SPACES OF OPERATORS, THE ψ-DAUGAVET PROPERTY, AND NUMERICAL INDICES

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Abstract. Suppose $\psi : [0, \infty) \to [0, \infty)$ is a strictly increasing function. A Banach space X is said to have the ψ-Daugavet Property if the inequality $||I_X + T|| \geq \psi(||T||)$ holds for every compact operator $T : X \to X$. We show that, if $1 < p < \infty$ and $K(\ell_p) \hookrightarrow X \hookrightarrow B(\ell_p)$, then X has the ψ-Daugavet Property with $\psi(t) = (1 + c_p t^q)^{1/q}$ (here $q = \max\{2, p\}$ and c_p is an absolute constant). We also prove that a C^*-algebra A is commutative if and only if $1 + ||T|| = \sup\{||I_A + \omega T|| \mid |\omega| = 1\}$ for any $T : A \to A$. Together, these results allow us to distinguish between some types of von Neumann algebras by considering spaces of operators on them.

1. Introduction

Suppose $\psi : [0, \infty) \to [1, \infty)$ is a strictly increasing function. A Banach space X is said to have the *ψ-Daugavet Property* (*ψ-DP*) if the inequality $||I_X + \Phi|| \geq \psi(||\Phi||)$ holds for every compact operator $\Phi : X \to X$. X has the *Pseudo-Daugavet Property* (*PDP*, in short) if it has the ψ-DP for some ψ. In particular, X has the classical *Daugavet Property* (*DP*) if it has the ψ-DP for $\psi(t) = 1 + t$. Recently, the Daugavet property of Banach spaces was subject of intense investigation (see e.g. [8], [14], [18]). In particular, the connection between the DP and certain geometric properties of the unit ball of X was clarified in [8]. By contrast, the study of the PDP is still in its infancy. In [1] (see also [15]), it was shown that, for $p \in (1, 2) \cup (2, \infty)$, $L_p(0, 1)$ has the ψ-DP with $\psi(t) = (1 + a_p t^2)^{1/2}$, where a_p is a constant. In [14], this result was generalized to non-atomic non-commutative L_p-spaces. In this paper, we establish that certain spaces of operators on ℓ_p have the PDP. We say that X is a *large subspace* of $B(\ell_p)$ if it contains all compact operators on ℓ_p. In particular, $B(\ell_p)$ is a large subspace of itself. In Theorems 2.1 and 2.2, we show that large subspaces of on $B(\ell_p)$ $(1 < p < \infty)$ have the PDP. Section 2 is devoted to proving this result.

In Section 3, we consider a different variation on the Daugavet theme. Recall that the *numerical radius* of an operator $T \in B(X)$ is defined as

$$\nu(T) = \sup\{|f(Tx)| \mid f \in X^*, x \in X, ||f|| = ||x|| = f(x) = 1\}.$$

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It is easy to observe that $\nu(T) \leq ||T||$. If A is a unital C^*-algebra and $T \in A \hookrightarrow B(H)$, then, by Theorem 3.4 of [2],

$$\nu(T) = \sup\{|f(T)| \,|\, f \in A^*, ||f|| = f(\mathbf{1}) = 1\}.$$

The *numerical index* of a Banach space X is the largest number $n(X) = c$ such that $\nu(T) \geq c||T||$ for any $T : X \to X$. It follows from Lemma 9.2 of [2] that a Banach space X has numerical index 1 if and only if $1 + ||\Phi|| = \sup\{||I_X + \omega\Phi|| \,|\, |\omega| = 1\}$ for any $\Phi : X \to X$.

It is known (see [7]) that a C^*-algebra has numerical index 1 or 1/2, depending on whether it is commutative or non-commutative. We show that, if A is a non-commutative C^*-algebra, then there exists an operator $\Phi : A \to A$ s.t. $||\Phi|| = 1$ and $||I_X + \omega\Phi|| < 1.975$ whenever $|\omega| = 1$. Together with Theorems 2.1, 2.2, and some results of [14], this result allows us to distinguish between some types of von Neumann algebras by the structure of linear operators on them (Corollary 3.2).

In Section 4, we discuss real finite dimensional spaces of numerical index 1. In particular, we show that for any such n-dimensional space, the Banach-Mazur distance to the Hilbert space is at least $cn^{1/4}$ (Theorem 4.1).

2. Large spaces of operators and the ψ-Daugavet Property

In this section, we prove that large subspaces of $B(\ell_p)$ $(1 < p < \infty)$ have the ψ-DP for $\psi(t) = (1 + c_p t^q)^{1/q}$, where $q = \max\{2, p\}$. More generally, suppose X is a large subspace of $B(\ell_p)$, and (Y, Z) is a pair $(C(K, X), C(K, B(\ell_p)))$ (K is a compact Hausdorff space), $(c_0(X), c_0(B(\ell_p)))$, $(L_1(\mu, X), L_1(\mu, B(\ell_p)))$, or $(L_\infty(\mu, X), L_\infty(\mu, B(\ell_p)))$. Denote by J_Y the canonical embedding of Y into Z (sometimes we identify $J_Y y$ with y).

THEOREM 2.1. *For* $1 < p < \infty$, *there exists a constant* $c_p > 0$ *such that* $||J_Y + \Phi|| \geq (1 + c_p||\Phi||^q)^{1/q}$ *whenever* $q = \max\{2, p\}$, Y *and* Z *are as above, and* $\Phi : Y \to Z$ *is a compact operator.*

If $p = 2$, a better result can be obtained.

THEOREM 2.2. *Suppose* $p = 2$, Y *and* Z *are as above, and* $\Phi : Y \to Z$ *is compact. Then* $||J_Y + \Phi|| \geq 1 + ||\Phi||/(8\sqrt{2})$.

To prove these theorems, we need a series of lemmas. First, we introduce some notation. Suppose (Ω, μ) is a measure space, and S is a measurable subset of Ω. We denote by P_S the "natural" contractive projection from $L_p(\Omega, \mu)$ onto $L_p(S, \mu)$. If $\Omega = \mathbb{N}$, μ is the counting measure, and $S = \{i\}$, we shall, for the sake of brevity, use the notation P_i instead of $P_{\{i\}}$. If H is a Hilbert space and $\eta \in H$ is a non-zero vector, we shall denote by P_η the orthogonal projection onto span$[\eta]$.

The lemma below states that a compact operator on a large subspace $X \hookrightarrow B(\ell_p)$ $(1 < p < \infty)$ "ignores" "far out" columns and rows. More precisely, consider a Banach space Z and a compact operator $\Phi : X \to Z$. Note that the spaces $X \cdot P_i = \{TP_i \,|\, T \in X\}$ and $P_i \cdot X = \{P_i T \,|\, T \in X\}$ are isometric to ℓ_p and $\ell_{p'}$, respectively (here $1/p' + 1/p = 1$). Define operators $u_i : \ell_p \to Z$ and $v_i : \ell_{p'} \to Z$ by setting $u_i(TP_i) = \Phi(TP_i)$, $v_i(P_i T) = \Phi(P_i T)$.

