

# THE NON-COMMUTATIVE GURARII SPACE

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ABSTRACT. We construct a “universal space” for 1-exact finite dimensional operator spaces (an analogue of the Gurarii space).

## 1. INTRODUCTION

The main result of this paper is a construction of an “almost universal space for finite dimensional inclusions of 1-exact spaces.” Recall that an operator space  $X$  is called *c-exact* if, for every finite dimensional subspace  $E$  of  $X$ , and for every  $\varepsilon > 0$ , there exists a subspace  $F$  of  $M_n$  satisfying  $d_{cb}(E, F) < c + \varepsilon$ . Following [4], we say that a separable 1-exact space  $X$  is *non-commutative Gurarii* (NG for short) if it satisfies the following condition: suppose a finite dimensional operator space  $F$  is 1-exact,  $E$  is a subspace of  $F$ ,  $E'$  is a subspace of  $X$ , and  $u : E \rightarrow E'$  is an isomorphism. Then for any  $\delta > 0$  there exists a subspace  $F'$  of  $X$ , containing  $E'$ , and linear map  $\tilde{u} : F \rightarrow F'$ , such that  $\tilde{u}|_E = u$ , and  $\|\tilde{u}\|_{cb}\|\tilde{u}^{-1}\|_{cb} < (1 + \delta)\|u\|_{cb}\|u^{-1}\|_{cb}$ .

Following the classical theory of  $\mathcal{L}_\infty$ -spaces (found, for instance, in [8]), we say that an operator space  $X$  is an  $\mathcal{OL}_{\infty,c}$ -space ( $c \geq 1$ ) if, for every finite dimensional subspace  $E$  of  $X$ , there exists a finite dimensional  $C^*$ -algebra  $A$  (recall that  $A = (\sum_{k=1}^m M_{n_k})_\infty$ ), and a subspace  $F$  of  $X$ , containing  $E$ , s.t.  $d_{cb}(F, A) \leq c$ .  $X$  is an  $\mathcal{OL}_{\infty,c+}$ -space if it is an  $\mathcal{OL}_{\infty,c+\varepsilon}$ -space for any  $\varepsilon > 0$ . We say that  $X$  is an  $\mathcal{OL}_\infty$ -space if it is an  $\mathcal{OL}_{\infty,c}$ -space for some  $c \geq 1$ .

More generally, the classical  $\mathcal{L}_p$ -spaces have a non-commutative analogue, namely  $\mathcal{OL}_p$ -spaces. The investigation of the latter class of operator spaces has been very active lately, see e.g. [5] and the references contained therein.

Clearly, every  $\mathcal{OL}_{\infty,c+}$ -space is *c-exact*. By [2], every separable 1-exact space embeds into a separable  $\mathcal{OL}_{\infty,1+}$ -space completely isometrically. One can easily see that any NG space is an  $\mathcal{OL}_{\infty,1+}$ -space. We shall show that, moreover, every separable  $\mathcal{OL}_{\infty,1+}$ -space is contained in some separable NG space. More precisely:

**Theorem 1.1.** *Suppose  $Y$  is a separable  $\mathcal{OL}_{\infty,1+}$ -space. Then there exists a non-commutative Gurarii space, containing  $Y$  as a completely contractively complemented subspace.*

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This theorem is proved in Section 3. We also show the “almost uniqueness” of an NG space: any two NG spaces are  $c$ -completely isomorphic, for any  $c > 1$ . Furthermore, we establish that an NG space cannot be isometric to a unital Banach algebra.

The necessary ingredients are collected in Section 2.

Throughout the paper, we are freely using the standard operator space and Banach space terminology and results. The reader is referred to [3, 11, 12] for operator spaces, and [9] for Banach spaces.

## 2. PRELIMINARIES

Below we introduce some constructions involving operator spaces, to be used throughout the paper.

