{\xi \in H \,|\, \|p_n \ldots p_1 \xi\| = \|\xi\|\}$.

(ii) Suppose $\mathcal{I}$ is an index set, $(\mathcal{N}_\alpha)_{\alpha \in \mathcal{I}}$ are von Neumann algebras, and $u_\alpha \in \mathcal{N}_\alpha$ for each α. Let $\mathcal{N} = \oplus_{\alpha \in \mathcal{I}} \mathcal{N}_\alpha$, and $u = \sum_{\alpha \in \mathcal{I}} u_\alpha$. Then u is a product of n projections iff u_α is a product of n projections for each α.

(iii) Suppose $u \in \mathcal{N}$, and p is the projection onto $\mathbf{ker}\,(u - \mathbf{1}_\mathcal{N}) = \{\xi \in H \,|\, u\xi = \xi\}$. Let $\mathcal{N}_1 = p^\perp \mathcal{N} p^\perp$, and $u_1 = p^\perp u p^\perp$. Then the following are equivalent:

1. u is a product of n projections in $\mathcal{N}$
2. u_1 is a product of n projections in $\mathcal{N}_1$.

Indeed, (2) $\Rightarrow$ (1) follows directly from Observation (ii). To see the converse, suppose $u = p_n \ldots p_1$. Then, by (i), $u_1 = (p_n - p) \ldots (p_1 - p)$.

(iv) Because of (i) and (iii), we are especially interested in those $u \in \mathcal{N} \hookrightarrow B(H)$ for which $\|u\xi\| < \|\xi\|$ for any $\xi \in H \backslash \{0\}$. However, characterization of such elements is difficult, see [13]. To make the problem more manageable, we concentrate on the case $\|u\| < 1$.

Lemma 2.1. *Suppose* $u = q_n \dots q_1$, *for* $q_1, \dots, q_n \in \mathcal{P}(\mathcal{N})$. *Then there exist* $p_1, \dots, p_n \in \mathcal{P}(\mathcal{N})$ *such that* $u = p_n \dots p_1$, $\mathbf{ker}\, u = \mathbf{ker}\, p_1$, $\mathbf{ran}\, p_j = \overline{\mathbf{ran}\,(p_j p_{j-1} \dots p_1)}$, *and, for* $2 \leqslant j \leqslant n$, p_j *is surjective on* $\mathbf{ran}\,(p_{j-1} \dots p_1)$.

Proof. Let $p_1 = q_1 \wedge \mathbf{pr}(\mathbf{ker}\, u)^\perp$. Then $u = q_n \dots q_2 p_1$, and $\mathbf{ker}\, u = \mathbf{ker}\, p_1$. Therefore, $\mathbf{ker}\, q_k \cap \mathbf{ran}\,(q_{k-1} \dots q_2 p_1) = \{0\}$ for $2 \leqslant k \leqslant n$.

Suppose $p_1, \dots, p_k$ have already been defined ($1 \leqslant k < n$) in such a way that $\mathbf{ran}\, p_j = \overline{\mathbf{ran}\,(p_j p_{j-1} \dots p_1)}$ for $j \leqslant k$, and $u = q_n \dots q_{k+1} p_k \dots p_1$. Let $p_{k+1} = \mathbf{pr}(\mathbf{ran}\,(q_{k+1} p_k \dots p_1))$. Clearly, this projection has the desired properties: $u = q_n \dots q_{k+2} p_{k+1} \dots p_1$, and p_{k+1} is surjective on $\mathbf{ran}\,(p_k \dots p_1)$ (otherwise, $\mathbf{ker}\, u$ strictly contains $\mathbf{ker}\, p_1$, which contradicts our definition of p_1). ■

Henceforth, we will always assume that any product $u = p_n \dots p_2 p_1$, the projections $p_1, \dots, p_n$ satisfy the conditions of the lemma above.

Below we collect some facts about the relation $\ll_n$. Note first that, if $p, q \in \mathcal{P}(\mathcal{N})$ are such that $p \ll_n q$, and $e \in \mathcal{P}(\mathcal{N})$ is central, then $pe \ll_n qe$. Conversely, suppose $(e_\alpha)_{\alpha \in \mathcal{I}}$ is a family of central projections in $\mathcal{N}$, whose sum equals $\mathbf{1}$. If $p, q \in \mathcal{P}(\mathcal{N})$ satisfy $pe_\alpha \ll_n qe_\alpha$ for each α, then $p \ll_n q$.

Lemma 2.2. *Suppose* $p, q, r \in \mathcal{P}(\mathcal{N})$, *and* $r \leqslant q \ll_n p$. *Then* $r \ll_n p$.

Proof. Write $q = q_1 + \ldots + q_n$ (a sum of n mutually orthogonal projections, dominated by p). Our goal is to write r as a sum of n mutually orthogonal projections $r_1, \dots, r_n$, dominated by p. Let $r_1 = \mathbf{pr}(\mathbf{ran}\,(r q_1))$. Clearly, $r_1 \prec q_1 \prec p$.

Now suppose $r_1, \dots, r_k$ have been defined in such a way that, for $1 \leqslant i \leqslant k$, (1) $r_i \prec q_i$, and (2) $r_1 + \ldots + r_i = \mathbf{pr}(\mathbf{ran}\,(r(q_1 + \ldots + q_i)))$. Set $r_{k+1} = \mathbf{pr}(\mathbf{ran}\,(r(q_1 + \ldots + q_{k+1}))) - (r_1 + \ldots + r_k)$. Then $r_{k+1} r q_i = 0$ for $1 \leqslant i \leqslant k$, hence

$$\begin{aligned} r_{k+1} &= \mathbf{pr}(\mathbf{ran}\,(r_{k+1} r (q_1 + \ldots + q_{k+1}))) \\ &= \mathbf{pr}(\mathbf{ran}\,(r_{k+1} r q_{k+1})) \prec q_{k+1} \prec p, \end{aligned}$$

as desired. ■

Lemma 2.3. *Suppose* $p \in \mathcal{P}(\mathcal{N})$ *is* n*-majorant,* $q \in \mathcal{P}(\mathcal{N})$, *and* $\|p - q\| < 1$. *Then* q *is* n*-majorant.*

Proof. We have: $\|p - q\| = \|p^\perp - q^\perp\| < 1$. Then (see e.g. [17]) there exists a unitary $u \in \mathcal{N}$ s.t. $q = u^* p u$ and $q^\perp = u^* p^\perp u$. Now suppose $p^\perp = p_1 + \ldots + p_n$, where $p_1, \dots, p_n$ are equivalent to subprojections of p. Taking $q_k = u^* p_k u$ ($1 \leqslant k \leqslant n$), we are done. ■

Next we describe n-majoration in specific types of von Neumann algebras. By Section 1.22 of [14], for any $k \in \mathbb{N} \cup \{0\}$, any separably acting von Neumann algebra of type I_k can be represented as $L_\infty(\Omega, \mu, B(\ell_2^k))$, where μ is a σ-finite measure on Ω (we take ℓ_2^∞ to mean ℓ_2).

Lemma 2.4. *Suppose μ is a σ-finite measure on Ω, and $n \in \mathbb{N}$.*

(a) Suppose $\mathcal{N} = L_\infty(\Omega, \mu, M_k)$ ($k \in \mathbb{N}$). Then $p \in \mathcal{P}(\mathcal{N})$ is n-majorant iff $\operatorname{rank} p(t) \geqslant m = \lceil k/(n+1) \rceil$ for almost every $t \in \Omega$.

(b) Suppose $\mathcal{N} = L_\infty(\Omega, \mu, B(\ell_2))$. Then $p \in \mathcal{P}(\mathcal{N})$ is n-majorant iff $p(t)$ has infinite rank for almost every $t \in \Omega$. In this case, $p \succ p^\perp$.

(c) Suppose $\mathcal{N}$ is a von Neumann algebra of type I_∞, II_∞, or III. Then $p \in \mathcal{P}(\mathcal{N})$ is n-majorant iff $p \succ p^\perp$.

(d) If $\mathcal{N}$ is a type III von Neumann algebra, then $p \in \mathcal{P}(\mathcal{N})$ is n-majorant iff $p \neq 0$.

The following technical result must be known to specialists, but we haven't been able to find a reference to it in the literature.