LEMMA 2.3. *In the above notation,* $\lim_i \|u_i\| = \lim_i \|v_i\| = 0$.

Proof. Consider a map $u : \ell_{p'} \to B(\ell_p, Z)$, defined by $[u(e^*)](e) = \Phi(e^* \otimes e)$ ($e^* \in \ell_{p'}$, $e \in \ell_p$). The canonical basis (e_i^*) forms a weakly null sequence in $\ell_{p'}$, and the operator u is compact, hence the sequence $u_i = u(e_i^*)$ is null. We deal with the sequence (v_i) in a similar way. ■

LEMMA 2.4. *Suppose X is a large subset of $B(\ell_p)$ ($1 < p < \infty$), Φ is a compact operator from X to a Banach space Z, $T \in X$, $\|T\| \leq 1$, and $S \in B(\ell_p)$. Then for any $\varepsilon > 0$ there exist $b \in (0,1)$ and $\tilde{T} \in B(\ell_p)$ such that $\tilde{T} - bT$ has finite rank (hence $\tilde{T} \in X$), $\|\Phi(\tilde{T} - T)\| < \varepsilon$, $\|\tilde{T}\| \leq 1$, and $\|\tilde{T} + bS\|^q \geq 1 + \kappa\|S\|^q$. Here, $q = \max\{2, p\}$, and $0 < \kappa < 1$ is a constant depending on p.*

REMARK. For $2 \leq p < \infty$, b is a constant depending only on p. For $1 < p < 2$, $\alpha \min\{\|S\|^{2-p}, 1\} \leq b \leq \beta \min\{\|S\|^{2-p}, 1\}$, with α and β depending on p.

Proof of Theorem 2.1. First consider the case of $Y = L_1(\mu, X)$, $Z = L_1(\mu, B(\ell_p))$. By a standard discretization argument, we can assume that $Y = \ell_1^N(X)$, $Z = \ell_1^N(B(\ell_p))$ ($N \in \mathbb{N}$). Without loss of generality, we can assume that there exists $\mathbf{T} = (T, 0, \ldots, 0) \in Y$ ($T \in X$) s.t. $\|\mathbf{T}\| = \|T\| = 1$ and $\|\Phi(\mathbf{T})\| > \|\Phi\|/2$. Denote $\Phi(\mathbf{T})$ by $\mathbf{S} = (S_1, \ldots S_N)$ (with $S_i \in B(\ell_p)$ for $1 \leq i \leq n$). If $\sum_{i=2}^N \|S_i\| \geq 2\|\mathbf{S}\|/3$,

$$\|J_Y + \Phi\| \geq \|(J_Y + \Phi)\mathbf{T}\| \geq \|T\| - \|S_1\| + \sum_{i=2}^N \|S_i\| \geq 1 + \|\Phi\|/6. \tag{2.1}$$

If $\|S_1\| > \|\Phi\|/6$, fix $\varepsilon > 0$ and use Lemma 2.4 to find $b > 0$ and $\tilde{\mathbf{T}} = (\tilde{T}, 0, \ldots, 0) \in Y$ s.t. $\|\tilde{\mathbf{T}}\| = 1$, $\|\Phi(\tilde{\mathbf{T}} - b\mathbf{T})\| < \varepsilon$, and $\|\tilde{T} + bS_1\|^q \geq 1 + \kappa\|S_1\|^q$, with $q = \max\{2, p\}$ and $0 < \kappa < 1$. Then, for some $c > 0$,

$$\|J_Y + \Phi\| \geq \|(J_Y + \Phi)\tilde{\mathbf{T}}\| \geq \|\tilde{T} + bS_1\| - \varepsilon \geq (1 + c\|\Phi\|^q)^{1/q} - \varepsilon. \tag{2.2}$$

Either (2.1) and (2.2) holds, implying that the statement of the theorem is true for the pair $(\ell_1^N(X), \ell_1^N(B(\ell_p)))$.

Next consider the pair $(C(K,X), C(K, B(\ell_p)))$. Fix $\varepsilon > 0$, and find $f \in C(K,X)$ s.t. $\|f\| = 1$ and $\|[\Phi(f)](t)\| > \|\Phi\| - \varepsilon$. for some $t \in K$. Let $T = f(t)$, $S = [\Phi(f)](t)$, and define the function $g \in C(K)$ s.t. $\|g\| = g(t) = 1$ and $g(s) = 0$ whenever $\|f(s) - T\| > \varepsilon/(\|\Phi\| + 1)$. Define $h \in C(K,X)$ by setting $h(s) = (1 - g(s))f(s) + Tg(s)$. Then $f(s) - h(s) = (T - f(s))g(s)$, hence $\|f - h\| < \varepsilon/(\|\Phi\| + 1)$, and $\|\Phi(f - h)\| < \varepsilon$. Find $\tilde{T} \in X$ s.t. $\|\tilde{T}\| = 1$, $\|\Phi((\tilde{T} - bT)g)\| < \varepsilon$, and $\|\tilde{T} + bS\|^q \geq 1 + \kappa\|S\|^q$. Let $\tilde{f} \in C(K,X)$ be defined by $\tilde{f}(s) = \tilde{T}g(s) + b(1 - g(s))f(s)$. Then $\|\tilde{f}(s)\| \leq \|\tilde{T}\|g(s) + b\|f(s)\|(1 - g(s)) \leq 1$. Moreover, $\tilde{f} - bf = (\tilde{f} - bh) + b(h - f)$, hence $\|\Phi(\tilde{f} - f)\| < 2\varepsilon$. Therefore,

$$\|\Phi + J_Y\| \geq \|[(\Phi + J_Y)\tilde{f}](t)\| \geq \|\tilde{T} + bS\| - 2\varepsilon \geq \big(1 + \kappa(\|\Phi\| - \varepsilon)^q\big)^{1/q} - 2\varepsilon.$$

Since ε can be made arbitrarily small, we are done.

The pair $(c_0(X), c_0(B(\ell_p)))$ is dealt with in a similar manner.

Finally, we consider $(L_\infty(\Omega, \mu, X), (L_\infty(\Omega, \mu, B(\ell_p)))$. Fix $\varepsilon > 0$, and find $f \in L_\infty(\mu, X)$ s.t. $\|f\| = 1$ and $\|\Phi(f)\| > \|\Phi\| - \varepsilon$. Find $F \subset \Omega$ s.t. $\mu(F) > 0$ and there exists $S \in B(\ell_p)$ with $\|S\| > \|\Phi\| - \varepsilon$ and $\|\Phi(f)|_F - S\mathbf{1}_F\| < \varepsilon$ (see e.g.