**2.1. The functor  $\text{MIN}_n$ .** Recall that, if  $E$  is an operator space, and  $n \in \mathbb{N}$ , then  $\text{MIN}_n(E)$  is isometric to  $E$  as a Banach space level, and

$$\|x\|_{\text{MIN}_n(E) \otimes \mathcal{K}_0} = \sup\{\|(u \otimes I_{\mathcal{K}_0})x\| \mid u \in CB(E, M_n), \|u\|_{cb} \leq 1\}$$

for  $x \in E \otimes I_{\mathcal{K}_0}$  ( $\mathcal{K}_0$  denotes the space of infinite matrices with finitely many non-zero entries). More information of these spaces can be found in [7, 10]. Some useful properties of the functor  $\text{MIN}_n$  are summarized in the following lemma:

**Lemma 2.1.** *Suppose  $E$  and  $F$  are operator spaces.*

(1) *If  $u \in B(E, F)$ , then*

$$\|u\|_{CB(E, \text{MIN}_n(F))} = \|I_{M_n} \otimes u\|_{B(M_n(E), M_n(F))}.$$

*In particular,*

$$\|u\|_{CB(\text{MIN}_n(E), \text{MIN}_n(F))} = \|I_{M_n} \otimes u\|_{B(M_n(E), M_n(F))}.$$

(2) *If  $E$  is a subspace of  $F$ , then  $\text{MIN}_n(E)$  is a subspace of  $\text{MIN}_n(F)$ .*

*Proof.* (1) follows from Smith’s lemma (Proposition 2.2.2 of [3]). An application of Stinespring Extension Theorem yields (2). ■

For an operator space  $E$  and  $n \in \mathbb{N}$ , let  $\alpha_n(E) = \|id : \text{MIN}_n(E) \rightarrow E\|_{cb}$  (this number can be finite or infinite). Clearly,  $\alpha_n(E) \leq \alpha_n(F)$  if  $E$  is a subspace of  $F$ . Moreover,  $\alpha_1(E) \geq \alpha_2(E) \geq \dots$ . If  $E$  is finite dimensional, then  $\lim_n \alpha_n(E) = \text{ex}(E)$  (the exactness constant of  $E$ ).

It is easy to see that, if  $\alpha_n(E)$  is finite, then  $E$  embeds into  $\ell(\mathcal{I}, M_n)$  ( $\mathcal{I}$  is an appropriate index set)  $\alpha_n(E)$ -completely isomorphically. If  $E$  is finite dimensional, then we have:

**Lemma 2.2.** *Suppose  $E$  is a finite dimensional operator space, and  $\varepsilon > 0$ . Then there exists an operator space  $X$ ,  $\alpha_n(E)$ -completely isomorphic to a finite dimensional  $C^*$ -algebra  $A$ , and a subspace  $F$  of  $X$  satisfying  $d_{cb}(E, F) < 1 + \varepsilon$ .*

The above result follows immediately from the following technical lemma:

**Lemma 2.3.** *Suppose  $C_1, C_2 \geq 1$ , the operator spaces  $E$  and  $F$  are  $C_1 C_2$ -completely isomorphic, and  $F$  is a subspace of  $X$ . Then there exists an operator space  $Y$ , its subspace  $G$ , and operators  $u : G \rightarrow E$  and  $v : X \rightarrow Y$ , such that  $\|u\|_{cb} \leq C_1$ ,  $\|u^{-1}\|_{cb} \leq 1$ ,  $\|v\|_{cb} \leq C_2$ ,  $\|v^{-1}\|_{cb} \leq 1$ , and  $v(F) = G$ .*

*Proof.* Find an operator  $T : F \rightarrow E$  s.t.  $\|T\|_{cb} = C_1 C_2$  and  $\|T^{-1}\|_{cb} = 1$ . Let  $F_0 = F$ , and define the new operator space structure  $F_1$  on  $F$  in by setting

$$\|x\|_{F_1 \otimes \mathcal{K}_0} = \|(T \otimes I_{\mathcal{K}})(x)\|_{E \otimes \mathcal{K}_0}$$

for any  $x \in F \otimes \mathcal{K}_0$ . Let  $G = (F_0, F_1)_\theta$ , where  $\theta \in [0, 1]$  satisfies  $(C_1 C_2)^\theta = C_2$ . Then

$$\|id : F_0 \rightarrow G\|_{cb} \leq \|id : F_0 \rightarrow F_0\|_{cb}^{1-\theta} \|id : F_0 \rightarrow F_1\|_{cb}^\theta \leq C_2,$$

and similarly,  $\|id : G \rightarrow F_1\|_{cb} \leq C_1$ ,  $\|id : G \rightarrow F_0\|_{cb} \leq 1$ , and  $\|id : F_0 \rightarrow G\|_{cb} \leq 1$  ( $id$  denotes the formal identity map). Since  $F_1$  is completely isometric to  $E$ , we obtain the existence of an operator  $u$  as in the statement of the lemma. To define  $Y$ , note that  $G$  embeds completely isometrically into  $B(H)$ , for some Hilbert space  $H$ . By Stinespring Extension Theorem, there exists  $S \in CB(X, B(H))$  s.t.  $\|S\|_{cb} \leq C_2$ , and  $S|_F = id : F \rightarrow G$ . Define  $Y$  by setting