Lemma 2.5. *Suppose $\mathcal{N}$ is a von Neumann algebra, μ is a σ-finite measure on Ω, and $p \in \mathcal{P}(L_\infty(\Omega, \mu, \mathcal{N}))$. Then for every $\varepsilon > 0$ there exist disjoint measurable subsets $S_i \subset \Omega$ ($i \in \mathbb{N}$), and projections $p_i \in \mathcal{P}(\mathcal{N})$, such that $\|p - \sum_i \chi_{S_i} \otimes p_i\| < \varepsilon$.*

Proof. Suppose $\varepsilon \in (0, 1/2)$. It is known (see e.g. Section II.1 of [6]) that separably valued functions are dense in $L_\infty(\Omega, \mu, \mathcal{N})$. Thus, there exist (S_i) as in the statement of the lemma, and $a_i \in \mathcal{N}$ s.t. $\|p - a\| < \varepsilon/2$, where $a = \sum_i \chi_{S_i} \otimes a_i$. By passing from a_i to $(a_i + a_i^*)/2$, we can assume that all the a_i's are self-adjoint. Note that $\|p - \lambda\| = \max\{|\lambda|, |1-\lambda|\}$, hence $\sigma(a) \subset [-\varepsilon/2, \varepsilon/2] \cup [1-\varepsilon/2, 1+\varepsilon/2]$. Set

$$f(s) = \begin{cases} 0 & s \leqslant \varepsilon/2 \\ (s - \varepsilon/2)/(1-\varepsilon) & \varepsilon/2 \leqslant s \leqslant 1 - \varepsilon/2 \\ 1 & s \geqslant 1 - \varepsilon/2 \end{cases} ,$$

and let $q = f(a) = \sum_i \chi_{S_i} f(a_i)$. Then q is a projection, and $\|q - a\| = \max_{s \in \sigma(a)} |f(s) - s| \leqslant \varepsilon$. ∎

Proof of Lemma 2.4. (a) If p is n-majorant, then, for almost every $t \in \Omega$, $p(t)$ is n-majorant in M_k. Therefore, $n \operatorname{rank} p(t) \geqslant \operatorname{rank} p(t)^\perp = k - \operatorname{rank} p(t)$, which yields $\operatorname{rank} p(t) \geqslant m$.

Suppose $\operatorname{rank} p(t) \geqslant m$ for almost every t. By Lemmas 2.5 and 2.3, we can assume that $p = \sum_{i \in \mathbb{N}} \chi_{S_i} \otimes p^{(i)}$, where $p^{(i)} \in \mathcal{P}(M_k)$ for each i, and $\cup_i S_i = \Omega$. Then we can write $\mathbf{1}_{M_k} - p^{(i)} = \sum_{j=1}^n p_j^{(i)}$, with $\operatorname{rank} p_j^{(i)} \leqslant m$. Letting, for $1 \leqslant j \leqslant n$, $p_j = \sum_{i \in \mathbb{N}} \chi_{S_i} \otimes p_j^{(i)}$, we conclude that p is n-majorant.

(b) is proved in a similar fashion. If p is n-majorant, then $p(t)$ is n-majorant for almost every t, hence $p(t)$ is infinite almost everywhere. If this is the case, we can use Lemma 2.5, and assume that $p = \sum_{i \in \mathbb{N}} \chi_{S_i} \otimes p^{(i)}$, where $p^{(i)} \in \mathcal{P}(B(\ell_2))$ is infinite for each i, and $\cup_i S_i = \Omega$. Then $p \succ \mathbf{1} - p$.

(c) Suppose p is a projection in a II_∞ von Neumann algebra $\mathcal{N}$. Then there exists a central projection e in $p\mathcal{N}p$ such that $e\mathcal{N}e$ and $(p-e)\mathcal{N}(p-e)$ are von Neumann algebras of types II_1 and II_∞, respectively. It suffices show that $e = 0$. Then, an application of Halving Lemma will complete the proof.

Denote the central cover of e (in $\mathcal{N}$) by f. As e and $p - e$ have no equivalent central projections, Proposition 6.1.8 of [9] implies that the central cover of $p - e$ is disjoint from f. Therefore, the central cover of $p - e$ equals $\mathbf{1} - f$ (otherwise, there exists a central projection $g \in \mathcal{N}$, disjoint from the central cover of p; in particular, no subprojection of g is equivalent to a subprojection of p, which contradicts the n-majoration assumption).

Suppose, for the sake of contradiction, that $f \neq 0$. Write $\mathbf{1} - p = \sum_{k=1}^n p_k$, where the projections p_k are mutually orthogonal, and, for each k, $p_k \prec p$. Then $p_k f \prec pf = e$, hence $f = \sum_{k=1}^n p_k f + e$ is finite (as a sum of finitely many finite projections). This, however, contradicts the definition of type II_∞.

Finally, (d) follows directly from the Halving Lemma. ■

2.2. **Products of projections: sufficient conditions.** We first establish two lemmas, dealing with representing "nice" operators on "small" subspaces as products of projections. The first of these lemmas essentially comes from [12].

Lemma 2.6. *Suppose $\mathcal{N}$ is a von Neumann algebra, $p \in \mathcal{N}$ is a projection such that $p \prec p^\perp$, and $a \in \mathbb{N}$ satisfies $0 \leqslant a \leqslant \mathbf{1}$. Then there exists $q \in \mathcal{P}(\mathcal{N})$ such that $pap = pqp$, and $q \prec p$.*

Proof. The construction seems to be folklore (see e.g. [1, 12]), but we reproduce it here for the sake of completeness. Without loss of generality, assume that $\overline{\mathbf{ran}\, a} = \mathbf{ran}\, p$. Find a unitary $u \in \mathcal{N}$ s.t. $p_0 = u^* p u \leqslant p^\perp$. Then $q = a + (a - a^2)^{1/2} u + u^*(a - a^2)^{1/2} + u^*(\mathbf{1} - a)u$ is a projection, and $a = pqp$. Moreover, $v^* q v = p$, where $v = a^{1/2} + u^*(p - a)^{1/2} + (p - a)^{1/2} u - u^* a^{1/2} u$. ■

Corollary 2.7. *Suppose $\mathcal{N}$ is a von Neumann algebra, and $a \in \mathcal{N}$ is such that $0 \leqslant a \leqslant \mathbf{1}$, and $\overline{\mathbf{ran}\, a} \ominus \mathbf{ker}\,(\mathbf{1} - a)$ is n-majorated by $\mathbf{ker}\, a$. Then there exist $p_1, \ldots, p_{n+1} \in \mathcal{P}(\mathcal{N})$, such that $a = p_1 \ldots p_n p_{n+1} p_n \ldots p_1$.*

Proof. As noted above, we can assume that $\mathbf{ker}\,(\mathbf{1} - a) = 0$. Let $p = \mathbf{pr}(\mathbf{ran}\, a)$. By Observation (ii) in Subsection 2.1, we can assume that $\mathcal{N}$ is of type I_k ($k \in \mathbb{N} \cup \{\infty\}$), II_1, II_∞, or III. By Lemma 2.4, if $\mathcal{N}$ is infinite, then $p \prec p^\perp$. In this case, by Lemma 2.6, a is a product of 3 projections.

Now suppose $\mathcal{N}$ is a finite von Neumann algebra, and there exist mutually orthogonal projections $q_1', \ldots, q_n'$ s.t. $p = q_1' + \ldots + q_n'$, with $q_k' \prec p^\perp$ for each k. Then we can write $p = q_1 + \ldots + q_n$ (as sum of mutually orthogonal projections), such that, for each i, $q_i \sim q_i'$, and $q_i a = a q_i$. Indeed, by Theorem 1 of [8], there exists a subprojection q_1 of p, equivalent to p_1, s.t. $q_1 a = a q_1$. Then $(p - q_1)a = a(p - q_1)$. Moreover, by Proposition V.1.38 of [15], $p - q_1 \sim p - q_1' = q_2' + \ldots + q_n'$. By a repeated application of Theorem 1 of [8], we obtain $q_1, \ldots q_n$ with the desired properties. Note that $\mathbf{ran}\,(aq_i) \subset \mathbf{ran}\,q_i$ for any i, and $\overline{\mathbf{ran}\,a} = \mathbf{ran}\,(q_1 + \ldots + q_n)$, hence $\overline{\mathbf{ran}\,(aq_i)} = \mathbf{ran}\,q_i$.

Let $r'_0 = p^\perp$, and define inductively the sequences of projections $(r_k)_{k=1}^n$ and $(r'_k)_{k=0}^n$ as follows: on the k-th step, select $r_k \leqslant r'_{k-1} + q_k$ s.t. $q_k r_k q_k = aq_k$ and $r_k \sim q_k$ (this is possible, by Lemma 2.6). Let $r'_k = r_{k-1} + q_k - r_k$. By Proposition V.1.38 of [15], $r'_k \sim r_{k-1}$. Note that $r_k \leqslant p^\perp + q_1 + \ldots + q_k$, hence it is orthogonal to q_j for $j > k$. Moreover, by induction we show that r_k and r'_k are orthogonal to r_j, for $j < k$.