[12]). Furthermore, find $E \subset F$ with $\mu(E) > 0$ s.t. $||f|_E - T\mathbf{1}_E|| < \varepsilon/(||\Phi|| + 1)$, with $T \in X$ and $||T|| \leq 1$. Find $\tilde{T} \in X$ s.t. $||\tilde{T}|| = 1$, $||\Phi((\tilde{T} - bT)\chi_E)|| < \varepsilon$, and $||\tilde{T} + bS||^q \geq 1 + \kappa||S||^q$. Let $\tilde{f} = \tilde{T}\chi_E + bf\chi_{\Omega\setminus E}$. Then $\tilde{f} \in L_\infty(\mu, X)$, $||\tilde{f}|| \leq 1$, and $\tilde{f} - bf = ((\tilde{T} - bT) + b(T - f))\chi_E$, hence $||\Phi(\tilde{f} - bf)|| < 2\varepsilon$. Thus,

$$||\Phi + J_Y|| \geq ||((\Phi + J_Y)\tilde{f})|_E|| \geq ||\tilde{T} + bS|| - 3\varepsilon \geq \big(1 + \kappa(||\Phi|| - \varepsilon)^q\big)^{1/q} - 3\varepsilon.$$

Since ε can be arbitrarily small, we are done. ■

Proof of Lemma 2.4. (1) $2 \leq p < \infty$. Fix $\varepsilon > 0$, and find $x_0 \in \ell_p$ with finite support such that $||x_0|| = 2^{-1/p}$, and $||Sx_0||^p > ||S||^p/2 - \varepsilon$. Perturbing S slightly, we can assume that Sx_0 has finite support. More precisely, there exist sets I and J of cardinality n s.t. $x_0 \in E_1 = \mathrm{span}[e_i \,|\, i \in I]$ (here (e_i) denotes the canonical basis in ℓ_p) and $Sx_0 \in F_1 = \mathrm{span}[e_i \,|\, i \in J]$. Fix $\varepsilon > 0$. By Lemma 2.3, there exist "far out" sets $I', J' \subset \mathbb{N}$ of cardinality n s.t. $(I \cup J) \cap (I' \cup J') = \emptyset$ and $\sum_{i \in I' \cup J'}(||u_i|| + ||v_i||) < \varepsilon/4$ (with u_i and v_i as above). Let $E_2 = \mathrm{span}[e_i \,|\, i \in I']$, $F_2 = \mathrm{span}[e_i \,|\, i \in J']$, $E_3 = \mathrm{span}[e_i \,|\, i \notin I \cup I']$, and $F_3 = \mathrm{span}[e_i \,|\, i \notin J \cup J']$. Note that, by choosing I' first and J' second and perturbing S slightly, we can assume that $\mathrm{supp}\,(S(E_2) \cap (J \cup J') = \emptyset$. Denote by R_j and Q_j $(j = 1, 2, 3)$ the natural embedding of E_j into ℓ_p, and projection from ℓ_p onto F_j. Let $T_{ij} = Q_i T|_{E_j} : E_j \to F_i$. Then $T = \sum_{i,j=1}^{3} Q_i T_{ij} R_j$.

If $x_i \in E_i$ $(1 \leq i \leq 3)$, we denote $\sum R_i x_i$ by $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_E$. $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}_F$ is defined analogously. Abusing the notation slightly, we write

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}.$$

Let

$$\tilde{T} = \begin{pmatrix} bT_{11} & -bT_{11} & bT_{13} \\ aJ_1 & aJ_2 & 0 \\ bT_{31} & -bT_{31} & bT_{33} \end{pmatrix},$$

where $a = 2^{-(p-1)/p} = 2^{-1/p'}$ and J_k $(k = 1, 2)$ is the "coordinatewise" isometry from E_k onto F_2 (b will be chosen later). By our choice of E_2 and F_2, $||\Phi(bT - \tilde{T})|| < \varepsilon$. To select b, pick $x_i \in E_i$ such that $\sum_i ||x_i||^p = 1$. Let

$$y = \tilde{T}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_E = \begin{pmatrix} bT_{11}(x_1 - x_2) + bT_{13}x_3 \\ a(x_1 + x_2) \\ bT_{31}(x_1 - x_2) + bT_{33}x_3 \end{pmatrix}_F.$$

Then

$$\left\|\begin{pmatrix} T_{11}(x_1 - x_2) + T_{13}x_3 \\ 0 \\ T_{31}(x_1 - x_2) + T_{33}x_3 \end{pmatrix}_F\right\| \leq \left\|\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{13} \end{pmatrix}\begin{pmatrix} x_1 - x_2 \\ 0 \\ x_3 \end{pmatrix}_E\right\|$$

$$\leq ||T|| \cdot \left\|\begin{pmatrix} x_1 - x_2 \\ 0 \\ x_3 \end{pmatrix}_E\right\| \leq \big(||x_1 - x_2||^p + ||x_3||^p\big)^{1/p},$$

and therefore,

$$\begin{aligned}||y||^p \leq& ||x_1+x_2||^p/2^{p-1} + b^p(||x_1-x_2||^p + ||x_3||^p)\\ \leq& ||x_1+x_2||^p/2^{p-1} + b^p||x_1-x_2||^p + ||x_3||^p.\end{aligned}$$

However, b can be chosen in such a way that

$$||x_1+x_2||^p/2^{p-1} + b^p||x_1-x_2||^p \leq ||x_1||^p + ||x_2||^p$$

(see e.g. [10], p. 80). Then $||y|| \leq 1$. Since the selection of x_1, x_2, x_3 is arbitrary, we have demonstrated that $||\tilde{T}|| \leq 1$.

Finally we have to show that $||\tilde{T} + bS||^p - 1 \geq \kappa ||S||^p$, for some constant κ. Consider $x = \begin{pmatrix} x_0 \\ x_0' \\ 0 \end{pmatrix}_E$, where $x_0' = J_2^{-1}J_1x_0 \in E_2$. Recall that $||x_0|| = 2^{-1/p}$, $Sx_0 \in F_1$, and $S(E_2) \subset F_3$. Then $||x|| = 1$ and $(\tilde{T}+bS)x = \begin{pmatrix} bSx_0 \\ 2^{1/p}x_0 \\ bSx_0' \end{pmatrix}_F$, which implies

$$||\tilde{T}+bS||^p \geq ||(\tilde{T}+bS)(x)||^p \geq 1 + b^p||Sx_0||^p \geq 1 + b^p(||S||^p/2 - \varepsilon).$$

(2) $1 < p < 2$. The idea of the proof is the same as in the case of $2 \leq p < \infty$. However, the details differ. By [10], p. 80, there is a constant $C = C_p$ s.t.

$$\Big(C^2||x-y||^2 + ||x+y||^2\Big)^{1/2} \leq 2^{1/p'}\Big(||x||^p + ||y||^p\Big)^{1/p}$$

for any $x, y \in \ell_p$ (here $1/p' = 1 - 1/p$). Note that an operator $\Lambda : \ell_2^2 \to \ell_p^2 : (s,t) \mapsto (As, Bt)$ (here, A and B are positive) satisfies $||\Lambda|| \leq 2^{-1/p'}$ iff

$$A^r + B^r \leq 2^{-r/p'} = 2^{-\gamma} \ \big(\text{with}\, r = \frac{2p}{2-p}, \gamma = 2(p-1)/(2-p)\big).$$

Therefore,

$$A^p||x+y||^p + B^pC^p||x-y||^p \leq ||x||^p + ||y||^p$$

whenever the previous inequality is satisfied.