$$\|y\|_{Y \otimes \mathcal{K}_0} = \max\{\|y\|_{X \otimes \mathcal{K}_0}, \|(S \otimes I_{\mathcal{K}})(y)\|_{B(H) \otimes \mathcal{K}_0}\}$$

for  $y \in X \otimes \mathcal{K}_0$ . The formal identity map from  $X$  to  $Y$  now works as  $v$ .  $\blacksquare$

**2.2. Direct sums and quotients of operator spaces.** Recall first that, for a family  $(Z_i)$  of operator spaces, the  $\ell_1$  direct sum  $Z = (\sum_i Z_i)_1$  is defined as follows: any element  $z \in Z \otimes \mathcal{K}_0$  can be represented by a family  $(z_i)$ , with  $z_i \in Z_i \otimes \mathcal{K}_0$ . We set

$$\|z\|_{Z \otimes \mathcal{K}_0} = \sup \left\| \sum_i (T_i \otimes I_{\mathcal{K}}) Z_i \right\|_{B(H) \otimes \mathcal{K}_0},$$

where the supremum is taken over all Hilbert spaces  $H$  (we can restrict their density character appropriately), and all families of complete contractions  $T_i : Z_i \rightarrow B(H)$ . On the Banach space level, this definition yields the classical  $\ell_1$  sum of Banach spaces (this follows from Hahn-Banach theorem). If the family  $(Z_i)$  is finite, we also use the  $\oplus_1$  notation as an alternative.

If  $Z$  is a subspace of an operator space  $Y$ , then the *quotient operator space*  $Y/Z$  is defined by setting, for  $x \in Y/Z \otimes \mathcal{K}_0$ ,

$$\|x\|_{Y/Z \otimes \mathcal{K}_0} = \inf\{\|y\|_{Y \otimes \mathcal{K}_0} \mid x = (q \otimes I_{\mathcal{K}})y\},$$

where  $q : Y \rightarrow Y/Z$  is the quotient map.

Throughout the paper, we are going to use the following construction: suppose  $(Z_i)_{0 \leq i \leq N}$  ( $N$  may be either finite or infinite) is a family of operator spaces,  $(E_i)_{i \geq 1}$  is a family of subspaces of  $Z_0$ , and, for each  $i \in [1, N]$ , we have a c.b. map  $u_i : E_i \rightarrow F_i$ , where  $F_i$  is a subspace of  $Z_i$ , and  $u_i^{-1}$  is a complete contraction. We define the space

$$Z = \left( \sum_{i \geq 0} Z_i \right)_1 / E, \text{ where } E = \text{span}[(z, 0, \dots, 0, -u_i z, 0, \dots) \mid z \in E_i]$$

(it is easy to see that  $E$  is a closed subspace of  $(\sum Z_i)_1$ ).

**Lemma 2.4.** *In the above notation, denote by  $J_k$  ( $k \geq 0$ ) the formal identity map from  $Z_k$  into  $Z$ . Then:*

- (1)  $J_0$  is a complete isometry.
- (2) For  $k \geq 1$ ,  $J_k$  is a complete contraction, and  $\|J_k^{-1}\|_{cb} \leq \|u_k\|_{cb}$ .

*Proof.* Clearly,  $J_k$  is a complete construction for every  $k$ .

To show that  $J_0$  is a complete isometry, consider  $x \in Z_0 \otimes \mathcal{K}_0$ . Then

$$\|(J_0 \otimes I_{\mathcal{K}_0})(x)\|_{Z \otimes \mathcal{K}_0} = \inf_{(e_i)} \sup_{(T_i)} \|(T_0 \otimes I_{\mathcal{K}_0})(x - \sum_{i \geq 1} e_i) + \sum_{i \geq 1} (T_i u_i \otimes I_{\mathcal{K}_0})(e_i)\|,$$

where the infimum and the supremum are taken over all selections of  $e_i \in E_i \otimes \mathcal{K}_0$ , and all the complete contractions  $T_i : Z_i \rightarrow B(H)$ , respectively. Let  $T_0 : Z_0 \rightarrow B(H)$  be a complete isometry. For  $i \geq 1$ , Stinespring Extension Theorem guarantees the existence of complete contractions  $T_i : Z_i \rightarrow B(H)$  for which  $T_i|_{F_i} = T_0 u_i^{-1}$ . Then

$$\|(T_0 \otimes I_{\mathcal{K}_0})(x - \sum_{i \geq 1} e_i) + \sum_{i \geq 1} (T_i u_i \otimes I_{\mathcal{K}_0})(e_i)\| = \|(T_0 \otimes I_{\mathcal{K}_0})(x)\| = \|x\|,$$

which proves (1).

