UNCONDITIONAL BASIC SEQUENCES AND HOMOGENEOUS HILBERTIAN SUBSPACES OF NON-COMMUTATIVE L_p SPACES

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ABSTRACT. Suppose A is a von Neumann algebra with a normal faithful normalized trace τ. We prove that if E is a homogeneous Hilbertian subspace of $L_p(\tau)$ $(1 \le p < \infty)$ such that the norms induced on E by $L_p(\tau)$ and $L_2(\tau)$ are equivalent, then E is completely isomorphic to the subspace of $L_p([0,1])$ spanned by Rademacher functions. Consequently, any homogeneous subspace of $L_p(\tau)$ is completely isomorphic to the span of Rademacher functions in $L_p([0,1])$. In particular, this applies to the linear span of operators satisfying the canonical anti-commutation relations. We also show that the real interpolation space $(R, C)_{\theta,p}$ embeds completely isomorphically into $L_p(\mathcal{R})$ ($\mathcal{R}$ is the hyperfinite II_1 factor) for any $1 \le p < 2$ and $\theta \in (0,1)$.

0. INTRODUCTION AND THE MAIN RESULTS

The classical Khintchine Inequality states that, for $1 \le p < \infty$ and $a_1, \dots, a_n \in \mathbb{C}$,

$$\|\sum_{i=1}^n a_i r_i\|_{L_p} \sim \big(\sum_{i=1}^n |a_i|^2\big)^{1/2}.$$

Here, (r_i) are the Rademacher functions (corresponding to independent mean 0 Bernoulli random variables), and $\sim$ denotes "equivalence up to a multiplicative constant". This inequality has found numerous applications in functional analysis and probability.

The non-commutative analogue of this inequality was discovered by F. Lust-Piquard and G. Pisier (see [21], [22]). In its simplest form, their result can be stated as follows: suppose $1 \le p < \infty$, and $a_1, \dots, a_n \in S_p$ (the Schatten p class). Then

$$\|\sum_{i=1}^n r_i a_i\|_{L_p(S_p)} \sim \left\{ \begin{array}{c} \max\{\|(\sum_i a_i^* a_i)^{1/2}\|_{S_p}, \|(\sum_i a_i a_i^*)^{1/2}\|_{S_p}\} \\ \text{for } 2 \le p \le \infty \\ \inf\{\|(\sum_i b_i^* b_i)^{1/2}\|_{S_p} + \|(\sum_i c_i c_i^*)^{1/2}\|_{S_p} \mid a_i = b_i + c_i\} \\ \text{for } 1 \le p \le 2 \end{array} \right. .$$

Recently, the ubiquity of non-commutative "Khintchine-type" inequalities became obvious: there are many families $e_1, e_2, \dots \in L_p(N, \tau)$ (τ is a faithful normal semi-finite trace on a von Neumann algebra N, and $1 \le p < \infty$) such that

$$\|\sum_{i=1}^n r_i a_i\|_{L_p(S_p)} \sim \|\sum_{i=1}^n a_i \otimes e_i\|_{S_p[L_p(N,\tau)]} \tag{0.1}$$

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for any $a_1, \dots, a_n \in S_p$. In particular, the above equivalence holds if e_i corresponds to the left regular representation of the generators of a free group, Gaussian random variables, or elements of a free semi-circular system in the sense of Voiculescu (see Chapter 9 of [31]).

The goal of this paper is to better understand the structure of sequences of vectors in non-commutative L_p spaces, satisfying (0.1), and to give more examples of such sequences (Theorems A and C). This includes all the examples described above, as well as the generators of a CAR algebra (see Section 3 below).

In this paper, we work with operator spaces (see [6] or [31] for an introduction). We need to recall a few definitions, which will be used throughout this paper. An operator space is called *C-Hilbertian* if it is C-isomorphic (in the Banach space sense) to a Hilbert space. X is *C-homogeneous* if every linear operator $u : X \to X$ is completely bounded, and $\|u\|_{cb} \leq C\|u\|$. X is called *Hilbertian* (*homogeneous*) if it is C-Hilbertian (resp. C-homogeneous) for some C.

Suppose $e_1, e_2, \dots$ form a basis in an operator space E. We say that $(e_i)_{i=1}^{\infty}$ is *M-completely unconditional* if $\|I - 2P_S\|_{cb} \leq M$ for every $S \subset \mathbb{N}$, where the projection $P_S : E \to E$ is defined by $P_S(\sum_{i=1}^{\infty} \alpha_i e_i) = \sum_{i \in S} \alpha_i e_i$. The bases $(e_i) \subset E$ and $(e_i') \subset E'$ are called *C-completely equivalent* if the operator $u : E \to E' : e \mapsto e'$ satisfies $\|u\|_{cb}\|u^{-1}\|_{cb} \leq C$.

The general theory of non-commutative L_p spaces can be found in [30] and [31]. Some important facts are also mentioned in Section 1. Following [9], we denote by RC_p the operator space $(R \cap C, R + C)_{1/p}$, where R and C are the classical row and column spaces. Equivalently, $RC_p = R_p \cap C_p$ if $2 \leq p \leq \infty$, and $RC_p = R_p + C_p$ if $1 \leq p \leq 2$, where R_p and C_p denote the rows and columns of the Schatten class S_p.

The results of Lust-Piquard and Pisier mentioned above ([21], [22]) show that the space Rad_p (the span of independent Rademacher functions in $L_p([0,1])$) is completely isomorphic to RC_p for $1 \leq p < \infty$. However, there exist many more examples of subspaces of non-commutative L_p spaces completely isomorphic to RC_p. As in the classical setting, the cases of $1 \leq p < 2$ and $2 < p < \infty$ are radically different.

Theorem A. *Suppose $2 < p < \infty$, and N is a von Neumann algebra with a faithful normal semifinite trace τ.*

(1) *Any homogeneous subspace of $L_p(\tau)$ is completely isomorphic to R_p, C_p, or RC_p.*
(2) *If (e_i) is a completely unconditional basic sequence in $L_p(\tau)$ for finite τ, then it is either completely equivalent to the canonical basis in RC_p, or for any $\varepsilon > 0$ there exists a sequence (i_j) such that (e_{i_j}) is $(1+\varepsilon)$-completely equivalent to the canonical basis of ℓ_p, and $\mathrm{span}[e_{i_j} \mid j \in \mathbb{N}]$ is $(1+\varepsilon)$-completely complemented in $L_p(\tau)$.*
(3) *If, in addition, τ is finite, then any homogeneous subspace of $L_p(\tau)$ is completely isomorphic to RC_p.*

This theorem will be proved in Section 2, where other results concerning the case of $p \in (2, \infty)$ will also be mentioned.

$L_p(\tau)$ contains a wider variety of subspaces for $1 \leq p < 2$. Theorems B and C below will be proved in Section 3.

Theorem B. *Suppose $1 \leq p < 2$, and $\tau_{\mathcal{R}}$ is the trace on the hyperfinite II_1 factor*

$\mathcal{R}$.

1. *For any $\theta \in (0,1)$, the real interpolation space $(R_p, C_p)_{\theta,p}$ embeds completely isomorphically into $L_p(\tau_{\mathcal{R}})$.*
2. *$L_p(\tau_{\mathcal{R}})$ contains uncountably many homogeneous Hilbertian subspaces which are not completely isomorphic to each other.*

However, Hilbertian subspaces of $L_p(\tau)$ satisfying certain conditions are completely isomorphic to RC_p.