As before, find $x_0 \in \ell_p$ with finite support such that $||x_0|| = 2^{-1/p}$ and $\lambda = 2^{1/p}||Sx_0|| > ||S|| - \varepsilon$. Define E_i and F_i $(1 \leq i \leq 3)$ as in part (1) of the proof, and let

$$\tilde{T} = \begin{pmatrix} bT_{11} & -bT_{11} & bT_{13} \\ aJ_1 & aJ_2 & 0 \\ bT_{31} & -bT_{31} & bT_{33} \end{pmatrix},$$

where $K = \lambda C/2$, $a = 2^{-(p-1)/p}/(1+K^2)^{1/r}$, and $b = 2^{-(p-1)/p}C\big(K^2/(1+K^2)\big)^{1/r}$. As before, we may assume that $||\Phi(bT - \tilde{T})|| < \varepsilon$ and $S(E_2) \subset F_3$. To show that that $||\tilde{T}|| \leq 1$, consider $x_i \in E_i$ $(1 \leq i \leq 3)$ s.t. $\sum_i ||x_i||^p = 1$. Let

$$y = \tilde{T}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_E = \begin{pmatrix} bT_{11}(x_1 - x_2) + bT_{13}x_3 \\ a(x_1+x_2) \\ bT_{31}(x_1-x_2) + bT_{33}x_3 \end{pmatrix}_F.$$

Reasoning as before, we have

$$||y||^p \leq b^p(||x_1 - x_2||^p + ||x_3||^p) + a^p||x_1 + x_2||^p.$$

By the discussion above, $||\tilde{T}|| \leq 1$ whenever $a^r + (b/C)^r \leq 2^{-\gamma}$. This is satisfied, by our choice of a and b.

It remains to show that $||\tilde{T} + bS||^2 - 1 \geq \kappa\lambda^2$, for some constant κ. As in part (1), we apply $\tilde{T} + bS$ to $\begin{pmatrix} x_0 \\ x_0' \\ 0 \end{pmatrix}_E$ to obtain:

$$||\tilde{T} + bS||^p \geq (2a)^p||x_0||^p + b^p||Sx_0||^p \geq \frac{1 + 2^{-p}C^p\lambda^p K^{2p/r}}{(1 + K^2)^{p/r}} = (1 + K^2)^{p/2},$$

since $2^{-p}C^p\lambda^p = K^p$, $p + 2p/r = 2$, and $2 - p/r = p/2$. ■

Similarly, Theorem 2.2 follows from the lemma below.

LEMMA 2.5. *Suppose X is a large subspace of $B(\ell_2)$, $T \in X$, $||T|| = 1/(2\sqrt{2})$, $x_0 \in \ell_2$, $||x_0|| = 1/\sqrt{2}$, $S \in B(\ell_2)$, and $\varepsilon > 0$. Then there exists $\tilde{T} \in B(\ell_2)$ such that $\tilde{T} - T$ is finite rank (hence $\tilde{T} \in X$), $||\tilde{T}|| \leq 1$, $||\Phi(\tilde{T} - T)|| < \varepsilon$, and $||\tilde{T} + S|| \geq 1 + ||Sx_0||/(2\sqrt{2})$.*

Proof. The proof will proceed in two steps. First we construct an auxiliary operator T', and then proceed to construct $\tilde{T}$. Throughout the proof, we use the notation $x \otimes y$ ($x, y \in \ell_2$) for the rank 1 operator which maps $\eta \in \ell_2$ to $\langle \eta, x\rangle y$. For $z \in \ell_2$, P_z shall define the orthogonal projection onto span$[z]$.

Find "far our" $x_1, t_1 \in \ell_2$ s.t. $||x_1|| = 1/\sqrt{2}$, $||t_1|| = 1$, $t_1 \perp Sx_0$, $t_1 \perp Tx_0$, $t_1 \perp Sx_1$, and $x_1 \perp x_0$. Let $x = x_0 + x_1$. By changing x_1 to $-x_1$ if necessary, we can assume that $||Sx|| \geq ||Sx_0||$. Define

$$T' = (1 - P_{t_1})T(1 - P_{x_1}) - 2x_1 \otimes Tx_0 + x \otimes t_1$$

(that is, $T'\eta = (1 - P_{t_1})T(1 - P_{x_1})\eta - 2\langle \eta, x_1\rangle Tx_0 + \langle \eta, x\rangle t_1$ for $\eta \in \ell_2$). Reasoning as in the proof of Lemma 2.4, we may assume that $||\Phi(T' - T)|| < \varepsilon/2$. Observe that $T'x = t_1$. We shall show that $||T'|_{x^\perp}|| \leq 1/\sqrt{2}$ and $||T'|| \leq 1$. Indeed, if $\eta \perp x$, then

$$T'\eta = (1 - P_{t_1})T(1 - P_{x_1})\eta - 2\langle \eta, x_1\rangle Tx_0$$

is orthogonal to $t_1 = T'x$. Moreover,

$$||T'\eta|| \leq ||T|| \cdot ||\eta|| + 2||Tx_0|| \cdot ||x_1|| \cdot ||\eta|| \leq \Big(\frac{1}{2\sqrt{2}} + 2 \cdot \frac{1}{4} \cdot \frac{1}{\sqrt{2}}\Big)||\eta|| = \frac{||\eta||}{\sqrt{2}}.$$

Now we need to modify T'. Let $y = Sx/||Sx||$. Find "far out" $x_2, t_2 \in \ell_2$ s.t. $||x_2|| = ||t_2|| = 1$, $x_2 \perp x$, $y \perp t_2$, $T'^*y \perp x_2$, $t_2 \perp t_1$, $x_2 \perp S^*y$ (i.e. $Sx_2 \perp y$), and $Sx_2 \perp t_1$. Let

$$\tilde{T} = (1 - P_{t_2})T'(1 - P_{x_2}) + \frac{1}{\sqrt{2}}x_2 \otimes (y + t_2) - T'^*y \otimes t_2.$$

By Lemma 2.3, we can assume $||\Phi(\tilde{T} - T')|| < \varepsilon/2$, hence $||\Phi(\tilde{T} - T)|| < \varepsilon$. Observe that $\tilde{T}x_2 = (y + t_2)/\sqrt{2}$ and $\tilde{T}x = t_1$, since $T'x = t_1$ is orthogonal to Sx (indeed, we have chosen t_1 to be orthogonal to Sx_0 and Sx_1). If $\eta \in \ell_2$ is orthogonal to both x and x_2, then

$$\tilde{T}\eta = (1 - P_{t_2})T'\eta - \langle T'\eta, y\rangle t_2.$$

Thus,

$$||\tilde{T}\eta||^2 = ||(1 - P_{t_2})T'\eta||^2 + |\langle T'\eta, y\rangle|^2 \leq ||\eta||^2.$$

It is easy to see that $\langle \tilde{T}\eta, \tilde{T}x\rangle = 0$. Moreover,

$$\sqrt{2}\langle \tilde{T}\eta, \tilde{T}x_2\rangle = \langle (1 - P_{t_2})T'\eta - \langle T'\eta, y\rangle t_2, y + t_2\rangle = \langle T'\eta, y\rangle - \langle T'\eta, y\rangle = 0.$$

Finally, $\langle y, t_1\rangle = \langle t_2, t_1\rangle = 0$, hence $\langle \tilde{T}x_2, \tilde{T}x\rangle = 0$. Thus, $||\tilde{T}|| = 1$.