It remains to prove that  $\|J_k^{-1}\|_{cb} \leq \|u_k\|_{cb}$  for  $k \geq 1$ . To this end, take  $x \in Z_k \otimes \mathcal{K}_0$ . Then

$$\begin{aligned} & \|(J_k \otimes I_{\mathcal{K}_0})(x)\|_{Z \otimes \mathcal{K}_0} \\ &= \inf_{(e_i)} \sup_{(T_i)} \|(T_k \otimes I_{\mathcal{K}_0})(x) + \sum_{i \geq 1} ((T_0 \otimes I_{\mathcal{K}_0})(e_i) - (T_i u_i \otimes I_{\mathcal{K}_0})(e_i))\|, \end{aligned}$$

where  $e_i \in E_i \otimes \mathcal{K}_0$ , and the maps  $T_i : Z_i \rightarrow B(H)$  are complete contractions. Now suppose  $T : Z_k \rightarrow B(H)$  is a complete isometry, and let  $T_k = T/\|u_k\|_{cb}$ . Let  $T_0 : Z_0 \rightarrow B(H)$  be a complete contraction for which  $T_0|_{E_k} = T_k u_k$ . For  $i \notin \{0, k\}$ , find a complete contraction  $T_i : Z_i \rightarrow B(H)$  so that  $T_i|_{F_i} = T_0 u_i^{-1}$ . Then

$$\begin{aligned} \|(J_k \otimes I_{\mathcal{K}_0})(x)\|_{Z \otimes \mathcal{K}_0} &\geq \|(T_k \otimes I_{\mathcal{K}_0})(x) + \sum_{i \geq 1} ((T_0 \otimes I_{\mathcal{K}_0})(e_i) - (T_i u_i \otimes I_{\mathcal{K}_0})(e_i))\| \\ &= \|(T_k \otimes I_{\mathcal{K}_0})(x)\| = \frac{\|x\|}{\|u_k\|_{cb}}. \end{aligned}$$

■

**2.3. The general form of a separable  $\mathcal{OL}_{\infty, c+}$ -space.** Following [6], we use “inductive limits” to describe separable  $\mathcal{OL}_{\infty, c+}$ -spaces. Suppose we are given a sequence of complete contractions  $u_i : E_i \rightarrow E_{i+1}$  ( $i \in \mathbb{N}$ ), such that  $\prod_i \|u_i^{-1}\|_{cb} < \infty$  (abusing the notation somewhat, we view  $u_i^{-1}$  as a map from  $u_i(E_i)$  to  $E_i$ ;  $u_i^{-1}$  need not be defined on the whole  $E_{i+1}$ ). Then the inductive limit  $\lim_{(u_i)} E_i$  of the sequence  $(E_i)$  contains a  $\prod_{i \geq j} \|u_i^{-1}\|_{cb}$ -completely isomorphic copy of  $E_j$ , for any  $j$ .

**Lemma 2.5.** *For a separable infinite dimensional operator space  $X$ , the following statements are equivalent:*

- (1)  $X$  is an  $\mathcal{OL}_{\infty, c+}$ -space.
- (2) For any sequence  $(\varepsilon_i)_{i=1}^{\infty}$  of positive numbers, satisfying  $\sum_i \varepsilon_i < \infty$ , there exists a sequence of finite dimensional spaces  $(Y_i)_{i=1}^{\infty}$ , such that  $d_{cb}(Y_i, A_i) \leq c$  ( $A_i$  is a finite dimensional  $C^*$ -algebra), and a sequence of complete contractions  $u_i : Y_i \rightarrow Y_{i+1}$ , such that  $\|u_i^{-1}\|_{cb} < 1 + \varepsilon_i$ , and  $X = \lim_{(u_i)} Y_i$ .
- (3) The statement of (2) holds for some sequence  $(\varepsilon_i)_{i=1}^{\infty}$  with  $\sum_i \varepsilon_i < \infty$ .