Theorem C. *Suppose N is a von Neumann algebra equipped with faithful normal finite trace τ, E is an H-homogeneous subspace of $L_p(\tau)$ $(1 \leq p \leq 2)$, and $\gamma\|e\|_p \geq \|e\|_2$ for any $e \in E$. Then E is $\alpha\gamma^3 H^2$-completely isomorphic to RC_p, where α is a constant.*

Consequently, the linear span of a set of generators of a CAR algebra is completely isomorphic to RC_p for $1 \leq p < \infty$ (Proposition 3.6). We use this fact to compute the projection constant of the span of n operators satisfying the canonical anticommutation relation (Corollary 3.7).

We conclude Section 3 by showing, in Theorem 3.8, that RC_q does not embed completely isomorphically into $L_p(\tau)$ if $1 \leq q < p$ or $p/(p-1) < q \leq \infty$ (see also the remark preceding Lemma 3.1 for more information about subspaces of L_p spaces).

In Section 4 we work with "commutative" L_p spaces. This situation is special, since, in addition to the methods applicable to non-commutative L_p spaces, we can also use the techniques of "change of density." We show that any homogeneous Hilbertian subspace of $L_p(\mu)$ (μ is a measure) is completely isomorphic to RC_p (Theorem 4.1). As a corollary, we prove an "interpolation/extrapolation" result for homogeneous Hilbertian subspaces of L_p: suppose E is a closed subspace of $L_p(\mu)$ $(2 \leq p < \infty$, μ is a finite measure). Denote by E_q the space E with the operator space structure inherited from $L_q(\mu)$. We show that if E_p is completely isomorphic to RC_p, then E_q is completely isomorphic to RC_q for $p/(p-1) \leq q \leq p$ (Proposition 4.4). We also show that any completely unconditional basis equivalent to the canonical basis of ℓ_p is, in fact, completely equivalent to it, and describe a class of completely complemented copies of ℓ_p (Proposition 4.7, Corollary 4.8).

For the sake of completeness, we devote the next section to outlining the definition of non-commutative L_p spaces, and recalling some pertinent results. More information can be found in [30], or in Chapter 7 of [31].

1. PRELIMINARIES

The "natural" operator space structure on a C^*-algebra A arises from the C^*-algebra structure on $M_n(A)$ for $n \in \mathbb{N}$. We define *the opposite* of A – called A^{op} – by considering the same algebra with a new multiplication $\circ$: $x \circ y = yx$. It is easy to check that

$$\|\sum a_n \otimes x_n\|_{M_n(A)} = \|\sum a_n^t \otimes x_n\|_{M_n(A^{op})},$$

where $a_n \in M_n$, $x_n \in A$, and a^t is the transpose of the matrix a.

To define the operator space structure on $L_p(\tau)$, let $L_1(\tau) = A_*^{op}$ (the predual of A^{op}), and $L_p(\tau) = (A, L_1(\tau))_{1/p}$ for $1 < p < \infty$. To give an alternative description of $L_p(\tau)$ a Banach space, denote by $\Lambda_p(\tau)$ the set of $a \in A$ for which $\tau(|a|^p) < \infty$. Then $L_p(\tau)$ $(1 \leq p < \infty)$ is the completion of $\Lambda_p(\tau)$ in the norm $\|a\|_p = \tau(|a|^p)^{1/p}$.

Trace duality allows us to identify $L_1(\tau)$ with A_*. For the sake of convenience we sometimes use the symbol $L_\infty(\tau)$ for A.

More generally, if E is an operator space, we let $L_p(\tau, E) = \big(A\overset{\smile}{\otimes}E, L_1(\tau)\overset{\frown}{\otimes}E\big)_{1/p}$. Here $\overset{\smile}{\otimes}$ and $\overset{\frown}{\otimes}$ denote the minimal and maximal operator space tensor products, respectively. It was shown in [30] that, if E_0 and E_1 are operator spaces and τ is a trace on a hyperfinite von Neumann algebra A, then

$$\big(L_{p_0}(\tau, E_0), L_{p_1}(\tau, E_1)\big)_\theta = L_p(\tau, E_\theta),$$

where $E_\theta = (E_0, E_1)_\theta$ and $1/p = (1-\theta)/p_0 + \theta/p_1$.

If we apply the above definitions to $B(\ell_2)$ equipped with the canonical trace tr, we obtain Schatten p-classes S_p $(1 \le p < \infty)$. S_∞ denotes the set of compact operators on ℓ_2. We use the notation $S_p[E]$ instead of $S_p(tr, E)$. Repeating the same steps for $B(\ell_2^n)$ instead of $B(\ell_2)$, we obtain the spaces S_p^n. Letting (E_{ij}) be the matrix units in S_p, we denote span$[E_{i1} \,|\, i \in \mathbb{N}]$ and span$[E_{1j} \,|\, j \in \mathbb{N}]$ by R_p and C_p, respectively. These spaces are 1-Hilbertian and 1-homogeneous. Their n dimensional subspaces are denoted by R_p^n and C_p^n.

It is worth noting that $L_p(\tau, S_p) = S_p[L_p(\tau)]$ for any von Neumann algebra with a semifinite normal faithful trace τ. This identity follows from the facts listed above in the hyperfinite case. The general case was proved by the first named author in [12].

If E is an operator space, then $C_p \otimes_h E \otimes_h R_p$ is completely isometric to $S_p[E]$ (here $\otimes_h$ is the Haagerup tensor product). In particular, $S_\infty[E] = S_\infty\overset{\smile}{\otimes}E$ and $S_1[E] = S_1\overset{\frown}{\otimes}E$. It was shown in [30] that $S_p^n[E]^* = S_{p'}^n[E^*]$ (with $1/p + 1/p' = 1$). Moreover, $S_p[E]^* = S_{p'}[E^*]$ if $p > 1$. Here, we are using so called "parallel duality": $\langle (a_{ij})_{i,j=1}^n, (b_{ij})_{i,j=1}^n \rangle = \sum_{i,j=1}^n \langle a_{ij}, b_{ij} \rangle$ for $(a_{ij}) \in S_{p'}[E^*]$, $(b_{ij}) \in S_p[E]$.

The connection between the norms of $S_p[E]$ and $M_n(E)$ is given in [30]. Consequently, if E and F are operator spaces and $u : E \to F$ is a linear map, then

$$\|u\|_{cb} = \|I_{S_p} \otimes u : S_p[E] \to S_p[F]\|. \tag{1.1}$$

We denote by $R_p[E]$ the linear span of $\sum E_{1i} \otimes e_i$ $(e_i \in E)$ in $S_p[E]$. Similarly, $C_p[E]$ stands for the linear span of $\sum E_{i1} \otimes e_i$. By the above, $R_p[E]$ $(C_p[E])$ can be identified with $E \otimes_h R_p$ (resp. $C_p \otimes_h E$). If τ is a semi-finite normal faithful trace on a von Neumann algebra A, we identify $S_p[L_p(\tau)]$ with $L_p(\tau')$, where $\tau' = \tau \otimes tr$ is a trace on $A\overline{\otimes}B(\ell_2)$. In particular, if (δ_i) is an orthonormal basis in R_p (in other words, $\delta_i = E_{1i}$), then

$$\|\sum a_i \otimes \delta_i\|_{L_p(\tau, R_p)} = \|(a_1, a_2, \dots)\|_{R_p[L_p(\tau)]} = \|(\sum_i a_i a_i^*)^{1/2}\|_{L_p(\tau)}.$$

Similarly, $\|\sum a_i \otimes \delta_i\|_{L_p(\tau, C_p)} = \|(\sum_i a_i^* a_i)^{1/2}\|_{L_p(\tau)}$.