Now let $p_1 = p$, $a_0 = a$, and, for $k \geqslant 1$,

$$p_{k+1} = r_1 + \ldots + r_k + q_{k+1} + \ldots + q_n, \;\; a_k = r_1 + \ldots + r_k + aq_{k+1} + \ldots + aq_n.$$

Then $p_k a_k p_k = a_{k-1}$ for any $k \leqslant n$. Therefore, $a = p_1 \ldots p_n p_{n+1} p_n \ldots p_1$. ■

Proof of Theorem 1.4. Lemma 2.7 yields (1) ⇒ (2). To show (2) ⇒ (1), suppose $a = p_1 \ldots p_n p_{n+1} p_n \ldots p_1$. Without loss of generality, $p_1 \wedge \ldots \wedge p_{n+1} = \mathbf{ker}\,(\mathbf{1} - a) = \{0\}$. Let $u = p_{n+1} p_n \ldots p_1$. Note that $a = u^* u$, hence $\mathbf{ker}\,a = \mathbf{ker}\,u$, and $\overline{\mathbf{ran}\,a} = \overline{\mathbf{ran}\,u^*}$. By Proposition 6.1.6 of [9], $\overline{\mathbf{ran}\,u} \sim \overline{\mathbf{ran}\,u^*}$. By Lemma 2.15, $\overline{\mathbf{ran}\,u^*} \ll_n \mathbf{ker}\,u$. Therefore, $\overline{\mathbf{ran}\,a} \ll_n \mathbf{ker}\,a$. ■

Lemma 2.8. *Suppose $\mathcal{N}$ is a von Neumann algebra, $p \in \mathcal{P}(\mathcal{N})$, and $u \in \mathcal{N}$ is a unitary, for which $upu^* \leqslant p^\perp$. Fix $\lambda \in (0,1)$, and let $n = \lceil \pi^2/(8(1-\lambda)) \rceil$. Then there exist $p_1, \ldots, p_n \in \mathcal{P}(\mathcal{N})$ such that $p_n \ldots p_1 p = \lambda up$.*

Proof. Note that, for each $\alpha \in \mathbb{R}$, $\cos^2 \alpha = 1 - \sin^2 \alpha \geqslant 1 - \alpha^2$. Therefore, by the choice of n,

$$\Big(\cos \frac{\pi}{2n} \Big)^n \geqslant \Big(1 - \frac{\pi^2}{4n^2} \Big)^{n/2} \geqslant 1 - \frac{\pi^2}{4n} \geqslant \lambda.$$

Therefore, there exist $0 = \phi_0 < \phi_1 < \ldots < \phi_{n-1} < \phi_n = \pi/2$, such that $\prod_{k=1}^n \cos(\phi_k - \phi_{k-1}) = \lambda$. For $1 \leqslant k \leqslant n$, denote by p_k the projection onto $\mathbf{ran}\,(\cos \phi_k p + \sin \phi_k up)$.

Pick an orthonormal basis $(\xi_i^{(0)})_{i \in \mathcal{I}}$ in $\mathbf{ran}\,p$. For $1 \leqslant k \leqslant n$, let $\xi_i^{(k)} = \cos \phi_k \xi_i^{(0)} + \sin \phi_k u \xi_i^{(0)}$. It is easy to verify that

$$\langle \xi_i^{(k)}, \xi_j^{(\ell)} \rangle = (\cos \phi_k \cos \phi_\ell + \sin \phi_k \sin \phi_\ell) \delta_{ij} = \cos(\phi_k - \phi_\ell) \delta_{ij}$$

(here δ_{ij} is Kronecker's delta). In particular, $p_k \xi_i^{(k-1)} = \cos(\phi_k - \phi_{k-1}) \xi_i^{(k)}$. Therefore,

$$p_n \ldots p_1 \xi_i^{(0)} = \prod_{k=1}^n \cos(\phi_k - \phi_\ell) \xi_i^{(n)},$$

and we are done. ■

Lemma 2.9. *Suppose $\mathcal{N}$ is a von Neumann algebra, $p \in \mathcal{N}$ is a projection, such that $p \prec p^\perp$. Suppose, furthermore, that $u \in \mathcal{N}$ is a unitary, and $u(\mathbf{ran}\,p) \subset \mathbf{ker}\,p$. Then, for any $\lambda \in (0,1)$, $\mathbf{M}(\lambda up) \leqslant 2\lceil \pi^2/(8(1 - \sqrt{\lambda})) \rceil$.*

Proof. Find a unitary $v \in \mathcal{N}$ such that $q = vpv^* \leqslant p^\perp$. Apply Lemma 2.8 twice - first to $\sqrt{\lambda} vp$, then to $\sqrt{\lambda} uv^* q$. ∎

Corollary 2.10. *Suppose $\mathcal{N}$ is a von Neumann algebra, $p \in \mathcal{P}(\mathcal{N})$, and $p^\perp$ is n-majorant. Suppose, furthermore, that $u \in \mathcal{N}$ is a unitary, and $u(\mathbf{ran}\, p) = \mathbf{ran}\, p$. Then, for any $\lambda \in (0,1)$, $\mathbf{M}(\lambda up) \leqslant 2n\lceil \pi^2/(8(1-\sqrt{\lambda}))\rceil$.*

Proof. As before, we can consider the cases of $\mathcal{N}$ being infinite, and $\mathcal{N}$ being finite, separately. If $\mathcal{N}$ is infinite, then $p \prec p^\perp$ by Lemma 2.4, and an application of Lemma 2.9 completes the proof. If $\mathcal{N}$ is finite, use Theorem 1 of [8] to write p as a sum of mutually orthogonal projections $p_1, \ldots, p_n$, s.t. $p_k \prec p^\perp$ for each k, and $up_k = p_k u$. Then $pu = \prod_{k=1}^n (p_k u + p - p_k)$, and, by Lemma 2.9, $\mathbf{M}(p_k u + p - p_k) \leqslant 2\lceil \pi^2/(8(1-\sqrt{\lambda}))\rceil$. ∎

Proof of Theorem 1.3. By [12], $\sigma(a) \subset (-1,1]$ whenever a is a self-adjoint product of projections. Suppose $\sigma(a) \subset [\lambda, 1]$. Consider a function

$$\phi(x) = \begin{cases} x & x \geqslant 0 \\ 0 & x < 0 \end{cases},$$

and let $a_+ = \phi(a)$, $a_- = a - a_+$. By the spectral theorem, $p_+ = \overline{\mathbf{ran}\, a_+}$ and $p_- = \overline{\mathbf{ran}\, a_-}$ are mutually orthogonal. By Corollary 2.7, a_+ (viewed as acting on $p_+ + p^\perp$) is a product of $2n+1$ projections. By Theorem 1.1(1), a_- (viewed as acting on $p_- + p^\perp$) is a product of $\gamma n/(1-\lambda)$ projections. This yields an upper estimate for $\mathbf{M}(a)$. ∎

Lemma 2.11. *Suppose $p, q \in \mathcal{P}(\mathcal{N})$ are such that $p \sim q$ and $p^\perp \sim q^\perp$. Fix $\lambda \in (0,1)$, and let $n = \lceil 2\pi^2/(1-\lambda)\rceil$. Then there exist $p_1, \ldots, p_n \in \mathcal{P}(\mathcal{N})$ such that $p_n \ldots p_1(\mathbf{ran}\, p) = \mathbf{ran}\, q$, and $\|p_n \ldots p_1 \xi\| \geqslant \lambda \|\xi\|$ for any $\xi \in \mathbf{ran}\, p$.*

Proof. Find partial isometries $v, w \in \mathcal{N}$ s.t. $v^*v = p$, $vv^* = q$, $w^*w = p^\perp$, and $ww^* = q^\perp$. Then $u = v + w$ is a unitary. Write $u = \exp(ia)$, where $a \in \mathcal{N}$ is self-adjoint, with $\|a\| \leqslant \pi$. Let $n = \lceil 2\pi^2/(1-\lambda)\rceil$. For $0 \leqslant k \leqslant n$, denote by p_k the projection onto $\mathbf{ran}\,(\exp(ika/n)p)$ (in this notation, $p_0 = p$, and $p_n = q$). For $1 \leqslant k \leqslant n$, $\|p_k - p_{k-1}\| \leqslant \|\exp(ia/n) - \mathbf{1}\| \leqslant \|a\|/n \leqslant \pi/n$. Therefore, for every norm 1 $\xi \in \mathbf{ran}\, p_{k-1}$, $\|p_k\xi - \xi\| \leqslant \pi/n$. Thus, $\|p_k \xi\|^2 = \|\xi\|^2 - \|p_k\xi - \xi\|^2 \geqslant 1 - \pi^2/n^2$. Therefore,

$$\|p_n \ldots p_1 \xi\| \geqslant \left(1 - \pi^2/n^2\right)^{n/2} \|\xi\| \geqslant (1 - \pi^2/(2n))\|\xi\| \geqslant \lambda \|\xi\|$$

for any $\xi \in \mathbf{ran}\, p$. ∎

Remark 2.12. For a von Neumann algebra $\mathcal{N}$ and $p \in \mathcal{P}(\mathcal{N})$, denote by $\mathcal{S}(p)$ the set of all $q \in \mathcal{P}(\mathcal{N})$ s.t. $p \sim q$ and $p^\perp \sim q^\perp$. The above proof shows that $\mathcal{S}(p)$ is path-connected, and moreover, any such p and q can be connected by a path of length at most π. See [4] for more on this topic.