*Proof.* The implication (2)  $\Rightarrow$  (3) is trivial, and (3) implies (1) by the discussion above. It remains to show that (1)  $\Rightarrow$  (2). Find a sequence  $(\delta_i)$  of positive numbers satisfying  $(1 + \delta_{i+1})^2 < 1 + \delta_i < \sqrt{1 + \varepsilon_i}$  (for any  $i$ ). By definition, there exists a sequence  $(X_i)$  of finite dimensional subspaces of  $X$ , such that  $X_1 \hookrightarrow X_2 \hookrightarrow \dots$ ,  $X = \overline{\cup_i X_i}$ , and  $d_{cb}(X_i, A_i) < c(1 + \delta_i)$ , where, as before,  $A_i$  is a finite dimensional  $C^*$ -algebra. By Lemma 2.2, for any  $i$  there exists an operator space  $Y_i$  s.t.  $d_{cb}(A_i, Y_i) \leq c$ , and there exists  $T_i : Y_i \rightarrow X_i$  s.t.  $\|T_i\|_{cb} = (1 + \delta_i)^{-1}$ , and  $\|T_i^{-1}\|_{cb} \leq (1 + \delta_i)^2$ . Then  $u_i = T_{i+1}^{-1} T_i$  is a complete contraction, and  $\|u_i^{-1}\|_{cb} < (1 + \delta_i)^2 < 1 + \varepsilon_i$ .  $\blacksquare$

### 3. PROOF OF THE MAIN RESULT

In this section, we establish Theorem 1.1, and show some properties of NG spaces. A word about notation: if  $u : X \rightarrow Y$  is a linear map, we shall use  $u^{-1}$  for an operator from  $u(Y)$  to  $X$ .

*Proof of Theorem 1.1.* Fix a decreasing sequence  $(\delta_i)_{i \in \mathbb{N}}$  of positive numbers, s.t.  $\sum_i \delta_i < \infty$ . By Lemma 2.5, there exists a sequence  $(Y_i)_{i=0}^{\infty}$  of finite dimensional  $C^*$ -algebras, and operators  $u_i : Y_{i-1} \rightarrow Y_i$  ( $i \geq 1$ ), such that  $Y = \lim_{(u_i)} Y_i$ ,  $\|u_i\|_{cb} < (1 + \delta_i)^{-1/2}$ , and  $\|u_i^{-1}\|_{cb} < 1 + \delta_i$ . Find an increasing sequence  $(n_i)$  such that  $Y_i$  embeds into  $M_{n_i}$  completely isometrically. For simplicity, assume that  $Y_0$  is 1-dimensional, and take  $n_0 = 1$ .

Fix a sequence  $(c_i)_{i=1}^{\infty} \subset (1, \infty)$ , increasing without a bound. We are going to construct a sequence of separable spaces  $(X_i)_{i=0}^{\infty}$ , such that:

- (1)  $\alpha_{n_i}(X_i) = 1$ .
- (2) There exists a complete contraction  $v_i : X_{i-1} \rightarrow X_i$ , such that  $\|v_i^{-1}\|_{cb} = \|u_i^{-1}\|_{cb} < 1 + \delta_i$ .

- (3) There exists a completely isometric embedding  $J_i : Y_i \rightarrow X_i$ , and a completely contractive projection  $P_i$  from  $X_i$  onto  $J_i(Y_i)$ , satisfying  $J_i u_i = P_i v_i J_{i-1}$ .
- (4) If  $E$  is a subspace of  $X_{i-1}$  of dimension less than  $n_i^2$ , and the operator  $T : E \rightarrow M_{n_i}$  is such that  $\|T\|_{cb} < c_i$ , and  $T^{-1}$  is a complete contraction, then there exists a complete contraction  $S : M_{n_i} \rightarrow X_i$  s.t.  $\|S^{-1}\|_{cb} < (1 + \delta_i)\|T\|_{cb}$ , and  $ST = v_i|_E$ .