For a von Neumann algebra A with a faithful normal semi-finite trace τ and $1 \le p \le \infty$, identify a sequence of $(a_1, a_2, \dots)$ with an element of $R_p[L_p(\tau)] + C_p[L_p(\tau)]$ $(1 \le p \le 2)$, or $R_p[L_p(\tau)] \cap C_p[L_p(\tau)]$ $(2 \le p \le \infty)$, and denote by $|||\cdot|||_p$ the corresponding norm. In other words,

$$\begin{aligned}|||(a_1, a_2, \dots)|||_p &= \max\{\|(\sum_i a_i^* a_i)^{1/2}\|_{L_p}, \|(\sum_i a_i a_i^*)^{1/2}\|_{L_p}\} \\ &\quad \text{for } 2 \le p \le \infty \\ |||(a_1, a_2, \dots)|||_p &= \inf\{\|(\sum_i b_i^* b_i)^{1/2}\|_{L_p} + \|(\sum_i c_i c_i^*)^{1/2}\|_{L_p} \,|\, a_i = b_i + c_i\} \\ &\quad \text{for } 1 \le p \le 2.\end{aligned} \tag{1.2}$$

By [30], the space described above coincides with $L_p(\tau, RC_p)$ if the von Neumann algebra A is hyperfinite.

By the results of F. List-Piquard and G. Pisier [21, 22], the span Rad_p of independent Rademacher functions in $L_p(\mu)$ (μ is the standard Lebesgue measure on $[0, 1]$) is completely isomorphic to RC_p for $1 \le p < \infty$. More precisely, there exist constants A and B such that, for any $a_1, a_2, \dots \in L_p(\tau)$,

$$\begin{aligned} A|||(a_i)|||_p &\le \|\sum_i r_i a_i\|_{L_p(\mu, L_p(\tau))} \le |||(a_i)|||_p \text{ for } 1 \le p \le 2, \\ |||(a_i)|||_p &\le \|\sum_i r_i a_i\|_{L_p(\mu, L_p(\tau))} \le B\sqrt{p}\,|||(a_i)|||_p \text{ for } 2 \le p < \infty.\end{aligned} \tag{1.3}$$

2. THE CASE OF $2 \le p < \infty$

In this section we deal with the space $L_p = L_p(N, \tau)$ for $2 \le p < \infty$. Here, as before, N is a von Neumann algebra with a faithful normal semi-finite trace τ. As in the commutative case, we establish a *lower p-estimate* and an *upper 2-estimate* for completely unconditional basic sequences.

Theorem 2.1. *Suppose $(e_i)_{i=1}^\infty$ is an M-completely unconditional basis in $E \hookrightarrow L_p$ ($2 \le p < \infty$). Then*

$$\frac{1}{M} \inf_i \|e_i\| \cdot \big(\sum_i \|a_i\|^p\big)^{1/p} \le \|\sum_i e_i \otimes a_i\|_{S_p[L_p(\tau)]} \le BM\sqrt{p} \sup_i \|e_i\| \cdot |||(a_i)|||_p$$

for any $a_1, a_2, \dots \in S_p$ (here, B is the constant from (1.3)).

In some cases, we can improve the lower estimate.

Theorem 2.2. *Suppose $(e_i)_{i=1}^\infty$ is an M-completely unconditional basis in $E \hookrightarrow L_p$ ($2 \le p < \infty$), and there exist functionals $\phi, \psi \in L_{p/2}^*$ such that $\|\phi\| = \|\psi\| = 1$ and $\min\{\phi(e_i e_i^*), \psi(e_i^* e_i)\} \ge \gamma^2 > 0$ for every i (in particular, this condition is satisfied if τ is normalized and $\|e_i\|_2 = \tau(e_i^* e_i)^{1/2} = \tau(e_i e_i^*)^{1/2} \ge \gamma > 0$), and $\|e_i\|_p \le 1$. Then, for any $a_1, a_2, \dots \in S_p$,*

$$\frac{\gamma}{M} |||(a_i)|||_p \le \|\sum_i e_i \otimes a_i\|_{S_p[L_p(\tau)]} \le BM\sqrt{p} \sup_i \|e_i\| \cdot |||(a_i)|||_p.$$

Corollary 2.3. *Suppose E is an H-homogeneous subspace of L_p ($2 \leq p < \infty$), and $\gamma\|e\|_2 \geq \|e\|_p$ for any $e \in E$. Then E is $\alpha\gamma^3 H^2\sqrt{p}$-completely isomorphic to RC_p, where α is a constant.*

Proof. Pick an orthonormal (with respect to $\|\cdot\|_2$) basis (e_i) in E. Then (e_i) is $H\gamma$-completely unconditional. Indeed, if S is a subspace of $\mathbb{N}$ and P is the corresponding projection, then

$$\|I - 2P\|_{cb} \leq H\|I - 2P : E \to E\| \leq H\gamma.$$

The statement of this corollary then follows from the preceding theorem (and (1.1)). ■

Proof of Theorem A. (1) follows from Corollary 3.12 of [17].

(2) By Theorem 2.4 of [37], either $\inf_i \|e_i\|_2 > 0$, or (e_i) has a subsequence equivalent to the canonical basis in ℓ_p. In the former case, the sequence (e_i) is completely equivalent to the orthonormal basis in RC_p, by Theorem 2.2.

Otherwise, note that $\lim_i \|he_i\|_1 = \lim_i \|e_i h\|_1 = 0$ for every $h \in L_{p'}$ (here, $1/p + 1/p' = 1$). Indeed, for every $\varepsilon > 0$ there exists a partial isometry u and mutually orthogonal projections $p_1, \ldots, p_n$ s.t. $\|h - h'\|_{p'} < \varepsilon$, where $h' = u(\sum_j \alpha_j p_j)$, with $\alpha_j \in \mathbb{C}$. Then $\|(h - h')e_i\|_1 \leq \|h - h'\|_{p'}\|e_i\|_p < \varepsilon$, and $\lim_i \|h'e_i\|_1 \leq \lim_i \|h'\|_2\|e_i\|_2 = 0$. Since ε is arbitrary, we conclude that $\lim_i \|he_i\|_1 = 0$. The equality $\lim_i \|e_i h\|_1 = 0$ is established similarly.

Therefore, by Proposition 2.11 of [35], (e_i) contains (for every $\varepsilon > 0$) a subsequence which is $(1+\varepsilon)$-completely equivalent to the canonical basis of ℓ_p, and whose span is $(1+\varepsilon)$-completely complemented in $L_p(\tau)$.

(3) follows from (2) and the fact that ℓ_p is not homogeneous (see [8]). ■

Remark. Clearly, the spaces R_p, C_p, RC_p, and ℓ_p can be found in S_p.

To prove Theorems 2.1 and 2.2, we need several lemmas.