It remains to estimate $||\tilde{T} + S||$. Note that

$$(\tilde{T}+S)(x+x_2) = t_1+(y+t_2)/\sqrt{2}+Sx+Sx_2 = t_1+t_2/\sqrt{2}+(1/\sqrt{2}+||Sx||)y+Sx_2.$$

Since $||Sx|| \geq ||Sx_0||$ and the four summands above are mutually orthogonal, we obtain:

$$||(\tilde{T} + S)(x + x_2)||^2 \geq 1^2 + \Big(\frac{1}{\sqrt{2}}\Big)^2 + \Big(\frac{1}{\sqrt{2}} + ||Sx_0||\Big)^2 = 2 + \sqrt{2}||Sx_0|| + ||Sx_0||^2.$$

Since $||x + x_2|| = \sqrt{2}$, we have:

$$||\tilde{T} + S||^2 \geq 1 + \frac{||Sx_0||}{\sqrt{2}} + \frac{||Sx_0||^2}{2} \geq \Big(1 + \frac{||Sx_0||}{2\sqrt{2}}\Big)^2. \quad \blacksquare$$

REMARK. We do not know what is the best value of c for which $||I+\Phi|| \geq 1+c||\Phi||$ for any compact $\Phi : B(\ell_2) \to B(\ell_2)$. We shall show that any c for which the above inequality holds cannot exceed $1/2$. Indeed, consider a rank 1 projection $P \in B(\ell_2)$. For $T \in B(\ell_2)$ define $\Phi(T) = -PTP$. Clearly $||\Phi|| = 1$. We shall show that $||I + \varepsilon\Phi|| \leq 1 + \varepsilon/2$ for every $\varepsilon \in [0, 1]$.

Consider $\varepsilon = 1$ first. Select $T \in B(\ell_2)$ with $||T|| = 1$. Then

$$2(T + \Phi(T)) = T(1 - P) + (1 - P)T + \big(PT(1 - P) + (1 - P)TP\big).$$

Since the norm of each of the three summands on the right does not exceed 1, we conclude that $||T + \Phi(T)|| \leq 3/2$, and therefore, $||I + \Phi|| \leq 3/2$.

For an arbitrary $\varepsilon \in [0, 1]$, the triangle inequality implies

$$||I + \varepsilon\Phi|| \leq \varepsilon||I + \Phi|| + (1 - \varepsilon)||I|| = \frac{3}{2}\varepsilon + (1 - \varepsilon) = 1 + \frac{\varepsilon}{2}.$$

3. A characterization of commutative C^*-algebras

In this section we characterize commutative C^*-algebras. For the general results in the theory of C^*-algebras and von Neumann algebras, the reader is referred to [9] and [19]. Recall that an operator $\Phi : A \to A$ (A is a C^*-algebra) is called *hermitian* if $\Phi(u^*) = (\Phi(u))^*$ for any $u \in A$.

THEOREM 3.1. *(1) If A is a commutative C^*-algebra, then $n(A) = 1$, and consequently,* $\sup\{||I_A + \omega\Phi|| \,|\, |\omega| = 1\} = 1 + ||\Phi||$ *whenever A is a commutative C^*-algebra and $\Phi : A \to A$.*

(2) If A is a non-commutative C^*-algebra and $\varepsilon > 0$, then there exists a hermitian operator $\Phi : A \to A$ such that $||\Phi|| = 1$ and $||I_A + \omega\Phi|| < 2 - 1/40 + \varepsilon$ whenever $|\omega| = 1$.

Before proving this theorem, we summarize some consequences of the above results to classification of von Neumann algebras, which we assume, for the sake of convenience, to be acting on separable Hilbert spaces. Recall that any von Neumann algebra can be written as an ℓ_∞ direct sum of von Neumann algebras of types I, II, and III. In turn, any type I von Neumann algebra can be represented as an $(\sum_i L_\infty(\mu_i)\overline{\otimes}B(H_i))_\infty$. The summands $L_\infty(\mu_i, B(H_i))$ are called *type I_n M-summands*, where $n = \dim H_i$. We say that a von Neumann algebra N is *atomless* if for every non-trivial projection $p \in N$ there exists a non-trivial projection $q \in N$ dominated by p; otherwise, N is called *atomic*. It is well known that von Neumann algebras of types II and III are atomless. A von Neumann algebra $L_\infty(\mu, B(H))$ is atomless iff the measure μ is atomless.

THEOREM 3.2. *Suppose N is a von Neumann algebra.*

(1) N is commutative if and only if $\sup ||I_N + \omega\Phi|| \,|\, |\omega| = 1\} = 1 + ||\Phi||$ *for every $\Phi : N \to N$.*

(2) N is atomless if and only if it has the Daugavet property, that is, $||I_N + \Phi|| = 1 + ||\Phi||$ for any compact operator $\Phi : N \to N$.

(3) N has atomic type I_∞ M-summands, but no atomic type I_n M-summands for $n < \infty$, if and only if N fails the Daugavet property, yet $||I_N + \Phi|| \geq 1 + ||\Phi||/13$ for any compact operator $\Phi : N \to N$.

(4) N has atomic type I_n M-summands ($n < \infty$) if and only if there exists a finite rank projection $P : N \to N$ for which $||I_N - P|| = 1$.

Proof. (1) follows immediately from Theorem 3.1. (2) follows from [14]. If N has atomic type I_n M-summands for $n < \infty$, then there exists a finite rank projection $P : N \to N$ for which $||I_N - P|| = 1$. This establishes (4). It remains to prove (3). By Theorem 9.4.1 of [9], we can write $N = N_1 \oplus_\infty N_2$, where $N_1 = \ell_\infty^M(B(\ell_2))$ $(1 \leq M \leq \infty)$ and N_2 is an atomless von Neumann algebra. Suppose $\Phi : N \to N$ is compact, and show that

$$||I_N + \Phi|| \geq 1 + ||\Phi||/13. \tag{3.1}$$

Write $\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}$, where $\Phi_{ij} : N_j \to N_i$. Since we are considering operators on $N_1 \oplus_\infty N_2$,

$$||\Phi|| \leq \max\{||\Phi_{11}|| + ||\Phi_{12}||, ||\Phi_{21}|| + ||\Phi_{22}||\}.$$

By [14], N_2 has the Daugavet Property, hence

$$||I_N + \Phi|| \geq ||I_{N_2} + \Phi_{22}|| = 1 + ||\Phi_{22}||.$$

On the other hand, $||I_N + \Phi|| \geq 1 + ||\Phi_{21}||$. This proves (3.1) if $||\Phi_{21}|| + ||\Phi_{22}|| \geq ||\Phi||$.