Once this sequence is constructed, we define  $X$  as  $\lim_{(v_i)} X_i$ . Clearly,  $X$  contains  $Y$  as a completely contractively complemented subspace. To show that  $X$  is an NG space, suppose  $E, E', F, u$ , and  $\delta$  are as in the definition of such space. Without loss of generality, assume that  $\|u\|_{cb} = 1$ , and let  $c = \|u^{-1}\|_{cb}$ . Fix  $\sigma > 0$  for which  $(1 + \sigma)^7 < 1 + \delta$ . There exists  $w : F \rightarrow G$  ( $G$  is a subspace of  $M_n$ ), s.t.  $\|w^{-1}\|_{cb} = 1$ , and  $\|w\|_{cb} < 1 + \sigma$ . For  $i \geq 1$  let  $\tilde{v}_i = \dots \circ v_{i+1} \circ v_i$  be the embedding of  $X_{i-1}$  into  $X$ . By (2),  $\tilde{v}_i$  is completely contractive, and  $\|\tilde{v}_i^{-1}\|_{cb} < \lambda_i = \prod_{j \geq i} (1 + \delta_j)$ .

Select  $i$  so large that: (1)  $n_i \geq n$ , and  $c_i > c(1 + \sigma)^3$ ; (2)  $\lambda_i < 1 + \sigma$ ; (3) there exists an automorphism  $U$  of the space  $X$  such that  $\|U^{-1}\|_{cb} < (1 + \sigma)^2$ ,  $\|U\|_{cb} = 1/\lambda_i$ , and  $U(E') \subset \tilde{v}_i(X_{i-1})$  (the existence of such  $i$  follows from the ‘‘small perturbation principle’’, see e.g. Section 2.13 of [12]).

Consider a subspace  $E_0 = \tilde{v}_i^{-1}(U(E'))$  of  $X_{i-1}$ , and define  $T : E_0 \rightarrow M_n$  by setting  $T = wu^{-1}U^{-1}\tilde{v}_i|_{E_0}$ . Then  $T^{-1}$  is a complete contraction, and  $\|T\|_{cb} < c(1 + \sigma)^3$ . By (4), there exists a complete contraction  $S : M_n \rightarrow X_i$  s.t.  $ST = v_i|_{E_0}$ , and  $\|S^{-1}\|_{cb} < c(1 + \sigma)^4$ . Then  $\tilde{u} = U\tilde{v}_{i+1}Sw$  extends  $u$ ,  $\|\tilde{u}\|_{cb} < (1 + \sigma)^3$ , and  $\|\tilde{u}^{-1}\|_{cb} < c(1 + \sigma)^4$ .

To construct a sequence  $(X_i)$  satisfying (1) - (4) above, let  $X_0 = Y_0$ . Suppose  $X_{i-1}$ , satisfying the desired properties, has already been constructed. We shall use it to describe  $X_i$ .

By Section 2.13 of [12], there exists a family  $(E_j, T_j)_{j \in \mathbb{N}}$  ( $E_j$  is a subspace of  $X_{i-1}$  of dimension less than  $n_i^2$ , and  $T_j : E_j \rightarrow M_{n_i}$  satisfies  $\|T_j\|_{cb} \leq c_i$ ,  $\|T_j^{-1}\|_{cb} = 1$ ) is such that, for every  $E$  and  $T$  as in (4) above, and for every  $\varepsilon > 0$ , there exist automorphisms  $U$  and  $V$  of  $X_{i-1}$  and  $M_{n_i}$ , respectively, such that  $U(E) = E_j$ ,  $V^{-1}T_jU|_E = T$ , and  $\|U\|_{cb}, \|U^{-1}\|_{cb}, \|V\|_{cb}, \|V^{-1}\|_{cb} < 1 + \varepsilon$ . It suffices to construct  $X_i$  in such a way that, for every  $j$ , there exists a complete contraction  $S_j : M_{n_i} \rightarrow X_i$ , extending  $v_i|_E$ , s.t.  $\|S_j^{-1}\|_{cb} < (1 + \delta_i)\|T_j\|_{cb}$ .

Let  $X'_i = X_{i-1} \oplus_1 (\sum_j M_{n_i})_1 / F$ , where  $F$  is the linear span of the vectors  $(e, 0, \dots, 0, -T_j e, 0, \dots)$  ( $e \in E_j \hookrightarrow X_{i-1}$ ,  $T_j e$  lives in the  $j$ -th copy of  $M_{n_i}$ ). Denote by  $U_0 (U_j)$  the embedding of  $X_{i-1}$  (respectively, of the  $j$ -th copy of  $M_{n_i}$ ) into  $X'_i$ . By Lemma 2.4,  $U_0$  is a complete isometry, and  $U_j$  is a complete contraction satisfying  $\|U_j^{-1}\|_{cb} \leq \|T_j\|_{cb}$ .