Lemma 2.4. *Suppose N is a von Neumann algebra with a normal faithful semi-finite trace τ, $a_1, a_2, \ldots, a_k \in S_p$ ($2 \leq p \leq \infty$), and $e_1, e_2, \ldots, e_k \in L_p(\tau)$. Denote by (δ_i) an orthonormal basis in C_p. Then*

$$\inf_i \|e_i\|_{L_p}\big(\sum_i \|a_i\|^p\big)^{1/p} \leq \|\sum \delta_i \otimes e_i \otimes a_i\|_{C_p[S_p[L_p]]}$$
$$\leq \sup_i \|e_i\|_{L_p}\|\sum \delta_i \otimes a_i\|_{C_p[L_p]}.$$

Proof. We prove the right-hand side inequality first. Suppose $\sup_i \|e_i\|_p \leq 1$, and show that $\|\sum \delta_i \otimes e_i \otimes a_i\|_p \leq \|\sum \delta_i \otimes a_i\|_p$. Consider the map $T : \ell_\infty^k(L_p(\tau)) \times S_p^k[C_p] \to C_p^k[S_p[L_p(\tau)]]$, defined by

$$T\big((e_1, \ldots, e_k) \times \sum_{i=1}^k \delta_i \otimes a_i\big) = \sum_{i=1}^k \delta_i \otimes e_i \otimes a_i.$$

Note that

$$\|T : \ell_\infty^k(N) \times S_\infty[C_\infty^k] \to C_\infty[S_\infty[N]]\| \leq 1.$$

Indeed, denote by E_{ij} the (i,j) matrix unit. Then

$$T\big((e_1,\dots,e_k)\times\sum_{i=1}^k\delta_i\otimes a_i\big)=\big[\sum_i E_{ii}\otimes e_i\big]\cdot\big[\sum_i E_{i1}\otimes a_i\big]$$
$$\le\|\sum_i E_{ii}\otimes e_i\|\cdot\|\sum_i E_{i1}\otimes a_i\|=\sup_i\|e_i\|\cdot\|\sum_i\delta_i\otimes a_i\|.$$

Moreover,

$$\|T:\ell_\infty^k(L_2(\tau))\times S_2[C_2^k]\to C_2[S_2[L_2(\tau)]]\|\le 1.$$

Moreover $C_p^k[S_p]$ can be identified with S_p completely isometrically via the map $E_{i1}\otimes E_{mn}\mapsto E_{i+k(m-1),n}$. Therefore, $C_p^k[S_p]=\big(C_\infty^k[S_\infty],C_2^k[S_2]\big)_{2/p}$. By complex interpolation (see Theorem 4.4.1 of [1]),

$$\|T:\ell_\infty^k(L_p[\tau])\times S_p[C_p^k]\to S_p[L_p(\tau)]\|\le 1.$$

Now consider the left-hand side inequality. Denote the canonical basis in ℓ_p by σ_i, and consider a linear map $T:C_p\to\ell_p$, defined by $T\delta_i=\sigma_i$. Note that $T:C_\infty\to\ell_\infty$ is completely contractive, and $T:C_2\to\ell_2$ is a complete isometry. Thus, $\|T:C_p\to\ell_p\|_{cb}\le 1$, and therefore,

$$\inf_i\|e_i\|\cdot\big(\sum\|a_i\|^p\big)^{1/p}\le\big(\sum\|e_i\|\cdot\|a_i\|^p\big)^{1/p}$$
$$=\big(\sum\|e_i\otimes a_i\|^p\big)^{1/p}\le\|\sum\delta_i\otimes e_i\otimes a_i\|.\quad\blacksquare$$

Lemma 2.5. *Suppose N, τ, (δ_i), and p are as in Lemma 2.4, and there exists a contractive linear functional $\psi\in S_{p/2}^*$ such that $\psi(e_i^*e_i)\ge\gamma^2$ (or $\psi(e_ie_i^*)\ge\gamma^2$) for every i. Then, for $a_1,a_2,\ldots\in S_p$, $\|\sum\delta_i\otimes e_i\otimes a_i\|_{C_p[S_p[L_p]]}\ge\gamma\|\sum\delta_i\otimes a_i\|_{C_p[L_p]}$ (resp. $\|\sum\delta_i\otimes e_i\otimes a_i\|_{R_p[S_p[L_p]]}\ge\gamma\|\sum\delta_i\otimes a_i\|_{R_p[L_p]}$).*

Proof. Suppose $\|\psi\|\le 1$, $\psi(e_i^*e_i)\ge\gamma^2$. Then

$$\|\sum\delta_i\otimes e_i\otimes a_i\|^2_{C_p[S_p[L_p]]}=\|(\sum e_i^*e_i\otimes a_i^*a_i)^{1/2}\|^2_{S_p[L_p(\tau)]}$$
$$=\|\sum e_i^*e_i\otimes a_i^*a_i\|_{S_{p/2}[L_{p/2}(\tau)]}\ge\|\psi\otimes I_{S_{p/2}}(\sum e_i^*e_i\otimes a_i^*a_i)\|_{S_{p/2}[L_{p/2}(\tau)]}$$
$$\ge\gamma^2\|\sum a_i^*a_i\|_{S_{p/2}}=\gamma^2\|(\sum a_i^*a_i)^{1/2}\|^2_{S_p}.$$

The case of $\psi(e_ie_i^*)\ge\gamma^2$ is dealt with in a similar fashion. $\blacksquare$

Proof of Theorem 2.1. Since the sequence (e_i) is M-completely unconditional,

$$\frac{1}{M}\mathrm{Ave}_\pm\|\sum\pm e_i\otimes a_i\|_{S_p[L_p(\tau)]}\le\|\sum e_i\otimes a_i\|_{S_p[L_p(\tau)]}$$
$$\le M\mathrm{Ave}_\pm\|\sum\pm e_i\otimes a_i\|_{S_p[L_p(\tau)]}.$$

By (1.3) and Lemma 2.4,

$$M\mathrm{Ave}_\pm\|\sum\pm e_i\otimes a_i\|_{S_p[L_p(\tau)]}\le MB\sqrt{p}\,|||(e_i\otimes a_i)|||_p\le MB\sqrt{p}\,\sup_i\|e_i\|\cdot|||(a_i)|||_p$$

and

$$\frac{1}{M}\mathrm{Ave}_{\pm}\|\sum \pm e_i \otimes a_i\| \geq \frac{1}{M}|||(e_i \otimes a_i)|||_p \geq \frac{1}{M}\inf_i \|e_i\|\big(\sum \|a_i\|^p\big)^{1/p}.$$

The proof is complete. ■

Remark. A weaker version of the last inequality can be obtained without resorting to Lemma 2.4. Indeed, by Corollary 5.2 of [34], $L_p(\tau)$ has the modulus of uniform convexity $\delta(\varepsilon) \geq \varepsilon^p/(p2^p)$. Thus, by Theorem 1.e.16 of [20], the cotype p constant of $L_p(\tau)$ is at least $c/(p2^p)$, and therefore,

$$\mathrm{Ave}_{\pm}\|\sum \pm e_i \otimes a_i\| \geq \frac{c}{p2^p}\big(\sum \|e_i \otimes a_i\|^p\big)^{1/p} \geq \frac{c}{p2^p}\inf_i \|e_i\|\big(\sum \|a_i\|^p\big)^{1/p}.$$

Proof of Theorem 2.2. The right-had side follows immediately from Theorem 2.1. To prove the left-hand side, apply (1.3) and Lemma 2.5:

$$\mathrm{Ave}_{\pm}\|\sum \pm e_i \otimes a_i\|_{S_p[L_p(\tau)]} \geq |||(e_i \otimes e_i)|||_p \geq \gamma|||(a_i)|||_p. \quad ■$$

3. The case of $1 \leq p \leq 2$

We start the proof of Theorem B by describing the real interpolation functor (adapted to the non-commutative setting by Q. Xu in [39]).