Proof of Theorem 1.1(1). By Section 2.1, it suffices to consider the case of $\|a\| < 1$. Denote the initial and terminal projections of a by p and q, respectively. Let $\lambda = \|a\|^{1/2}$. By Lemma 2.11, there exists $b \in \mathcal{N}$ s.t. for any $\xi \in \mathbf{ran}\, p$, we have $bp \in \mathbf{ran}\, q$, $\|bp\| \geqslant \lambda\|\xi\|$, and $\mathbf{M}(b) \leqslant \lceil 2\pi^2/(1-\lambda)\rceil \leqslant \lceil 4\pi^2/(1-\|a\|)\rceil$. As $p^\perp \sim q^\perp$, there exists a partial isometry v s.t. $vv^* = p^\perp$ and $v^*v = q^\perp$. Then $a' = (bp+v)^{-1}a \in \mathcal{N}$ fixes $\mathbf{ran}\, p$, $ba' = a$, and $\|a'\| \leqslant \|(b|_{\mathbf{ran}\, p})^{-1}\|\|a\| \leqslant \lambda$.

Write $a' = \|a'\|up \cdot \|a'\|^{-1}|a'|$, where $u \in \mathcal{N}$ is a unitary, fixing $\mathbf{ran}\, p$. By Lemma 2.7, $\mathbf{M}(\|a'\|^{-1}|a'|) \leqslant 2n+1$. By Lemma 2.10,

$$\mathbf{M}(\|a'\|up) \leqslant 2n\lceil \pi^2/(8(1-\|a'\|^{1/2}))\rceil \leqslant 2n\lceil \pi^2/(2(1-\|a\|))\rceil.$$

Together, the estimates on $\mathbf{M}(b)$, $\mathbf{M}(\|a'\|up)$, and $\mathbf{M}(\|a'\|^{-1}|a'|)$ yield an upper estimate on $\mathbf{M}(a)$. ■

2.3. Necessary conditions and lower bounds.

Lemma 2.13. *Suppose $p_1, \dots, p_n$ are projections in a von Neumann algebra $\mathcal{N}$. Then $\mathbf{ran}\,(p_n \dots p_1)^\perp$ is equivalent to $\mathbf{ker}\,(p_n \dots p_1)$.*

Corollary 2.14. *Suppose $p_1, \dots, p_n \in \mathcal{P}(\mathcal{N})$. Then there exists a unitary $u \in \mathcal{N}$, mapping $\mathbf{ker}\,(p_n \dots p_1)$ onto $\mathbf{ran}\,(p_n \dots p_1)^\perp$, and $\mathbf{ker}\,(p_n \dots p_1)^\perp$ onto $\overline{\mathbf{ran}\,(p_n \dots p_1)}$.*

Proof. By Proposition 6.1.6 of [9],

$$\overline{\mathbf{ran}\,(p_n \dots p_1)} \sim \overline{\mathbf{ran}\,((p_n \dots p_1)^*)} = \mathbf{ker}\,(p_n \dots p_1)^\perp.$$

An application of the previous lemma finishes the proof. ■

Proof of Lemma 2.13. This statement can be proved by induction over n. The case of $n = 1$ is trivial. We handle the case of $n = 2$ by showing that, for any two projections $p, q \in \mathcal{N}$, $\mathbf{ker}\,(pq)$ is equivalent to $(\mathbf{ran}\,(pq))^\perp$. By passing to the orthogonal complement of $p \wedge q$ if necessary, we can assume that $p \wedge q = 0$. Let $p_1 = \mathbf{pr}(\mathbf{ran}\,(pq)) = p - p \wedge q^\perp$ (cf. Proposition 2.5.14 of [9]), and $q_1 = \mathbf{pr}(\mathbf{ker}\,(pq))^\perp = \mathbf{pr}(\mathbf{ran}\,(qp)) = q - q \wedge p^\perp$. Then p_1 is injective on $\mathbf{ran}\, q_1$, and $\mathbf{ran}\,(p_1 q_1) = \mathbf{ran}\,(pq)$ is dense in $\mathbf{ran}\, p_1$. We show that

$$\overline{p_1^\perp(\mathbf{ran}\, q_1^\perp)} = \mathbf{ran}\, p_1^\perp. \tag{2.1}$$

Indeed, suppose $\xi \in \mathbf{ran}\, p_1^\perp$ is orthogonal to $p_1^\perp q_1^\perp \eta$ for any η. Then

$$\langle q_1^\perp \eta, \xi\rangle = \langle q_1^\perp \eta, p_1^\perp \xi\rangle = \langle p_1^\perp q_1^\perp \eta, \xi\rangle = 0,$$

hence $\xi \in \mathbf{ran}\, q_1$, and (2.1) follows. Therefore, $p_1^\perp \prec q_1^\perp$. Similarly, we show that $p_1^\perp \succ q_1^\perp$. Thus, $p_1^\perp \sim q_1^\perp$.

Now suppose the statement is true for n, and $p_1, p_2, \dots, p_n, p_{n+1}$ are projections in $\mathcal{N}$. By Lemma 2.1, we can assume that, for $1 \leqslant j \leqslant n+1$, $\mathbf{ker}\,(p_j \dots p_1) =$

$\mathbf{ker}\,p_1$ (hence, $\mathbf{ker}\,p_j \cap \mathbf{ran}\,(p_{j-1}\dots p_1) = \{0\}$), and $\mathbf{ran}\,p_j = \overline{\mathbf{ran}\,(p_j\dots p_1)}$. Let $r = p_n \wedge p_{n+1}^{\perp}$, and $p'_n = p_n r^{\perp}$. By assumption, $\mathbf{ker}\,p_{n+1} \cap \mathbf{ran}\,(p_n\dots p_1) = \emptyset$, hence

$$\mathbf{ker}\,(p'_n p_{n-1}\dots p_1) = \mathbf{ker}\,(p'_n p_n p_{n-1}\dots p_1) = \mathbf{ker}\,(p_n\dots p_1) = \mathbf{ker}\,p_1.$$

Note that $\overline{\mathbf{ran}\,(p'_n p_{n-1}\dots p_1)} = \mathbf{ran}\,p'_n$. By the induction hypothesis, $\mathbf{ker}\,p'_n \sim \mathbf{ker}\,p_1$. Moreover, $p_{n+1}(p_n - p'_n) = 0$, hence $p_{n+1}p_n\dots p_1 = p_{n+1}p'_n p_{n-1}\dots p_1$. In particular, $\overline{\mathbf{ran}\,(p_{n+1}p'_n)} = \mathbf{ran}\,p_{n+1}$. Finally, $p_{n+1}|_{\mathbf{ran}\,p'_n}$ is surjective, hence $\mathbf{ker}\,(p_{n+1}p'_n) = \mathbf{ker}\,p'_n$. However, as shown above, $\mathbf{ran}\,(p_{n+1}p'_n)^{\perp} \sim \mathbf{ker}\,(p_{n+1}p'_n)$. Taken together, all the equivalences of the last paragraph complete the proof. ∎

Lemma 2.15. *If $n \geqslant 2$, and $p_1, \dots, p_n \in \mathcal{P}(\mathcal{N})$ are such that $p_1 \wedge \dots \wedge p_n = \{0\}$, then $\overline{\mathbf{ran}\,(p_n\dots p_1)} \ll_{n-1} \mathbf{ker}\,(p_n\dots p_1)$.*

Proof. By Lemma 2.1, we assume that, for each k, $\mathbf{ker}\,p_k \cap \mathbf{ran}\,(p_{k-1}\dots p_1) = \{0\}$ (hence $\mathbf{ker}\,(p_k\dots p_1) = \mathbf{ker}\,p_1$), and $\mathbf{ran}\,p_k = \overline{\mathbf{ran}\,(p_k\dots p_1)}$.

The lemma is proved by induction over n. If $n = 2$, we have to show that, given $p_1, p_2 \in \mathcal{P}(\mathcal{N})$ with $p_1 \wedge p_2 = \{0\}$, we have $\overline{\mathbf{ran}\,(p_2p_1)} \prec \mathbf{ker}\,(p_2p_1)$. By Kaplansky Formula,

$$\begin{aligned}\mathbf{pr}(\mathbf{ker}\,(p_2p_1)) = \mathbf{pr}(\mathbf{ker}\,p_1) &= (p_1 \vee p_2)^{\perp} + (p_1 \vee p_2 - p_1) \\ &\sim (p_1 \vee p_2)^{\perp} + p_2 \geqslant p_2 = \mathbf{pr}(\mathbf{ran}\,(p_2p_1)).\end{aligned}$$

Now suppose the statement of the lemma holds for n, and prove it for $n+1$. By Lemma 2.13, $\mathbf{ker}\,p_k \sim \mathbf{ker}\,p_1$ for $2 \leqslant k \leqslant n+1$. Let $r = p_1 \wedge \dots \wedge p_n$, and $q = \mathbf{pr}(\mathbf{ran}\,(p_n\dots p_1)) - r$. Clearly, r commutes with p_k ($1 \leqslant k \leqslant n$). Applying the induction hypothesis to the projections $p_k r^{\perp}$ ($1 \leqslant k \leqslant n$), we conclude that $q \ll_{n-1} r^{\perp} - r^{\perp}p_1 \leqslant p_1^{\perp}$. Moreover, $r \wedge p_{n+1} = \{0\}$, hence $\mathbf{ker}\,(p_{n+1}^{\perp} r) = \{0\}$. This implies $r \prec p_{n+1}^{\perp} \sim p_1^{\perp}$. Thus, by Lemma 2.2,

$$\mathbf{pr}(\mathbf{ran}\,(p_{n+1}\dots p_1)) \leqslant \mathbf{pr}(\mathbf{ran}\,(p_{n+1}(q+r))) \prec q + r \ll_n p_1^{\perp}.$$

∎

Corollary 2.16. *If $p_1, \dots, p_n \in \mathcal{P}(\mathcal{N})$, then $\overline{\mathbf{ran}\,(p_n\dots p_1)} \ominus (\wedge_{k=1}^{n} p_k) \ll_{n-1} \mathbf{ker}\,(p_n\dots p_1)$.*

This result can be reformulated as follows: suppose $a = p_n\dots p_1$. Then $\overline{\mathbf{ran}\,a} \ominus \mathbf{ker}\,(1-a) \ll_{n-1} \mathbf{ker}\,a$.