Now let  $\tilde{X}_i = \text{MIN}_{n_i}(X'_i)$ . Keeping the notation  $U_0$  and  $U_j$  for maps into  $\tilde{X}_i$ , we see that these maps are still completely contractive. Moreover, by Smith’s Lemma,

$\|U_j^{-1}\|_{cb} = \|U_j^{-1} \otimes I_{M_{n_i}}\|$ , hence, by Lemma 2.1,  $\|U_j^{-1}\|_{cb} = \|T_j\|_{cb}$  even if its domain is viewed as a subspace of  $\tilde{X}_i$ , and similarly,  $U_0 : X_{i-1} \rightarrow \tilde{X}_i$  is a complete isometry.

Finally, let  $\tilde{X}'_i = \tilde{X}_i \oplus_1 Y_i/G$ , where  $G = \text{span}[(-J_{i-1}y, u_iy) \mid y \in Y_{i-1}]$ . Let  $X_i = \text{MIN}_{n_i}(\tilde{X}'_i)$ . As before, we show that  $y \mapsto [(0, y)]$  is a completely isometric embedding of  $Y_i$  into  $X_i$ , and the complete contraction  $w : \tilde{X}_i \rightarrow X_i : \tilde{x} \mapsto [(\tilde{x}, 0)]$  is such that  $\|w^{-1}\|_{cb} < 1 + \delta_i$ . Then  $S_j = wU_j$  is a complete contraction with  $\|S_j^{-1}\|_{cb} < (1 + \delta_i)\|T_j\|_{cb}$ .

To define the projection  $P_i$ , note that  $P_{i-1} : X_{i-1} \rightarrow J_{i-1}(Y_{i-1})$  has extends to a completely contractive projection  $\tilde{P}_i$  from  $\tilde{X}_i$  onto  $J_{i-1}(Y_{i-1})$ . Define the  $Q_i \in CB(\tilde{X}_i \oplus_1 Y_i)$  by setting  $Q_i((\tilde{x}_i, y_i)) = (\tilde{P}_i\tilde{x}_i, y_i)$ . Clearly,  $Q_i$  is a completely contractive projection, and  $G$  belongs to its range. Thus,  $Q_i$  gives rise to a completely contractive projection on  $\tilde{X}'_i$ . The corresponding projection on  $X_i$  is also completely contractive, by Lemma 2.1. We call it  $P_i$ .  $\blacksquare$

Below, we establish the ‘‘almost uniqueness’’ of an NG space, and prove that such a space cannot be isometric to a unital Banach algebra.

**Proposition 3.1.** *Suppose  $X_1$  and  $X_2$  are non-commutative Gurarii spaces,  $x_1 \in X_1$ ,  $x_2 \in X_2$ , and  $\|x_1\| = \|x_2\| = 1$ . Then for every  $\varepsilon > 0$ , there exists a complete isomorphism  $u : X_1 \rightarrow X_2$  with  $\|u\|_{cb}\|u^{-1}\|_{cb} < 1 + \varepsilon$ , and  $ux_1 = x_2$ .*

*Proof.* For  $j = 1, 2$ , find a sequence  $(x_{ji})_{i \in \mathbb{N}}$ , which is dense in  $X_j$ . Fix  $\varepsilon > 0$ , and find a sequence  $(\varepsilon_i)$  of positive numbers such that  $\prod_i (1 + \varepsilon_i)^4 < 1 + \varepsilon$ . Clearly, it suffices to verify the existence of the following family of subspaces and operators between them:

- (1) For  $j = 1, 2$ ,  $E_{j0} = \text{span}[x_j] \hookrightarrow E_{j1} \hookrightarrow E_{j2} \hookrightarrow \dots \hookrightarrow X_j$ . For any  $i$ ,  $x_{ji} \in E_{ji}$ , and therefore,  $X_j = \overline{\cup_i E_{ji}}$ . Finally,  $\dim E_{1,i} = \dim E_{2,i}$ .
- (2)  $u_i \in B(E_{1,2i}, E_{2,2i})$  ( $i \geq 0$ ),  $v_i \in B(E_{2,2i-1}, E_{1,2i-1})$  ( $i \geq 1$ ) satisfy: (a)  $u_0x_1 = x_2$ , and (b) for  $i \geq 1$ ,  $v_i|_{E_{2,2i-2}} = u_{i-1}^{-1}$ ,  $u_i|_{E_{1,2i-1}} = v_i^{-1}$ ,  $\|v_i\|_{cb}, \|v_i^{-1}\|_{cb} \leq \prod_{j < i} (1 + \varepsilon_j)^2$ ,  $\|u_i\|_{cb}, \|u_i^{-1}\|_{cb} \leq \prod_{j < i} (1 + \varepsilon_j)^2(1 + \varepsilon_i)$ .