Now suppose $||\Phi_{11}|| + ||\Phi_{12}|| \geq ||\Phi||$. We conclude that $||I_N + \Phi|| \geq 1 + ||\Phi_{12}||$ and, by Theorem 2.2,

$$||I_N + \Phi|| \geq ||I_{N_2} + \Phi_{11}|| \geq 1 + ||\Phi_{11}||/12.$$

Then

$$\begin{aligned} ||I_N + \Phi|| \geq & 1 + \max\{\frac{||\Phi_{11}||}{12}, ||\Phi_{12}||\} \geq 1 + \frac{12}{13} \cdot \frac{||\Phi_{11}||}{12} + \frac{1}{13} \cdot ||\Phi_{12}|| \\ = & 1 + \frac{||\Phi_{11}|| + ||\Phi_{12}||}{13} \geq 1 + \frac{||\Phi||}{13}. \quad \blacksquare \end{aligned}$$

Part (1) of Theorem 3.1 was proved in [3]. For the sake of completeness, we reproduce the proof below. Suppose first A is a unital commutative C^*-algebra (hence $A = C(K)$ for a compact Hausdorff set K), and $\Phi : A \to A$ is a linear map with $||\Phi|| > c > 0$. We shall show that $\nu(T) > c$. By the famous Russo-Dye theorem, there exists a unimodular function $x \in C(K)$ s.t. $||\Phi(x)|| > c$. Let $y = \Phi(x)$. Find a point $k \in K$ s.t. $|y(k)| > c$. By considering ωx instead of x if necessary (with $|\omega| = 1$), we may assume that $x(k) = 1$. Consider $f \in A^*$, given by $f(x) = x(k)$. Then $||f|| = ||x|| = f(x) = 1$ and $|f(Tx)| > c$.

If A is non-unital, we consider the commutative von Neumann algebra A^{**}. We have shown that $n(A^{**}) = 1$. However, by [3], $n(A) \geq n(A^{**})$.

Part (2) of Theorem 3.1 is trickier. We shall first prove it for von Neumann algebras, and then use approximation arguments for general non-commutative C^*-algebras.

LEMMA 3.3. *Suppose N is a non-commutative von Neumann algebra. Then N contains two equivalent mutually orthogonal projections.*

Proof. Note first that there exists a projection $e \in N$ s.t. $eN(\mathbf{1} - e) \neq 0$ (here $\mathbf{1}$ denotes the identity in N). Indeed, otherwise $N = eNe + (\mathbf{1} - e)N(\mathbf{1} - e)$ for any projection $e \in N$. Then e commutes with N and, since projections span N, N must be commutative.

Pick a non-zero $x \in eN(\mathbf{1} - e)$. The initial and final projections of x are mutually orthogonal (being dominated by $\mathbf{1} - e$ and e, respectively), and equivalent. $\blacksquare$

To prove part (2) of Theorem 3.1 for a non-commutative von Neumann algebra N, consider a partial isometry $u \in N$ such as its initial and final projections (u^*u and uu^*, respectively) are mutually orthogonal (such a u exists, by the previous lemma). Note that $u^2 = 0$. Consider an operator $\Phi : N \to N$ defined by $\Phi(x) = (u^*x + xu)/2$. Clearly, $||\Phi|| \leq 1$. To show the converse, consider $x = u + u^*$. Then $||x|| = 1$ and $\Phi(x) = u^*u$, and therefore, $||\Phi|| \geq 1$. Also, $\Phi(x^*) = (\Phi(x))^*$.

It remains to show that $||I_N + \omega\Phi|| \leq \sqrt{3}$. Indeed, $(I_N + \omega\Phi)x = (a_\omega x + x b_\omega)/2$, where $a_\omega = \mathbf{1} + \omega u^*$ and $b_\omega = \mathbf{1} + \omega u$. However,

$$||a_\omega|| = \left\| \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \right\| \leq \sqrt{3}$$

(cf. [14]) and, similarly, $||b_\omega|| \leq \sqrt{3}$. This completes the proof for non-commutative von Neumann algebras.

To prove Theorem 3.1 for general C^*-algebras, we need to “approximate” elements of A^{**} by elements of A with “similar” numerical radii.

LEMMA 3.4. *Suppose* $T \in B(H)$, $||T|| = 1$, *and* $\nu(T) = 1 - \alpha$ *for some* $0 < \alpha \leq 1/2$. *Then* $||\mathbf{1} + T|| \leq 2 - \alpha^2/10$.

LEMMA 3.5. *Suppose* ϕ *is a state on a* C^**-algebra* A, *and* $\psi \in A^*$ *is hermitian. Then* $||\phi + i\psi||^2 \geq 1 + ||\psi||^2/2$.

Proof. By passing to the second dual if necessary, we may assume that A is a von Neumann algebra, and ϕ and ψ are normal functionals. Write $\psi = \psi_1 - \psi_2$, with ψ_1, ψ_2 positive and $||\psi|| = ||\psi_1|| + ||\psi_2||$. Denote by p_1 the support projection of ψ_1, and let $p_2 = \mathbf{1} - p_1$ (note that p_2 dominates the support projection of ψ_2). Let $\alpha_i = \psi_i(p_i) = ||\psi_i||$ (see e.g. p. 140 of [19]) and $\alpha = \max\{\alpha_1, \alpha_2\}$ Consider

$$x = (1 - i\alpha)p_1 + (1 + i\alpha)p_2 = -i\alpha p_1 + i\alpha p_2 + \mathbf{1}.$$

Clearly, $||x|| = \left(1 + \alpha^2\right)^{1/2}$. However,

$$(\phi + i\psi)(x) = 1 + i\alpha\phi(p_1 - p_2) - i\psi(\mathbf{1}) + \alpha\psi(p_1 - p_2).$$

Note that $\psi_i(p_j) = 0$ for $i \neq j$, hence

$$\psi(p_1 - p_2) = \psi_1(p_1) + \psi_2(p_2) = \alpha_1 + \alpha_2 = ||\psi||.$$

Thus, $||(\phi + i\psi)(x)|| \geq 1 + \alpha||\psi||$. Since $||\psi||/2 \leq \alpha \leq ||\psi||$, we have

$$||\phi + i\psi|| \geq \frac{1 + \alpha||\psi||}{(1 + \alpha^2)^{1/2}} \geq \frac{1 + \alpha||\psi||}{(1 + \alpha||\psi||)^{1/2}} = \left(1 + \alpha||\psi||\right)^{1/2} \geq \left(1 + ||\psi||^2/2\right)^{1/2}. \quad \blacksquare$$

COROLLARY 3.6. *Suppose* A *is a unital* C^**-algebra*, $f \in A^*$, $||f|| = 1$, *and* $||f - g|| \geq \alpha$ *whenever* g *is a state on* A, *with* $\alpha < 1/3$. *Then* $\Re f(\mathbf{1}) \leq 1 - \alpha^2/9$.