Proof. Note that the projection $p = (\wedge_{k=1}^{n} p_k)^{\perp}$ commutes with p_k for each k. Let $p'_k = pp_k$. Then $\wedge_{k=1}^{n} p'_k = \{0\}$, $\mathbf{ker}\,(p_n\dots p_1) = \mathbf{ker}\,(p'_n\dots p'_1)$, and $\overline{\mathbf{ran}\,(p_n\dots p_1)} \ominus (\wedge_{k=1}^{n} p_k) = \overline{\mathbf{ran}\,(p'_n\dots p'_1)}$. An application of Lemma 2.15 completes the proof. ∎

Proof of Theorem 1.1(2). Combine Corollaries 2.14 and 2.16. ∎

Remark 2.17. The conditions $\|u\| < 1$, $\mathbf{ker}\, u \sim (\mathbf{ran}\, u)^\perp$ are not sufficient to guarantee that u is a product of projections. Indeed, suppose $\mathcal{N} = \ell_\infty(\mathcal{R})$, where $\mathcal{R}$ is a II_1 factor. Fix $\lambda \in (0,1)$. By Theorem 1.2, for every $n \in \mathbb{N}$ there exists $p_n \in \mathcal{P}(\mathcal{R})$ s.t. $-\lambda p_n$ cannot be represented as a product of less than n projections. Consider $u = (u_n)_{n\in\mathbb{N}}$. Clearly, $\|u\| = \lambda$, and $\mathbf{ker}\, u \sim (\mathbf{ran}\, u)^\perp$, yet u is not a product of projections.

Proof of Proposition 1.2. (1) Suppose $-\lambda p = p_n \dots p_1$. Pick a norm one $\xi \in \mathbf{ran}\, p$. For $1 \leqslant k \leqslant n$, let $u_0 = p$, and $u_k = p_k \dots p_1$. Note that $u_{k-1} - u_k = p_k^\perp p_{k-1} \dots p_1$, hence $\|(u_{k-1} - u_k)\xi\|^2 = \alpha_k^2$, where $\alpha_k = (\|u_{k-1}\xi\|^2 - \|u_k\xi\|^2)^{1/2}$. But

$$1 - \lambda^2 = \|\xi\|^2 - \|p_n \dots p_1 \xi\|^2 = \sum_{k=1}^n \alpha_k^2.$$

Thus,

$$1 + \lambda = \| \sum_{k=1}^n (u_{k-1} - u_k)\xi\| \leqslant \sum_{k=1}^n \alpha_k \leqslant \sqrt{n}(\sum_{k=1}^n \alpha_k^2)^{1/2} = \sqrt{n}\sqrt{1-\lambda^2},$$

hence $n \geqslant (1+\lambda)/(1-\lambda)$.

Now suppose $-\lambda p = p_k \dots p_1$, and show that $p^\perp$ is k-majorant. Clearly, $p_1 \wedge \dots \wedge p_k = \{0\}$. For notational simplicity, let $p_0' = p_0 = p$. For $1 \leqslant i \leqslant k$, define $p_i' = p \wedge p_1 \wedge \dots \wedge p_i$, and $q_i = p_{i-1}' - p_i'$. Note that the projections q_i are mutually orthogonal, and $p = \sum_{i=1}^k q_i$. Moreover, for each i, $p_i\xi \neq \xi$ for any $\xi \in \mathbf{ran}\, q_i$ (otherwise, we would have $\xi \in \mathbf{ran}\,(p \wedge p_1 \wedge \dots \wedge p_{i-1} \wedge p_i)$), hence $p^\perp p_i|_{\mathbf{ran}\, q_i}$ is injective. Thus, q_i is equivalent to $\mathbf{pr}(p^\perp p_i q_i)$, and therefore, $p^\perp$ is k-majorant.

(2) Suppose p is as in the statement of the theorem, and $-\lambda p = p_n \dots p_1 = p_n \dots p_1 p$. As above, let $u_0 = p$, and $u_k = p_k \dots p_1 p$ for $1 \leqslant k \leqslant n$. For such k, let $v_k = u_{k-1} - u_k = p_k^\perp p_{k-1} \dots p_1 p$. Note that $p \prec p_k$ for each k, hence $\tau(p_k^\perp) = 1 - \tau(p_k) \geqslant 1 - \tau(p)$.

Denote the matrix units in M_n by E_{ij} ($1 \leqslant i,j \leqslant n$). Let $M_n(\mathcal{N})$ be the von Neumann algebra of all $n \times n$ matrices with entries in $\mathcal{N}$. Equip $M_n(\mathcal{N})$ with the trace τ_n defined by $\tau_n(\sum_{i,j=1}^n E_{ij} \otimes a_{ij}) = \sum_{i=1}^n \tau(a_{ii})$. We identify $\mathcal{N}$ with $E_{11} \otimes \mathcal{N} \hookrightarrow M_n(\mathcal{N})$ (that is, with the upper left corner of $M_n(\mathcal{N})$)). Define $v, w \in M_n(\mathcal{N})$ by setting $v = \sum_{i=1}^n E_{i1} \otimes v_i$, and $w = \sum_{i=1}^n E_{1i} \otimes p_i^\perp$. Then $(1+\lambda)p = wv$, hence $\|(1+\lambda)p\|_2 \leqslant \|w\|_2 \|v\|$ (here, $\|a\|_2 = (\tau_n(a^*a))^{1/2}$ for $a \in M_n(\mathcal{N})$). But $\|(1+\lambda)p\|_2 = (1+\lambda)\tau(p)^{1/2}$. Moreover,

$$ww^* = \sum_{k=1}^n p_k^\perp p_{k-1} \dots p_1 p p_1 \dots p_{k-1} p_k^\perp \leqslant \sum_{k=1}^n p_k^\perp,$$

hence

$$\|w\|_2^2 = \tau_n(w^*w) = \tau(ww^*) \leqslant \sum_{k=1}^n \tau(p_k^\perp) \leqslant n(1 - \tau(p)).$$

Finally, $\|v\| \leqslant \sqrt{1-\lambda^2}$. To see this, view $\mathcal{N}$ as acting on a Hilbert space H. If $\xi \in \mathbf{ran}\, p$, then the reasoning of part (1) shows that

$$\|v\xi\|^2 = \sum_i \|(u_{i-1} - u_i)\xi\|^2 = \sum_i (\|u_{i-1}\xi\|^2 - \|u_i\xi\|^2) = (1-\lambda^2)\|\xi\|^2.$$

As $v = v(\sum_i E_{ii} \otimes p)$, we obtain the desired estimate for $\|v\|$.

By the above inequalities, $(1+\lambda)^2\tau(p) \leqslant n(1-\tau(p))(1-\lambda^2)$, which yields the lower estimate for n. ∎

2.4. **Closure of products of projections.** For the proof of Theorem 1.5, we need a technical lemma (it may be known to specialists).

Lemma 2.18. *Suppose $\mathcal{N}$ is a separably acting von Neumann algebra without summands of finite type I. Then there exists a double indexed sequences of projections $p_n^{(k)}$ in $\mathcal{N}$ ($k \in \mathbb{N}$, $1 \leqslant n \leqslant 2^k$) such that:*

1. *For each k, the projections $(p_n^{(k)})_{n=1}^{2^k}$ are mutually orthogonal, equivalent to each other, and $\sum_{n=1}^{2^k} p_n^{(k)} = \mathbf{1}$. If $\mathcal{N}$ is properly infinite, then $p_n^{(k)} \sim \mathbf{1}$ for any n or k.*
2. *The sequence $(p_{2^k}^{(k)})_{k\in\mathbb{N}}$ converges to 0 in the strong operator topology.*

Proof. By the type decomposition, we can assume that either $\mathcal{N}$ is finite (hence of type II_1), or it is properly infinite. Part (1) can be obtained by a repeated applications of the standard "halving" results (see e.g. Propositions V.1.35 and V.1.36 of [15]). Moreover, we have $p_n^{(k)} = p_{2n-1}^{(k+1)} + p_{2n}^{(k+1)}$ for any $1 \leqslant n \leqslant 2^k$.