However, the existence of such sequences of operators and subspaces follows from the definition of an NG space.  $\blacksquare$

**Proposition 3.2.** *If  $X$  is a non-commutative Gurarii space, and  $A$  is a unital Banach algebra, then  $d(X, A) \geq 1 + e^{-1}$ .*

*Proof.* Suppose  $A$  is a Banach algebra with the identity  $I$  (we are assuming  $\|I\| = 1$ ), and  $v : X \rightarrow A$  is a contraction. Suppose, for the sake of contradiction, that  $c = \|v^{-1}\| < 1 + e^{-1}$ . Fix  $\lambda > 1$  s.t.

$$(3.1) \quad \left(1 + \frac{1}{e\lambda}\right) \frac{1}{\lambda} > c.$$

Denote the elements of the canonical basis of  $\ell_\infty^2$  by  $\delta_1$  and  $\delta_2$ . Applying the defining property of an NG space with  $F = \ell_\infty^2$ ,  $E = \text{span}[\delta_1]$ , and  $u : E \rightarrow X : \delta_1 \mapsto v^{-1}I/\|v^{-1}I\|$ , we obtain the existence of a map  $\tilde{u} : F \rightarrow X$  s.t.  $\|\tilde{u}\|, \|\tilde{u}^{-1}\| < \lambda$ , and  $\tilde{u}|_E = u$ . Let  $T = v\tilde{u}$ , and note that  $\|T\| < \lambda$  and  $\|T^{-1}\| < c\lambda$ . Set  $a = T\delta_2$ , and note that  $\|a\| \geq 1/(c\lambda)$ . Therefore, by [1],

$$\sup_{\omega \in \mathbb{T}} \|dI + \omega a\| \geq d + \nu(a) \geq d + \frac{\|a\|}{e} \geq \frac{1}{c} \left(1 + \frac{1}{e\lambda}\right)$$

(here,  $\mathbb{T}$  is the unit circle,  $d = 1/\|v^{-1}I\|$ , and  $\nu(a)$  is the numerical radius of  $a$ ).

On the other hand,  $dI + \omega a = T(\delta_1 + \omega\delta_2)$ , hence, for any  $\omega \in \mathbb{T}$ ,

$$\|dI + \omega a\| \leq \|T\| \cdot \|\delta_1 + \omega\delta_2\| < \lambda.$$

This contradicts (3.1). ■

**Remark 3.3.** We do not know whether an NG space is (completely) isomorphic to a  $C^*$ -algebra. However, by [6], an NG space is completely isometric to a quotient of the CAR algebra.

Finally, we observe that  $\delta > 0$  in the definition of an NG space is necessary. More precisely:

**Proposition 3.4.** *There is no separable 1-exact operator space such that, for any 1-exact finite dimensional operator space  $F$  is 1-exact, its subspace  $E$ , and any linear isomorphism  $u : E \rightarrow E'$  ( $E'$  is a subspace of  $X$ ) there exists a subspace  $F'$  of  $X$ , and a linear map  $\tilde{u} : F \rightarrow F'$ , satisfying  $\tilde{u}|_E = u$ , and  $\|\tilde{u}\|_{cb}\|\tilde{u}^{-1}\|_{cb} = \|u\|_{cb}\|u^{-1}\|_{cb}$ .*

*Proof.* Suppose, for the sake of contradiction, that  $X$  is such a space. As in the proof of Theorem 1.1, we can show that  $X$  contains an isometric copy of  $C([0, 1])$ . Emulating Proposition 3.1, we prove that for every  $x_1, x_2 \in X$  with  $\|x_1\| = \|x_2\| = 1$  there exists a complete isometry  $u \in CB(X)$  s.t.  $ux_1 = x_2$ . This, however, is impossible (see the proof of Theorem 10 of [4]). ■

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