Proof. Since the restriction of a state to a unital sub-C^*-algebra is again a state, we can (passing to the double dual) assume that f is a normal functional. Represent $f = \phi + i\psi$, where $\phi, \psi \in A^*$ are hermitian. Write $\phi = \phi_+ - \phi_-$, where ϕ_+ and ϕ_- are positive and mutually orthogonal (hence $||\phi_+|| + ||\phi_-|| = ||\phi||$). Let

$$\varepsilon = 1 - \Re f(\mathbf{1}) = 1 - \phi(\mathbf{1}) = 1 - \phi_+(\mathbf{1}) + \phi_-(\mathbf{1}) = 1 - ||\phi_+|| + ||\phi_-||.$$

We need to prove that $\alpha \leq 3\sqrt{\varepsilon}$. Suppose otherwise. Note that

$$\begin{aligned} \|\frac{\phi_+}{||\phi_+||} - \phi\| &= \|\Big(\frac{1}{||\phi_+||} - 1\Big)\phi_+ - \phi_-\| \\ &= \Big(\frac{1}{||\phi_+||} - 1\Big)||\phi_+|| + ||\phi_-|| = 1 - ||\phi_+|| + ||\phi_-|| = \varepsilon. \end{aligned}$$

However, $\phi_+/||\phi_+||$ is a state, hence, by previous lemma,

$$1 = \|f\| = \|\phi + i\psi\| \geq \|\frac{\phi_+}{||\phi_+||} + i\psi\| - \|\frac{\phi_+}{||\phi_+||} - \phi\| \geq \Big(1 + ||\psi||^2/2\Big)^{1/2} - \varepsilon.$$

Therefore,

$$||\psi||^2 \leq 2\big((1 + \varepsilon)^2 - 1\big) = 4\varepsilon + 2\varepsilon^2.$$

But

$$\alpha \leq \| \frac{\phi_+}{||\phi_+||} - f \| \leq \| \frac{\phi_+}{||\phi_+||} - \phi \| + \|\psi\| = \varepsilon + \bigl(4\varepsilon + 2\varepsilon^2\bigr)^{1/2} \leq 3\sqrt{\varepsilon}. \quad \blacksquare$$

Proof of Lemma 3.4. Suppose, for the sake of contradiction, that $||\mathbf{1} + T|| > 2-\alpha^2/10$. Find $f \in B(H)^*$ s.t. $||f|| = 1$ and $f(\mathbf{1}+T) > 2-\alpha^2/10$. Since $|f(\mathbf{1})| \leq 1$, we have $|f(T)| > 1 - \alpha^2/10$. However, $|g(T)| \leq 1 - \alpha$ whenever g is a state, thus $||f - g|| \geq \alpha - \alpha^2/10$. Thus, by Corollary 3.6, $\Re f(\mathbf{1}) \leq 1 - (\alpha - \alpha^2/10)^2/9 < 1 - \alpha^2/10$. However, $|f(T)| \leq 1$, and therefore $f(\mathbf{1}+T) \leq \Re f(\mathbf{1}) + |f(T)| < 2 - \alpha^2/10$. $\blacksquare$

Now we are ready to prove part (2) of Theorem 3.1. Suppose A is a non-commutative C^*algebra. As before, we can find a partial isometry u in the enveloping von Neumann algebra $A^{**} \hookrightarrow B(H)$ s.t. the projections $p = uu^*$ and $q = u^*u$ are mutually orthogonal. Then $\nu(u) = 1/2$. Indeed, pick $\xi \in B(H)$ with $||\xi|| = 1$, and let $\xi_1 = p\xi$, $\xi_2 = q\xi$, $\xi_3 = \xi - \xi_1 - \xi_2$. Then

$$|\langle \xi, u\xi \rangle| = |\langle \xi_1, \xi_2 \rangle| \leq ||\xi_1|| \cdot ||\xi_2|| \leq \bigl(||\xi_1||^2 + ||\xi_2||^2\bigr)/2 \leq \frac{1}{2}.$$

On the other hand, $\nu(u) \geq ||u||/2 = 1/2$.

Fix $\varepsilon > 0$. By Theorem V.5.3 of [6], there exists a net $(u_\alpha) \subset A$ s.t. $||u_\alpha|| \leq 1$, $u_\alpha \to u$ in the weak operator topology, and $\nu(u_\alpha) \leq 1/2 + \varepsilon$. Consider the maps $T_\alpha : A \to A$, defined by $T_\alpha x = (u_\alpha^* x + x u_\alpha)/2$. Clearly, T_α is hermitian, and $||T_\alpha|| \leq 1$. Moreover, $(I_A + \omega T_\alpha)x = \bigl((\mathbf{1} + \omega u_\alpha^*)x + x(\mathbf{1} + \omega u_\alpha)\bigr)/2$. By Lemma 3.6, $||\mathbf{1} + \omega u_\alpha|| \leq 2 - (1/2 - \varepsilon)^2/10$, hence $||I_A + \omega T_\alpha|| \leq 2 - (1/2 - \varepsilon)^2/10$. It remains to prove that $\lim_\alpha ||T_\alpha|| = 1$. To this end, consider $x = u + u^* \in A^{**}$. We know that $||x|| = 1$ and

$$2T_\alpha^{**}x = u_\alpha^* u + u_\alpha^* u^* + u u_\alpha + u^* u_\alpha.$$

Pick $\xi \in \operatorname{ran} p$ with $||\xi|| = 1$. Then

$$\begin{aligned} 2\langle T_\alpha^{**}x\xi, \xi\rangle =& \langle u_\alpha^* u\xi, \xi\rangle + \langle u_\alpha^* u^*\xi, \xi\rangle + \langle u u_\alpha \xi, \xi\rangle + \langle u^* u_\alpha \xi, \xi\rangle \\ &\to_\alpha \langle u^* u\xi, \xi\rangle + \langle (u^*)^2\xi, \xi\rangle + \langle u^2\xi, \xi\rangle + \langle u^* u\xi, \xi\rangle = 2. \end{aligned}$$

Thus, $\lim_\alpha ||T_\alpha|| = 1$. $\blacksquare$

4. Finite dimensional spaces of numerical index 1

In this section we study *real* finite dimensional Banach spaces having numerical index 1, that is, spaces X satisfying $\max\{||I_X + T||, ||I_X - T||\} = 1 + ||T||$ for any $T : X \to X$. Equivalently, these are spaces with numerical index 1. We prove that these spaces are "very different" from a Hilbert space. Recall that the Banach-Mazur distance between Banach spaces X and Y is defined as $d(X, Y) = \inf\{||u|| \cdot ||u^{-1}|| \,|\, u : X \to Y \text{ is an isomorphism}\}$.