To tackle (2), recall that $\mathcal{N}$ is a WOT closed subalgebra of $B(H)$, where H is a separable Hilbert space. Suppose $(\xi_i)_{i\in\mathbb{N}}$ is a dense subset of the unit sphere H. It suffices to show that there exists a sequence $(n_k)_{k\in\mathbb{N}}$ such that $p_{n_k}^{(k)}\xi_i \to 0$ for every i. More precisely, we shall select sequences (n_k) and (k_s) so that $\|p_{n_s}^{(k_s)}\xi_i\| \leqslant 2^{-s}$ whenever $1 \leqslant i \leqslant s$.

By the Pythagorean theorem,

$$\sum_{n=1}^{2^k} \|p_n^{(k)}\xi_i\|^2 = 1 \tag{2.2}$$

for each i and k. More generally, for any i, $k > m$, and $1 \leqslant \ell \leqslant 2^m$,

$$\sum_{n=2^{k-m}(\ell-1)+1}^{2^{k-m}\ell} \|p_n^{(k)}\xi_i\|^2 = \|p_\ell^{(m)}\xi_i\|^2. \tag{2.3}$$

We select k_1 and n_1 using (2.2). Suppose $n_1, \ldots, n_s, k_1, \ldots, k_s$ satisfying our inequalities have been selected. Let $k_{s+1} = k_s + s + 2$, and find $n_{s+1} \in (2^{k_{s+1}-k_s}n_s, 2^{k_{s+1}-k_s}n_s]$ with the desired properties. To this end, for $1 \leqslant i \leqslant s+1$, let

$$S_i = \{n \in (2^{k_{s+1}-k_s}n_s, 2^{k_{s+1}}n_s] \mid \|p_n^{(k)}\xi_i\| \leqslant 2^{-(s+1)}\},$$

and $S_i^c = (2^{k_{s+1}-k_s} n_s, 2^{k_{s+1}-k_s} n_s] \backslash S_i$. By (2.3),

$$\sum_{n=2^{k-k_s}(n_s-1)+1}^{2^{k-k_s}n_s} \|p_n^{(k)}\xi_i\|^2 \leqslant 2^{-2s}$$

for $1 \leqslant i \leqslant s$, and

$$\sum_{n=2^{k-k_s}(n_s-1)+1}^{2^{k-k_s}n_s} \|p_n^{(k)}\xi_{s+1}\|^2 \leqslant 1.$$

Therefore, $|S_i^c| < 4$ for $1 \leqslant i \leqslant s$, and $|S_{s+1}^c| < 2^{2(s+1)}$. But

$$|(2^{k_{s+1}-k_s} n_s, 2^{k_{s+1}-k_s} n_s]| = 2^{k_{s+1}-k_s} > 4^{s+1} + 4s > \sum_{i=1}^{s+1} |S_i^c|,$$

and therefore, by the pigeon-hole principle, there exists $n_s \in \cup_{i=1}^{s+1} S_i$. ■

Proof of Theorem 1.5. First suppose $\mathcal{N}$ has no summands of type I_n. It suffices to show that any $u \in \mathcal{N}$ with $\|u\| < 1$ belongs to the $\boldsymbol{\tau}$-closure of the products of projections.

Write $\mathcal{N} = \mathcal{N}_{[f]} \oplus \mathcal{N}_{[i]}$, where the von Neumann algebras $\mathcal{N}_{[f]}$ and $\mathcal{N}_{[i]}$ are finite and properly infinite, respectively. The identities of the summands are denoted by $\mathbf{1}_{[i]}$ and $\mathbf{1}_{[f]}$, respectively. By Lemma 2.18, there exist mutually orthogonal projections $(p_{[f],n}^{(k)})_{n=1}^{2^k} \in \mathcal{N}_{[f]}$ and $(p_{[i],n}^{(k)})_{n=1}^{2^k} \in \mathcal{N}_{[f]}$, so that

(1) $p_{[f],m}^{(k)} \sim p_{[f],n}^{(k)}$ if $m \neq n$, and $p_{[i],n}^{(k)} \sim \mathbf{1}_{[i]}$ for any n;
(2) $\sum_{n=1}^{2^k} p_{[i],n}^{(k)} = \mathbf{1}_{[i]}$, and $\sum_{n=1}^{2^k} p_{[f],n}^{(k)} = \mathbf{1}_{[f]}$;
(3) the sequences $(p_{[i],2^k}^{(k)})_{k \in \mathbb{N}}$ and $(p_{[f],2^k}^{(k)})_{k \in \mathbb{N}}$ converge to 0 in the SOT.

For $k \in \mathbb{N}$ and $\alpha \in \{i, f\}$, let $v_{[\alpha],k} = p_{[\alpha],2^k}^{(k)\perp} u p_{[\alpha],2^k}^{(k)\perp}$. Clearly, $v_{[i],k} + v_{[f],k} \to u$ in the topology $\boldsymbol{\tau}$. It remains to approximate $v_{[\alpha],k}$ in the norm topology by products of projections.

Denote the initial and final projections of $v_{[\alpha],k}$ by $\overline{p}_{[\alpha],k}^{(in)}$ and $\overline{p}_{[\alpha],k}^{(fi)}$, respectively.

First consider $v_{[f],k}$. By Proposition V.1.38 of [15], $\mathbf{1}_{[f]} - \overline{p}_{[\alpha],k}^{(in)} \sim \mathbf{1}_{[f]} - \overline{p}_{[\alpha],k}^{(fi)}$. That is, $\mathbf{ker}\, v_{[i],k} \sim (\mathbf{ran}\, v_{[i],k})^\perp$. Moreover, $\mathbf{ran}\, p_{2^k}^{(k)} \hookrightarrow \mathbf{ker}\, v_{[i],k}$, hence $\mathbf{ker}\, v_{[i],k}$ is 2^k-majorant. By Theorem 1.1, $v_{[f],k}$ is a product of projections.

Now consider $v_{[i],k}$. Then $\mathbf{ker}\, v_{[i],k} \succ p_{[i],2^k}^{(k)} \sim \mathbf{1}_{[i]}$, hence $\mathbf{ker}\, v_{[i],k}$ is 1-majorant. Moreover, $(\mathbf{ran}\, v_{[i],k})^\perp \succ p_{[i],2^k}^{(k)} \sim \mathbf{1}_{[i]}$, hence $\mathbf{ker}\, v_{[i],k} \sim (\mathbf{ran}\, v_{[i],k})^\perp$. By Theorem 1.1, $v_{[i],k}$ is a product of projections.

Now suppose $z\mathcal{N}$ is a type I_n von Neumann algebra for some central projection $z \in \mathcal{N}$ and $n \in \mathbb{N}$. It suffices to show that $-z$ does not belong to the WOT closure of the products of projections in $z\mathcal{N}$. We shall find a WOT continuous linear functional

f on $z\mathcal{N}$ s.t. $f(z) = n$, and, for any product of projections u, either $|f(u)| \leqslant n-1$, or $u = z$.

By Theorems 9.3.2 and 9.4.1 of [9], $z\mathcal{N}$ is unitarily equivalent (via a unitary U) to $M_n \otimes L_\infty(\Omega, \mu) \otimes I_K$, where K is a Hilbert space, and μ is a σ-finite measure on a set Ω. Denote by $(e_i)_{i=1}^n$ the canonical basis in ℓ_2^n, and fix norm one vectors $\xi \in L_2(\mu)$ and $\eta \in K$. Define f by setting, for $a \in z\mathcal{N}$,

$$f(a) = \sum_{i=1}^n \langle (U^* a U) e_i \otimes \xi \otimes \eta, e_i \otimes \xi \otimes \eta \rangle.$$

Clearly, $f(z) = n$. Moreover, if $u \in z\mathcal{N}$ is a product of projections, different from z, then $(U^* u U)(\omega)|_{\ell_2^n \otimes \mathbb{C}\eta}$ has norm not exceeding 1, and rank less than n, for almost every $\omega \in \Omega$. Then, $|f(u)| \leqslant n-1$. ■

3. Products of idempotents

In this section, we consider products of idempotents in von Neumann algebras. A subspace F, affiliated with a von Neumann algebra $\mathcal{N}$, is called a *complement* of E if there exists an idempotent $p \in \mathcal{N}$ s.t. $\mathbf{ran}\, p = E$ and $\mathbf{ker}\, p = F$. A complement need not be unique. However, all the complements of a given E are equivalent to each other. Indeed, suppose F_1 and F_2 are two complements of E, corresponding to the idempotents p_1 and p_2, respectively. Then $\mathbf{1} - p_1$ is a bijection from F_2 to F_1.