Suppose (E_0, E_1) is a compatible pair of operator spaces. For $t \in \mathbb{R}$ and $p \in [1, \infty]$, consider an embedding $i_t : E_0 \cap E_1 \to E_0 \oplus_1 E_1$ and a quotient $q_t : E_0 \oplus_1 E_1 \to E_0 \oplus_1 E_1/S_t$, where $i_t(x) = (x, tx)$ and S_t is the closure of $\{(x, -tx) \,|\, x \in E_0 \cap E_1\}$. The closure of $q_t i_t(E_0 \cap E_1)$, equipped with the quotient operator space structure, is denoted by $K_t(R_p, C_p)$. Note that, for $x \in K_t(R_p, C_p)$,

$$\|x\|_t = K(t, x, E_0, E_1) = \inf\{\|x_0\|_{E_0} + t\|x_1\|_{E_1} \,|\, x = x_0 + x_1\}$$

(this is the classical K-functional – see e.g. [1]). More generally,

$$\|x\|_t \overset{2}{\sim} K(t, x, S_p[E_0], S_p[E_1]) = \inf\{\|x_0\|_{S_p[E_0]} + t\|x_1\|_{S_p[E_1]} \,|\, x = x_0 + x_1\} \quad (3.1)$$

for $x \in S_p[K_t(E_0, E_1)]$.

To introduce the real interpolation functor $(\cdot,\cdot)_{\theta,p}$ for $1 \leq p \leq \infty$ and $0 < \theta < 1$, in the category of operator spaces, define maps $j_t = q_t i_t : E_0 \cap E_1 \to K_t(E_0, E_1)$. Consider operator spaces $F_n = K_{2^n}(E_0, E_1)$ $(n \in \mathbb{Z})$ and $\ell_p(F_n)$. The last space is isometric to $(\sum_{n=-\infty}^{\infty} F_n)_p$ on the Banach space level, and has operator space structure defined by complex interpolation between $(\sum F_n)_\infty$ and $(\sum F_n)_1$. By Corollary 1.4 of [30],

$$\|x\|_{S_p[\ell_p(F_n)]} = \big(\sum_{n=-\infty}^{\infty} \|x_n\|_{S_p[F_n]}^p\big)^{1/p}$$

for $x = (x_n)$ with $x_n \in S_p[F_n]$.

The *real interpolation space* $(E_0, E_1)_{\theta,p}$ is defined as norm closure of $J(E_0 \cap E_1) \subset \ell_p(F_n)$, where J maps $x \in E_0 \cap E_1$ into $A^{-\theta n}(j_{2^n}(x))_{n=-\infty}^{\infty}$. By (3.1),

$$\|x\|_{S_p[(E_0,E_1)_{\theta,p}]} \overset{2}{\sim} \Big(\sum_{n=-\infty}^{\infty} 2^{-\theta p n} (K(2^n, x, S_p[E_0], S_p[E_1])^p)\Big)^{1/p}. \tag{3.2}$$

This definition is "natural," since many classical results remain true in the non-commutative case. In particular, by [39], if $T : E_0 \cap E_1 \to F_0 \cap F_1$ is a linear map, then

$$\|T : (E_0, E_1)_{\theta,p} \to (F_0, F_1)_{\theta,p}\|_{cb} \le 4 \|T : E_0 \to F_0\|_{cb}^{1-\theta} \|T : E_1 \to F_1\|_{cb}^{\theta} \tag{3.3}$$

Moreover, if X is an operator space, then

$$X \overset{C}{\sim} (X, X)_{\theta,p}, \tag{3.4}$$

with C depending only on θ and p (the proof uses (3.2) and Theorem 3.4.1 of [1]).

Proof of Theorem B(1). Consider the von Neumann algebra $N = L_\infty(0, \infty) \overline{\otimes} B(\ell_2)$ (with its "natural" trace τ). We shall denote by $L_2^{C_p}(\tau)$ the space $L_2(\tau)$, equipped with the operator space structure of C_p. $L_2^{R_p}(\tau)$ is defined in the same manner. By [11], there exists a C-completely isomorphic embedding of $Y = L_p(\tau) + L_2^{C_p}(\tau) + L_2^{R_p}(\tau)$ into $L_p(\mathcal{R}, \tau_{\mathcal{R}})$, where $\tau_{\mathcal{R}}$ is the trace on the hyperfinite II_1 factor $\mathcal{R}$.

Fix $s > 0$, and let

$$X = \{1_{(0,s)} \otimes x \,|\, x_{ij} = 0 \,\text{if}\, i \ne 1\}$$

be a subspace of Y. Note that X is completely contractively complemented in Y via the projection $P = P_s \otimes P_{row}$, where P_{row} is the projection onto the first row of $B(\ell_2)$, and

$$P_s(f) = \frac{1}{s} \int_0^s f(t)dt \cdot 1_{(0,s)}.$$

Thus, for $x \in S_p[X]$ and $s < 1$,

$$\begin{aligned} \|x\|_{S_p[X]} &\overset{3}{\sim} \inf_{x = 1_{(0,s)} \otimes (x_1 + x_2 + x_3)} \{ s^{1/p} \|x_1\|_{S_p[R_p]} + s^{1/2} \|x_2\|_{S_p[C_p]} + s^{1/2} \|x_3\|_{S_p[R_p]} \} \\ &= \inf\{ s^{1/p} \|x_1\|_{S_p[R_p]} + s^{1/2} \|x_2\|_{S_p[C_p]} \,|\, x = 1_{(0,s)} \otimes (x_1 + x_2)\}, \end{aligned}$$

since $s^{1/p} \ge s^{1/2}$. Therefore,

$$s^{-1/p} \|x\|_{S_p[X]} \overset{3}{\sim} K(t, x, S_p[R_p], S_p[C_p])$$

(with $t = s^{-(2-p)/2p}$), and the space $K_t(R_p, C_p)$ embeds into $L_p(\tau_{\mathcal{R}})$ $3C$-completely isomorphically for $t \ge 1$. Interchanging R_p and C_p, we obtain the result for $t \le 1$.

Recalling the construction of $(R_p, C_p)_{\theta,p}$ $(0 < \theta < 1)$, as well as the fact that $\ell_p(L_p(\tau_{\mathcal{R}}))$ embeds into $L_p(\tau_{\mathcal{R}})$ completely isometrically, we conclude that $(R_p, C_p)_{\theta,p}$ embeds into $L_p(\tau_{\mathcal{R}})$ completely isomorphically. ■