THEOREM 4.1. *Suppose X is an n-dimensional Banach space ($n > 1$) of numerical index 1. Then $d(X, \ell_2^n) \geq cn^{1/4}$, where c is a universal constant.*

REMARK. It was shown in [11] that any infinite dimensional real Banach space with the RNP and numerical index 1 contains ℓ_1. Thus, no isomorphic copy of ℓ_2 can have numerical index 1.

The proof of Theorem 4.1 follows from the lemma below.

LEMMA 4.2. *Suppose X is as in the statement of Theorem 4.1. Then either X or X^* contains an isometric copy of ℓ_1^m with $m \geq c\sqrt{n}$, where $c > 0$ is an absolute constant.*

Proof. By [13], the unit balls of X and X^* are polytopes. Moreover, let $(e_i)_{i=1}^M$ and $(f_j)_{j=1}^N$ be the sets of extreme points (vertices) of the unit balls of X and X^*, respectively. Then $|f_j(e_i)| = 1$ for any i and j. By [4], $\ln N \cdot \ln M \geq K^2 n$, where K is an absolute constant. Assume that $N \geq M$, and prove that X contains an isometric copy of ℓ_1^m (the case of $N < M$ is dealt with in the same way). We can assume that the set $(e_i)_{i=1}^n$ is linearly independent. Therefore, for any i and j satisfying $1 \leq i < j \leq n$ there exists k s.t. $f_k(e_i) \neq f_k(e_j)$.

Select $c > 0$ for which

$$\sum_{i=0}^{m-1} \binom{n}{i} \leq e^{K\sqrt{n}} \text{ for } m = \lceil c\sqrt{n} \rceil. \tag{4.1}$$

Indeed, if $i < m < n/3$, we have

$$\binom{n}{i+1} \Big/ \binom{n}{i} = \frac{n \cdot \ldots \cdot (n-i-1)/(i+1)!}{n \cdot \ldots \cdot (n-i)/i!} = \frac{n-i-1}{i} > 2,$$

hence

$$\sum_{i=0}^{m-1} \binom{n}{i} \leq \sum_{i=0}^{m-1} 2^{-m+i} \binom{n}{m} \leq \binom{n}{m} \leq \frac{n^m}{(m/e)^m} = \Big(\frac{ne}{m}\Big)^m.$$

The inequality (4.1) holds if $m(\ln n + 1 - \ln m) \leq K\sqrt{n}$, which, in turn, is satisfied if $m \leq \lceil c\sqrt{n} \rceil$ for some positive constant c.

By [17], there exists a subset $A \subset \{1, 2, \ldots, n\}$ of cardinality $m = \lceil c\sqrt{n} \rceil$ s.t. for any $B \subset A$ there exists j $(1 \leq j \leq N)$ for which

$$f_j(e_i) = \begin{cases} 1 & i \in B \\ -1 & i \in A \backslash B \end{cases}. \tag{4.2}$$

We claim that $\|\sum_{i \in A} \alpha_i e_i\| = \sum_{i \in A} |\alpha_i|$ for any set of scalars $(\alpha_i)_{i \in A}$. Indeed, consider $B \subset A$, and find j for which (4.2) is satisfied. Then

$$m = \sum_{i \in A} \|e_i\| \geq \|\sum_{i \in B} e_i - \sum_{i \in A \backslash B} e_i\| \geq f_j(\sum_{i \in B} e_i - \sum_{i \in A \backslash B} e_i) = m.$$

Now consider arbitrary scalars $(\alpha_i)_{i \in A}$. Let $B = \{i | \alpha_i \geq 0\}$, and $\alpha = \max |\alpha_i|$. Then

$$\begin{aligned} \sum_{i \in A} \alpha_i e_i = & \sum_{i \in B} \big(\alpha - (\alpha - \alpha_i)\big) e_i - \sum_{i \in A \backslash B} \big(\alpha - (\alpha - |\alpha_i|)\big) e_i \\ = & \alpha\big(\sum_{i \in B} e_i - \sum_{i \in A \backslash B} e_i\big) - \sum_{i \in A} (\alpha - |\alpha_i|) e_i, \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{i \in A} |\alpha_i| \geq & \|\sum_{i \in A} \alpha_i e_i\| \geq \alpha \|\sum_{i \in B} e_i - \sum_{i \in A \backslash B} e_i\| - \sum_{i \in A} (\alpha - |\alpha_i|) \\ = & \alpha m - \sum_{i \in A} (\alpha - |\alpha_i|) = \sum_{i \in A} |\alpha_i|. \quad \blacksquare \end{aligned}$$

REMARK. Real finite dimensional spaces with numerical index 1 are sometimes called *CL-spaces*. It is known (see [5]) that any finite dimensional space with the 3.2.I.P. is a CL-space (X is said to have the 3.2 *intersection property*, or 3.2.*I.P.*, if any three pairwise intersecting balls in X have a point in common). By [5], any finite dimensional space with the 3.2.I.P. is of the form $E_1 \oplus_1 E_2$ or $E_1 \oplus_\infty E_2$, where E_1 and E_2 possess the 3.2.I.P. and have positive dimension.

One can see that $d(E, \ell_2^n) = \sqrt{n}$ if E is an n-dimensional space with the 3.2.I.P.. Indeed, by induction and duality, it suffices to show that

$$d(E_1 \oplus_1 E_2, \ell_2^n) \geq \left((d(E_1, \ell_2^{n_1}))^2 + (d(E_2, \ell_2^{n_2}))^2\right)^{1/2} \tag{4.3}$$

whenever E_1 and E_2 are spaces of dimension n_1 and n_2, respectively, and $n = n_1 + n_2$.

To prove (4.3), let $c_i = d(E_i, \ell_2^{n_i})$ for $i = 1, 2$. Consider a contraction $u : E_1 \oplus_1 E_2 \to \ell_2^n$, and let $F_i = u(E_i)$ $(i = 1, 2)$. Find $x_i \in F_i$ s.t. $||x_i|| = c_i/\sqrt{c_1^2 + c_2^2}$ and $||u^{-1}x_i|| \geq c_i^2/\sqrt{c_1^2 + c_2^2}$. Let $x = x_1 + \varepsilon x_2$, where $\varepsilon = \pm 1$ is chosen in such a way that $||x|| \leq (||x_1||^2 + ||x_2||^2)^{1/2} = 1$. However,

$$||u^{-1}x|| = ||u^{-1}x_1|| + ||u^{-1}x_2|| \geq \sqrt{c_1^2 + c_2^2}.$$

By [16], there exist 5-dimensional CL-spaces failing the 3.2.I.P. (indeed, S. Reisner constructed such a space with a 1-unconditional basis). Nevertheless, it seems likely that $d(E, \ell_2^n) \geq c\sqrt{n}$ for any n-dimensional CL-space. We were unable to prove this conjecture.

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