Moreover (see e.g. [3, 10]), if q is an idempotent in a C^*-algebra (not necessarily a von Neumann algebra) $\mathcal{N}$, then there exists a (unique) projection $p \in \mathcal{N}$ onto the range of q (equivalently, $pq = q$ and $qp = p$). Furthermore, there exists an invertible $u \in \mathcal{N}$ s.t. $q = pu$ and $up = p$. Indeed, consider

$$u = q + p^\perp(\mathbf{1} - q) = p^\perp + q = \mathbf{1} + qp^\perp(\mathbf{1} - q).$$

Then u is invertible ($u^{-1} = \mathbf{1} - qp^\perp(\mathbf{1} - q)$), and the above inequalities can be easily verified. In fact, u maps $\mathbf{ker}\, q$ onto $\mathbf{ker}\, p$ ($u(\mathbf{1} - q) = p^\perp$).

Extending this to a product of several idempotents, we obtain:

Lemma 3.1. *Suppose $q_1, \ldots, q_n$ are idempotents in a C^*-algebra $\mathcal{N}$. Then there exist projections $p_1, \ldots, p_n \in \mathcal{N}$, and an invertible element $v \in \mathcal{N}$, such that $\cap_{k=1}^n \mathbf{ran}\, p_k = \cap_{k=1}^n \mathbf{ran}\, q_k$, $q_n \ldots q_1 = v p_n \ldots p_1$, and $v|_{\cap_{k=1}^n \mathbf{ran}\, p_k} = I_{\cap_{k=1}^n \mathbf{ran}\, p_k}$.*

Proof. Proceed by induction over n. The base of induction (the case of $n = 1$) has already been established. Suppose the statement is true for $n-1$, and prove it for n. Suppose $q_1, \ldots, q_n$ are idempotents. Then there exist projections $p_1, \ldots, p_{n-1}$ and an invertible u s.t. $q_{n-1} \ldots q_1 = u p_{n-1} \ldots p_1$, and the set of fixed points of u contains $\cap_{k=1}^{n-1} \mathbf{ran}\, p_k = \cap_{k=1}^{n-1} \mathbf{ran}\, q_k$. Write $u^{-1} q_n u = w p_n$, where the projection p_n has the same range as $u^{-1} q_n u$, w is invertible, and $w|_{\mathbf{ran}\, p_n} = I_{\mathbf{ran}\, p_n}$. Therefore, $q_n \ldots q_1 = v p_n \ldots p_1$, with $v = uw$, has the desired properties. ■

Proof of Theorem 1.6(1). Suppose a is a product of n idempotents, belonging to a von Neumann algebra $\mathcal{N}$, acting on a separable Hilbert space H. By Lemma 3.1, we can write $a = vp_n \dots p_1$, where v is invertible, and $p_1, \dots, p_n \in \mathcal{P}(\mathcal{N})$ s.t. $a = vp_n \dots p_1$. Moreover, v fixes $E = \cap_{k=1}^n \mathbf{ran}\, p_k$. Let $F = H \ominus (E + \mathbf{ker}\, u)$ (clearly, $E + \mathbf{ker}\, u$ is closed). Then $F' = \overline{p_n \dots p_1(F)}$ is orthogonal to E (cf. Observation (iii) of Section 2.1). Let $G = v(F')$. As v is invertible, G and E are at positive angles. An application of Lemma 2.13 and Corollary 2.16 completes the proof. ∎

To prove Theorem 1.6(2), we need some auxiliary results.

Lemma 3.2. *Suppose $\mathcal{N}$ is a von Neumann algebra, and the projections $p_1, p_2 \in \mathcal{N}$ satisfy $\|p_1p_2\| < 1$. Then for any $u \in \mathcal{N}$ satisfying $u = p_1up_2$ there exists an idempotent $h \in \mathcal{N}$ such that $hp_2 = u$, and $p_1h = h$.*

Proof. By the proof of Theorem 2.1 of [10], there exists an idempotent $q \in \mathcal{N}$ s.t. $qp_1 = p_1$, $p_1q = q$, and $qp_2 = 0$ (in other words, $\mathbf{ran}\, q = \mathbf{ran}\, p_1$ and $\mathbf{ker}\, q \supset \mathbf{ran}\, p_2$). Then $h = qu(\mathbf{1} - q) + q$ is an idempotent ($h^2 = h$). Moreover, $qp_2 = 0$, hence $(\mathbf{1} - q)p_2 = p_2$, and $hp_2 = qup_2 = qp_1up_2 = p_1up_2 = u$. Finally, $p_1q = q$, hence $p_1h = h$. ∎

Proof of Theorem 1.6(2). Consider u and F as in the statement of the theorem. Then the spaces $G = \overline{u(F)}$ and $G' = \mathbf{pr}(E)^{\perp}(G)$ are affiliated with $\mathcal{N}$. Let p_0, p and q be the projections onto E, G', and $(E+G)^{\perp}$, respectively. Moreover, let p' be the idempotent whose range is E, and whose kernel is $G + (E+G)^{\perp}$ (it belongs to $\mathcal{N}$, by Theorem 2.1 of [10]). Note that $\mathbf{ker}\, p = \mathbf{ker}\, p'$, hence, by the remark before Lemma 3.1, there exists an invertible $w \in \mathcal{N}$, s.t. $wp = p'$, and $w|_{\mathbf{ker}\, p} = I_{\mathbf{ker}\, p}$.

Let $c = (2\|pu\mathbf{pr}(F)\|)^{-1}$, $u' = p_0 + cpu\mathbf{pr}(F)$, and $v = p_0 + c^{-1}p'$. Then $u = vu'$. By Theorem 1.1, u' is a product of $2\gamma n$ projections. It remains to show that v is a product of $2n$ idempotents.

There exist mutually orthogonal projections $p_1, \dots, p_n$ s.t. $p_1 + \dots + p_n = p$, and partial isometries w_k, with the property that $p_k = w_k^* w_k$, and $q_k = w_k w_k^* \leqslant q$. Let $p_i' = wp_i$. Note that $pp_i' = pwpp_i = pp'p_i = pp_i = p_i$, hence

$$p_j p_i' = p_j p p_i' = p_j p_i = \begin{cases} p_j & j = i \\ 0 & j \neq i \end{cases} .$$

Furthermore,

$$p_j' p_i' = w p_j p_i' = \begin{cases} p_j' & j = i \\ 0 & j \neq i \end{cases} .$$

Finally, $p_i' p_j = w p_i p_j = 0$ if $i \neq j$, $= p_j'$ otherwise.

Note that q_k is orthogonal to $p_0 + p$ for each k, hence, by Lemma 3.2, for $1 \leqslant k \leqslant n$ there exists an idempotent $h_{k1} \in \mathcal{N}$ s.t. $h_{k1}(p_0 + p) = w_k$. Moreover, $\mathbf{pr}(\mathbf{ran}\, p_k')$ is at positive angles with p_0, and orthogonal to $\sum_{i \neq k} p_i$, hence, by Lemma 3.2 again, there exists an idempotent $h_{k2} \in \mathcal{N}$ s.t. $h_{k2}(q_k + p_0 + \sum_{i \neq k} p_i) = vw_k^*$. Note that

$\mathbf{ran}\, h_{k1} = \mathbf{ran}\, q_k$ and $\mathbf{ran}\, h_{k2} = \mathbf{ran}\, p'_k$ (by definition, v takes $\mathbf{ran}\, p_k$ onto $\mathbf{ran}\, p'_k$). Then $h'_{k1} = h_{k1} + p_0 + \sum_{i \neq k} p_i$ is an idempotent. Furthermore, let $\tilde{p}_k = \mathbf{pr}(\mathbf{ran}\, p_0 p'_k)$. By [10], there exists an idempotent $p''_k \in \mathcal{N}$ s.t. $\tilde{p}_k p''_k = p''_k$, $p''_k \tilde{p}_k = \tilde{p}_k$, and

$$p''_k(p_0 - \tilde{p}_k + \sum_{i \neq k} p_i + p'_k + q) = 0.$$

Then $h'_{k2} = h_{k2} + p_0 - \tilde{p}_k + p''_k + \sum_{i \neq k} p_i$ is an idempotent, and

$$h'_{k2} h'_{k1} = p'_k v p_k + \sum_{i \neq k} p_i + p_0.$$

From the above formulae on the products of p_i's and p'_j's, $\prod_{k=1}^n (h'_{k2} h'_{k1}) = v$. Thus, v is a product of $2n$ idempotents. ■

Proposition 3.3. *Any product of idempotents in a von Neumann algebra $\mathcal{N}$ belongs to the norm closure of invertible elements of $\mathcal{N}$.*

Proof. Combine Theorem 1.6 with Theorem 1 of [7]. ■

Proposition 3.4. *Suppose $\mathcal{N}$ is a von Neumann algebra without finite type I summands, acting on a Hilbert space H. Then for every $c > 0$ there exists $u \in \mathcal{N}$ s.t. $\mathbf{ran}\, u \prec \mathbf{ker}\, u$, and $\|u\| = c$, yet u cannot be written as a product of less than three idempotents in $B(H)$.*