Remark. By the construction above, the space $K_t(R_p, C_p)$ is homogeneous and Hilbertian. In fact, there exists a completely isomorphic embedding $u : K_t(R_p, C_p) \to L_p(\mathcal{R}, \tau_{\mathcal{R}})$ such that, for every unitary v there exists an automorphism $\alpha_v : \mathcal{R} \to \mathcal{R}$ s.t. $\alpha_v(u(x)) = u(vx)$. Indeed, suppose $t \geq 1$. Find $s \in (0,1)$ s.t. $t = s^{-(2-p)/2p}$, and consider the space

$$X = \{1_{(0,s)} \otimes E_{21} \otimes x \,|\, x_{ij} = 0 \text{ if } i \neq 1\}$$

in the von Neumann algebra $N = L_\infty(0,\infty)\overline{\otimes} M_2 \overline{\otimes} B(\ell_2)$. In other words, an element $x \in X$ can be written as $1_{(0,s)} \otimes E_{21} \otimes y$, where $y = \sum_j y_j E_{1j}$, with $y_j \in \mathbb{C}$. By [11], there exists a completely isomorphic embedding $T : L_p(N) + L_2^{C_p}(N) + L_2^{R_p}(N) \to L_p(\tau_{\mathcal{R}})$ with the property that, for every unitary $w \in N$, there exists an automorphism β_w of $\mathcal{R}$ satisfying $T(w^* x w) = \beta_w(T(x))$ for any $x \in L_p(N) + L_2^{C_p}(N) + L_2^{R_p}(N)$. Fix a unitary $v = (v_{ij})_{i,j=1}^\infty \in B(\ell_2)$. We can consider v as a unitary operator on X. Indeed, if $y = (y_j)$ is an element of ℓ_2, then $(vy)_j = \sum_i v_{ij} y_i$. Consider

$$\tilde{v} = \mathbf{1}_{L_\infty} \otimes \begin{pmatrix} 1 & 0 \\ 0 & v^t \end{pmatrix} \in N,$$

where v^t is the transpose of v ($(v^t)_{ij} = v_{ji}$). Then

$$\tilde{v}^* x \tilde{v} = 1_{(0,s)} \otimes E_{12} \otimes \sum_j y_j E_{1j} v^t = 1_{(0,s)} \otimes E_{12} \otimes \sum_j (vy)_j E_{1j}.$$

Thus, we can view $\tilde{v}^* x \tilde{v}$ as “a copy of” vx. We complete the argument by setting $\alpha_v = \beta_{\tilde{v}}$, and denoting the restriction of T onto X by u.

Remark. We have shown that the interpolation space $(R_p, C_p)_{\theta,p}$ embeds completely isomorphically into $L_p(\tau_{\mathcal{R}})$ for $0 < \theta < 1$. Moreover, the space $K(t, R_p, C_p)$ embeds into $L_p(\tau_{\mathcal{R}})$ C-completely isomorphically, with C independent of t. However, neither R_p nor C_p embed into $L_p(\tau_{\mathcal{R}})$. Indeed, if R_p embeds into $L_p(\tau_{\mathcal{R}})$, then so does C_p. Then, $S_p = R_p[C_p]$ embeds into $L_p(\tau_{\mathcal{R}} \otimes \tau_{\mathcal{R}}) = L_p(\tau_{\mathcal{R}})$, which is impossible by [7, 17].

Several more results related to homogeneous Hilbertian subspaces of L_p ($1 \leq p \leq 2$) spaces should be quoted here. Theorem 3.8 below shows that the spaces R_q, C_q, and RC_q do not embed completely isometrically into $L_p(\tau)$ if $q \notin [p, p']$ (here, as usual, p' satisfies $1/p + 1/p' = 1$). Moreover, the spaces C_q and R_q do not embed into $L_p(\tau)$ when $1 \leq p < q \leq 2$ (see [32] and [33] for the case $p = 1$, [40] for $1 < p < 2$).

However, by [10], [13], and [40], for $1 \leq p < q \leq 2$ the spaces R_q, C_q, $R_q \cap C_q$, and $R_q + C_q$ are completely isomorphic to subspaces of quotients of $R_p \oplus C_p$. Consequently, these spaces embed completely isomorphically to an $L_p(N)$ space, where N is an injective von Neumann algebra. Therefore, R_q^n, C_q^n, $R_q^n \cap C_q^n$, and $R_q^n + C_q^n$ embed in S_p C-completely isomorphically (with $C = C(p,q)$ independent of n).

The following technical lemma is needed to prove part (2) of Theorem B.

Lemma 3.1. *For* $0 < \theta < 1$ *and* $n \ge 2$*,*

$$\|id : (R_p^n, C_p^n)_{\theta,\infty} \to (R_p^n, C_p^n)_{\theta,1}\|_{cb} \le c_p \log n,$$

where c_p *is independent on* n*.*

Proof. Suppose $k \in \mathbb{N}$ and $x \in M_k \otimes \ell_2^n$ is such that $\|x\|_{M_k((R_p^n,C_p^n)_{\theta,\infty})} = 1/2$, and show that $\|x\|_{M_k((R_p^n,C_p^n)_{\theta,\infty})} \le c\log_2 n$. By [30], it suffices to prove that, for every $a, b \in S_2^k$ with $\|a\| = \|b\| = 1$,

$$\|a \cdot x \cdot b\|_{S_1^k[(R_p^n,C_p^n)_{\theta,1}]} \overset{2}{\sim} \sum_{i=-\infty}^{\infty} 2^{-i/2} K(2^i, a \cdot x \cdot b, S_1^k[R_p^n], S_1^k[C_p^n]) \le c\log_2 n. \quad (3.5)$$

By (3.2),

$$\|x\|_{M_k((R_p^n,c_p^n)_{\theta,\infty})} \overset{2}{\sim} \sup_{i\in\mathbb{Z}} 2^{-i/2} K(2^i, x, M_k(R_p^n), M_k(C_p^n)).$$

Thus, by (3.1), for $i \in \mathbb{Z}$ there exist $x_{1i}, x_{2i} \in M_k \otimes \ell_2^n$ s.t. $x = x_{1i} + x_{2i}$ and $\|x_{1i}\|_{M_k(R_p^n)} + 2^i\|x_{2i}\|_{M_k(C_p^n)} \le 2^{i\theta}$. Let $m = \lceil \log_2 \sqrt{n}\,\rceil$. Since $\|id : R_p^n \to C_p^n\|_{cb} = \|id : C_p^n \to R_p^n\|_{cb} \le \sqrt{n}$, we have

$$K(t, x, S_1^k[R_p^n], S_1^k[C_p^n]) = \begin{cases} \|x\|_{S_1^k[R_p^n]} & t \ge 2^m \\ t\|x\|_{S_1^k[C_p^n]} & t \le 2^{-m} \end{cases}.$$

For $-m \le i \le m$,

$$K(2^i, a \cdot x \cdot b, S_1^k[R_p^n], S_1^k[C_p^n]) \le \|a \cdot x_{1i} \cdot b\|_{S_1^k[R_p^n]} + 2^i\|a \cdot x_{2i} \cdot b\|_{S_1^k[C_p^n]} \le 2^{i\theta}.$$

For $i \ge m$,

$$K(2^i, a \cdot x \cdot b, S_1^k[R_p^n], S_1^k[C_p^n]) = \|x\|_{S_1^k[R_p^n]} = K(2^m, a \cdot x \cdot b, S_1^k[R_p^n], S_1^k[C_p^n]) \le 2^{m\theta}.$$

Similarly, for $i \le -m$,

$$\begin{aligned} &K(2^i, a \cdot x \cdot b, S_1^k[R_p^n], S_1^k[C_p^n]) = 2^i\|x\|_{S_1^k[C_p^n]} = \\ &2^i K(2^{-m}, a \cdot x \cdot b, S_1^k[R_p^n], S_1^k[C_p^n]) \le 2^{i+m(1-\theta)}. \end{aligned}$$

Thus,

$$\begin{aligned} &\sum_{i=-\infty}^{\infty} 2^{-i/2} K(2^i, a \cdot x \cdot b, S_1^k[R_p^n], S_1^k[C_p^n]) \le \\ &\sum_{i=-m}^{m} 1 + \sum_{i=-\infty}^{-m-1} 2^{(m+i)(1-\theta)} + \sum_{i=m+1}^{\infty} 2^{(m-i)\theta} = 2m + 1 + \sum_{j=1}^{\infty} (2^{-\theta j} + 2^{-(1-\theta)j}). \end{aligned}$$