Lemma 3.5. *Suppose $\mathcal{N}$ is a von Neumann algebra without finite type I summands, and $p_1, \ldots, p_n \in \mathcal{N}$ are mutually orthogonal, mutually equivalent projections, such that $p_1 + \ldots + p_n = \mathbf{1}$. Then $p_1 \mathcal{N} p_1$ has no finite type I summands.*

Proof. Suppose, for the sake of contradiction, there exists a central projection $z_1 \in p_1 \mathcal{N} p_1$ s.t. $z_1 p_1 \mathcal{N} p_1 = z_1 \mathcal{N}$ is of finite type I. For $2 \leqslant k \leqslant n$, there exists $u_k \in \mathcal{N}$ s.t. $u_k u_k^* = p_1$ and $u_k^* u_k = p_k$. Then $z = z_1 + \sum_{k=2}^n u_k^* z_1 u_k$ is a central projection in $\mathcal{N}$. Indeed, any $x \in \mathcal{N}$ can be written as $x = \sum_{k,\ell=1}^n p_k x p_\ell$. Note that

$$z p_k x p_\ell = u_k^* z u_k u_k^* u_k x u_\ell^* u_\ell = u_k^* (z p_1)(u_k x u_\ell^*) u_\ell = u_k^* (u_k x u_\ell^*) z_1 u_\ell = p_k x p_\ell z,$$

which shows the centrality of z. Then $z\mathcal{N}$ is isomorphic to $M_n(z_1 \mathcal{N})$, hence of finite type I. ■

Proof of Proposition 3.4. By [15], Proposition V.1.35, any von Neumann algebra $\mathcal{M}$ with no finite type I summands contains a projection p s.t. $p \sim p^\perp$. By Lemma 3.5, $p\mathcal{M}p$ has no finite type I summands. Denoting $\mathbf{1}$ by $p_0^{(0)}$, and applying these results, we obtain the existence of projections $(p_k^{(n)})$ ($n \in \mathbb{N}$, $k \in \{0, 1, 2, 3\}$), such that, for each $n \in \mathbb{N}$, the projections $(p_k^{(n)})$ ($k = 0, 1, 2, 3$) are mutually orthogonal, mutually equivalent, and $\sum_{k=0}^3 p_k^{(n)} = p_0^{(n-1)}$. For $n \in \mathbb{N}$ and $k = 2, 3$, there exist partial isometries $u_k^{(n)} \in \mathcal{N}$ s.t. $u_k^{(n)} u_k^{(n)*} = p_{k-1}^{(n)}$, and $u_k^{(n)*} u_k^{(n)} = p_k^{(n)}$. Define $u \in \mathcal{N}$ by setting $u = c \sum_{n=1}^\infty \sum_{k=2}^3 u_k^{(n)}$ As shown in [5], u cannot be a product of less than three idempotents. ■

Theorem 3.6. *Suppose $\mathcal{N}$ is a separably acting von Neumann factor, and $u \in \mathcal{N}$ is such that both $\mathbf{pr}(\mathbf{ker}\, u)$ and $\mathbf{pr}(\mathbf{ran}\, u)^\perp$ are equivalent to $\mathbf{1}$. Then u is a product of three idempotents.*

In the statement of the theorem, $\mathcal{N}$ is a factor of type I_∞, II_∞, or III. The type I_∞ case was handled in [5].

Lemma 3.7. *Suppose $\mathcal{N}$ is an infinite von Neumann factor, and the projections $p_1, p_2 \in \mathcal{N}$ are such that (1) $\|p_1 p_2\| < 1$, and (2) $p_1^\perp$ and $p_2^\perp$ are infinite. Then there exists an infinite projection $p \in \mathcal{N}$ s.t. $\|pp_i\| < 1$ for $i = 1, 2$.*

Proof. If p_1 is finite, then $p = p_1^\perp \wedge p_2^\perp$ is an infinite projection. Indeed,

$$p_2^\perp - p_2^\perp \wedge p_1^\perp \sim p_2^\perp \vee p_1^\perp - p_1^\perp \leqslant \mathbf{1} - p_1^\perp = p_1,$$

hence $p_2^\perp \prec p_2^\perp \wedge p_1^\perp + p_1$. If p is a finite projection, then $p_2^\perp$ is also finite, which yields a contradiction.

Thus, we can assume that both p_1 and p_2 are infinite, and $p_1^\perp \vee p_2^\perp$ is finite. Consider the polar decomposition $p_1 p_2 = UA$, where A is a positive operator, and U is a partial isometry. Suppose first that there exists $c > 0$ s.t. $q = \chi_{[c,1]}(A)$ is an infinite projection. Then $E = \mathbf{ran}\, q$ is a subspace of $F = \mathbf{ran}\, p_2$. As U is a partial isometry, $p_1(E)$ is orthogonal to $p_1(F \ominus E)$. Then $p_1^\perp(E)$ is orthogonal to $p_1^\perp(F \ominus E)$. Indeed, for $\xi \in E$ and $\eta \in F \ominus E$,

$$\langle p_1^\perp \xi, p_1^\perp \eta \rangle = \langle \xi, \eta \rangle - \langle p_1 \xi, p_1 \eta \rangle = 0.$$

Let $p = \mathbf{pr}(p_1^\perp(E))$. Then p is infinite (it is equivalent to q), $pp_1 = 0$, and $\|pp_2\| = \|p_1^\perp q\| \leqslant \sqrt{1 - c^2}$ (the last inequality follows from the Pythagorean Theorem). Thus, p has the desired properties.

Suppose $\chi_{[0.1,1]}(A)$ is a finite projection. Then $q = p_2 \chi_{[0,0.1]}(A)$ is infinite, and so is $q_1 = p_1 - \mathbf{pr}(\mathbf{ran}\,(p_1(p_2 - q)))$. Moreover, for any $\xi \in \mathbf{ran}\, q$,

$$\|p_1^\perp \xi\|^2 = \|\xi\|^2 - \|p_1 \xi\|^2 = \|\xi\|^2 - \|A\xi\|^2 \geqslant 0.99 \|\xi\|^2.$$

Therefore, $p_1^\perp$ is injective on $\mathbf{ran}\, q$, and $q_2 = \mathbf{pr}(\mathbf{ran}\,(p_1^\perp q))$ is infinite.

Consider a partial isometry u s.t. $u^* u = q_2$ and $uu^* = q_1$. Let $v = (uq_2 + q_2)/\sqrt{2}$ (this is a partial isometry), and $p = \mathbf{pr}(\mathbf{ran}\, v)$. Then $p \sim q_2$, hence it is infinite. For any $\xi \in \mathbf{ran}\, p$, $\|q_1 \xi\| = \|q_2 \xi\| = \|\xi\|/\sqrt{2}$. Therefore, $\|q_2 p\| = \|q_1 p\| = 2^{-1/2}$. Note that $p_1 - q_1$ is orthogonal to both q_1 and $q_2 \leqslant p_1^\perp$, hence $(p_1 - q_1)p = 0$, and $\|p_1 p\| = \|q_1 p\| = 2^{-1/2}$. Furthermore, $(p_2 - q)p = 0$, and therefore, $p_2 p = qp$. Since $\|qp\| \leqslant \|q_2 p\| + \|q - q_2\|$, it remains to establish that $\|q - q_2\| < 0.11$.

By Akhiezer-Glazman formula (see e.g. [11, 17]),

$$\|q - q_2\| = \max\{\|q_2^\perp q\|, \|q^\perp q_2\|\}. \tag{3.1}$$

Recall that q_2 was defined in such a way that $\|q_2^\perp q\| = \|p_1 q\| \leqslant 0.1$. Furthermore, for any norm one $\xi \in \mathbf{ran}\, q_2$ there exists $\eta \in \mathbf{ran}\, q$ s.t. $\xi = q_2\eta$, and $\|\eta\|^2 \leqslant 1/(1-10^{-2})$. Then

$$\|q^\perp \xi\|^2 \leqslant \|\xi - \eta\|^2 = \|\eta\|^2 - \|\xi\|^2 \leqslant 1/99,$$

and therefore, $\|q^\perp q_2\| \leqslant 99^{-1/2} < 0.11$. Plugging the estimates for $\|q^\perp q_2\|$ and $\|q_2^\perp q\|$ into (3.1), we complete the proof. ■

Proof of Theorem 3.6. Let $p_1 = \mathbf{pr}((\mathbf{ker}\, u)^\perp)$ and $p_2 = \mathbf{pr}(\mathbf{ran}\, u)$. By Lemma 3.7, there exists $p_3 \sim \mathbf{1}$ s.t. $\|p_3p_1\|, \|p_3p_2\| < 1$. Denote by v the partial isometry satisfying $vv^* = p_3$, $v^*v = p_1$. By Lemma 3.2, there exist idempotents $q_2, q_3 \in \mathcal{N}$ s.t. $q_2p_1 = v$, and $q_3p_3 = uv^*$. Then $q_3q_2p_1 = up_1 = u$. ■

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