This proves (3.5). ■

Proof of Theorem B(2). We have already shown that $(R_p, C_p)_{\theta,p}$ $(0 < \theta < 1)$ embeds into $L_p(\tau_{\mathcal{R}})$ completely isomorphically. By construction, the spaces $(R_p, C_p)_{\theta,p}$ are Hilbertian and homogeneous. Clearly, two homogeneous Hilbertian operator spaces are completely isomorphic iff any isomorphism between them is a complete isomorphism. By the reiteration theorem (see [39]),

$$\big((R_p, C_p)_{\theta_0,p}, (R_p, C_p)_{\theta_1,p}\big)_{\gamma,p} \sim (R_p, C_p)_{\theta,p},$$

where $\theta = (1-\gamma)\theta_0 + \gamma\theta_1$. Denote by E_θ and E_θ^n the space $(R_p, C_p)_{\theta,p}$ and its n-dimensional subspace, respectively (all such subspaces are completely isometric). Suppose, for the sake of contradiction, that $0 < \theta_0 < \theta_1 < 1$, and $\|id : E_{\theta_0} \to E_{\theta_1}\|_{cb}, \|id : E_{\theta_1} \to E_{\theta_0}\|_{cb} \le C$. We shall show that $\|id : E_{\theta_0} \to E_\theta\|_{cb} < \infty$ for any $\theta \in (0,1)$.

By (3.3) and (3.4), we conclude that E_θ is completely isomorphic to E_{θ_0} for $\theta \in (\theta_0, \theta_1)$. Now suppose $\theta_2 \in (\theta_1, 1)$, and let $B_n = \|id : E_{\theta_0}^n \to E_{\theta_2}^n\|_{cb}$. Then $\|id : E_{\theta_1}^n \to E_{\theta_2}^n\|_{cb} \ge B_n/C$. On the other hand, find $\gamma \in (0,1)$ so that $\theta_1 = (1-\gamma)\theta_2 + \gamma\theta_0$. By (3.3), there exists a constant K (depending only on p, θ_0, θ_1, and θ_2), for which

$$B_n/C \le \|id : E_{\theta_1}^n \to E_{\theta_2}^n\|_{cb} \le K\|id : E_{\theta_0}^n \to E_{\theta_2}^n\|_{cb}^\gamma = KB_n^\gamma,$$

which is implies $\|id : E_{\theta_0} \to E_{\theta_2}\|_{cb} = \sup_n B_n \le (CK)^{1/(1-\gamma)} < \infty$.

Similar reasoning implies that, for every $\gamma_1, \gamma_2 \in (0,1)$, $d_{cb}(E_{\gamma_1}, E_{\gamma_2}) \le f(\gamma_1, \gamma_2)$. On the other hand, by Lemma 3.1 and [39],

$$d_{cb}\big((R_p^n, C_p^n)_{\theta,p}, (R_p^n, C_p^n)_\theta\big) \le c(\log n)^2,$$

with c depending on θ and p. Fix $\varepsilon \in (0, 1/2)$. Then

$$d_{cb}((R_p^n, C_p^n)_\varepsilon, R_p^n) = d_{cb}((R_p^n, C_p^n)_{1-\varepsilon}, C_p^n) \le n^\varepsilon,$$

and therefore

$$d_{cb}((R_p^n, C_p^n)_{\varepsilon,p}, R_p^n) = d_{cb}((R_p^n, C_p^n)_{1-\varepsilon,p}, C_p^n) \le cn^\varepsilon(\log n)^2.$$

Hence,

$$\begin{aligned} d_{cb}(R_p^n, C_p^n) &\le d_{cb}(E_\varepsilon^n, R_p^n) d_{cb}(E_\varepsilon^n, E_{1-\varepsilon}^n) d_{cb}(E_{1-\varepsilon}^n, C_p^n) \\ &\le c^2 f(\varepsilon, 1-\varepsilon) n^{2\varepsilon}(\log n)^4. \end{aligned} \tag{3.6}$$

On the other hand, $R_p^n = (R_1^n, OH_n)_\alpha$, with $\alpha = 2(p-1)/p$. Thus,

$$d_{cb}(R_p^n, R_1^n) \le \big(d_{cb}(R_1^n, OH_n)\big)^\alpha = (\sqrt{n})^\alpha = n^{(p-1)/p}$$

and therefore

$$d_{cb}(R_p^n, OH_n) \ge \frac{d_{cb}(R_p^n, OH_n)}{d_{cb}(R_p^n, R_1^n)} = n^{(2-p)/2p}.$$

This contradicts (3.6) for sufficiently small ε and large n. ■

To prove Theorem C, we use some lemmas, similar in spirit to those from Chapter 2.

Lemma 3.2. *Suppose N is a von Neumann algebra with a normal faithful semi-finite trace τ, $a_1, a_2, \dots, a_k \in S_p$ $(1 \le p \le 2)$, and $e_1, e_2, \dots, e_k \in L_p(\tau)$. Denote by (δ_i) an orthonormal basis in C_p. Then*

$$\sup_i \|e_i\|_p \big(\sum \|a_i\|^p\big)^{1/p} \ge |||(e_i \otimes a_i)|||_p \ge \inf_i \|e_i\|_p |||(a_i)|||_p.$$

Proof. First we establish the right-hand side inequality. Suppose $|||(a_i)||| \le 1$. Consider p' such that $1/p + 1/p' = 1$. By the duality between $S_p[L_p(\tau)]$ and $S_{p'}[L_{p'}(\tau)]$,

$$|||(e_i \otimes a_i)|||_p = \sup\{\langle\langle (x_i), (e_i \otimes a_i)\rangle\rangle \mid x_i \in S_{p'}[L_{p'}(\tau)],\ |||(x_i)|||_{p'} \le 1\},$$

where $\langle\langle (\alpha_i), (\beta_i)\rangle\rangle = \sum_i \langle \alpha_i, \beta_i\rangle$. Pick $\varepsilon > 0$, and find $b_1, \dots, b_k \in S_{p'}$ such that $|||(b_i)|||_{p'} = 1$ and $\langle\langle (b_i), (a_i)\rangle\rangle > 1 - \varepsilon$. Let $f_i = (e_i^* e_i)^{p/2-1} e_i^* / \|e_i\|^p$. Then $\|f_i\|_{p'} = 1/\|e_i\|$ and $\langle\langle (f_i \otimes b_i), (e_i \otimes a_i)\rangle\rangle > 1 - \varepsilon$. By Lemma 2.4,

$$|||(f_i \otimes b_i)||| \le \sup_i \|f_i\| \cdot |||(b_i)||| = 1/\inf_i \|e_i\|,$$

hence $|||(e_i \otimes a_i)||| > (1-\varepsilon) \inf_i \|e_i\|$. Since ε can be chosen to be arbitrarily small, the inequality follows.

Now consider the left-hand side. Denote by (σ_i) the canonical basis in ℓ_p. It suffices to prove that, for any $k \in \mathbb{N}$,

$$\max_{1 \le i \le k} \|e_i\|_{L_p(\tau)} \| \sum_i \sigma_i \otimes a_i \|_{\ell_p^k(S_k)} \ge \| \sum_i \delta_i \otimes a_i \otimes e_i \|_{C_p^k[S_p[L_p(\tau)]]},$$

where (δ_i) is an orthonormal basis in C_p^k. Consider a bilinear map

$$T : \ell_\infty^k(L_p(\tau)) \times \ell_p^k(S_p) \to C_p^k[S_p[L_p(\tau)]],$$

defined by

$$T\Big((e_1, \dots, e_k) \times \sum_i \sigma_i \otimes a_i\Big) = \sum_i \delta_i \otimes a_i \otimes e_i.$$

It is easy to see that T is contractive for $p = 1, 2$. Therefore, by interpolation, T is contractive for any $p \in [1, 2]$. This concludes the proof. ■

Lemma 3.3. *Suppose $k \in \mathbb{N}$, $1 \le p \le 2$, N and τ are as in Theorem C. Denote by (δ_i) an orthonormal basis in C_p^k. Then for any $e_1, \dots e_k \in L_p(\tau)$ and $a_1, \dots, a_k \in S_p$,*

$$\sup_i \|e_i\|_2 \cdot \| \sum \delta_i \otimes a_i \|_{C_p[S_p]} \ge \| \sum \delta_i \otimes a_i \otimes e_i \|_{C_p[S_p[L_p]]}.$$

Proof. Suppose $\sup_i \|e_i\|_2 = 1$. Consider the map $T : C_p[S_p] \to C_p[S_p[L_p(\tau)]]$, defined by

$$T(\sum \delta_i \otimes a_i) = \sum \delta_i \otimes a_i \otimes e_i.$$

We shall show that T is a contraction. By interpolation, it suffices to prove this for $p = 1$ and $p = 2$. In the latter case, the inequality is obvious. If $p = 1$, note that $C_1[S_1] = C_\infty[S_\infty]^*$, hence it suffices to show that $\|T(\sum \delta_i \otimes a_i)\| \le 1$

if $\sum \delta_i \otimes a_i \in \operatorname{ext} B_{C_1^k[S_1]}$ (the set of extreme points of the closed unit ball of $C_1[S_1]$). Now use $E_{i1} \otimes E_{mn} \mapsto E_{i+k(m-1),n}$ (as in the proof of Lemma 2.4) to identify $C_1^k[S_1]$ with S_1, and recall that $\operatorname{ext} B_{S_1}$ consists of norm 1, rank 1 operators. Thus, $\sum \delta_i \otimes a_i \in \operatorname{ext} B_{C_1^k[S_1]}$ iff there exist vectors $f, g_1, \dots, g_k \in \ell_2$ s.t. $\|f\| = \sum_i \|g_i\|^2 = 1$, and, for any $i \in [1,k]$ and $\xi \in \ell_2$, $a_i\xi = \langle \xi, f\rangle g_i$. In this case, $a_i^* a_i = \|g_i\|^2 a$, where $a\xi = \langle \xi, f\rangle f$ for any $\xi \in \ell_2$. In this case,

$$\begin{aligned}
&\|\sum \delta_i \otimes a_i \otimes e_i\| = \|(\sum a_i^* a_i \otimes e_i^* e_i)^{1/2}\|_{S_1[L_1(\tau)]} \\
&= \|(\sum \|g_i\|^2 a \otimes e_i^* e_i)^{1/2}\|_{S_1[L_1(\tau)]} = \|(\sum \|g_i\|^2 e_i^* e_i)^{1/2}\|_{L_1(\tau)} \\
&\le \Big(\tau(\sum \|g_i\|^2 e_i e_i^*)\Big)^{1/2} \le (\sum \|g_i\|^2)^{1/2} = 1,
\end{aligned}$$

since $\tau(e_i e_i^*) \le 1$. This completes the proof. ■

Lemma 3.4. *Suppose p, N, τ, (a_i), and (e_i) are as in Lemma 3.3. If $\sup_i \|e_i\|_2 \le \gamma$, then $|||(e_i \otimes a_i)|||_p \le \gamma |||(a_i)|||_p$ for every $a_1, a_2, \ldots \in S_p$.*

Proof. By (1.2),

$$\begin{aligned}
&|||(e_i \otimes a_i)|||_p \\
&= \inf\{\|(\sum x_i^* x_i)^{1/2}\|_{S_p[L_p(\tau)]} + \|(\sum y_i y_i^*)^{1/2}\|_{S_p[L_p(\tau)]} \mid a_i \otimes e_i = x_i + y_i\} \\
&\le \inf\{\|(\sum b_i^* b_i \otimes e_i^* e_i)^{1/2}\|_{S_p[L_p(\tau)]} + \|(\sum c_i c_i^* \otimes e_i e_i^*)^{1/2}\|_{S_p[L_p(\tau)]} \mid a_i = b_i + c_i\} \\
&\le \gamma \inf\{\|(\sum b_i^* b_i)^{1/2}\|_{S_p} + \|(\sum c_i c_i^*)^{1/2}\|_{S_p} \mid a_i = b_i + c_i\} = \gamma |||(a_i)|||_p.
\end{aligned}$$

(we use Lemma 3.3 to establish the last inequality). ■

Theorem C can now be easily deduced from the following (cf. Corollary 2.3).

Theorem 3.5. *Suppose N is a von Neumann algebra with a faithful normal finite trace τ, $(e_i)_{i=1}^\infty$ is an M-completely unconditional basis in $E \hookrightarrow L_p(\tau)$ $(1 \le p \le 2)$, and $1 \le \|e_i\|_p \le \|e_i\|_2 \le \gamma$ for every i. Then*

$$\frac{A}{M} |||(a_i)|||_p \le \|\sum_i e_i \otimes a_i\|_{S_p[L_p(\tau)]} \le M\gamma |||(a_i)|||_p,$$

where A is the constant from (1.3).

Proof. Note first that

$$\begin{aligned}
\frac{1}{M} \operatorname{Ave}_\pm \|\sum \pm e_i \otimes a_i\|_{S_p[L_p(\tau)]} &\le \|\sum e_i \otimes a_i\|_{S_p[L_p(\tau)]} \\
&\le M \operatorname{Ave}_\pm \|\sum \pm e_i \otimes a_i\|_{S_p[L_p(\tau)]},
\end{aligned}$$

since the basis (e_i) is M-completely unconditional. By (1.3) and Lemma 3.2,

$$\frac{1}{M} \operatorname{Ave}_\pm \|\sum \pm e_i \otimes a_i\|_{S_p[L_p(\tau)]} \ge \frac{A}{M} |||(e_i \otimes a_i)|||_p \ge \frac{A}{M} \inf_i \|e_i\| \cdot |||(a_i)|||_p.$$

By Lemma 3.4,

$$M\mathrm{Ave}_{\pm}\|\sum \pm e_i \otimes a_i\|_{S_p[L_p(\tau)]} \leq M|||(e_i \otimes a_i)||| \leq \gamma M|||(a_i)|||_p. \quad \blacksquare$$

Remark. As in Chapter 2, we can obtain weaker versions of some of our estimates using the fact that $L_p(\tau)$ has type p. For $1 < p < 2$, we combine [20] and [34] to get

$$\mathrm{Ave}_{\pm}\|\sum \pm e_i \otimes a_i\| \leq \frac{c}{p-1}\big(\sum \|e_i \otimes a_i\|^p\big)^{1/p} \leq \frac{c}{p-1}\gamma\big(\sum \|a_i\|^p\big)^{1/p},$$

where c is an absolute constant.

Now we give some applications of Theorem C. Suppose N is a von Neumann algebra with a normal faithful normalized trace τ, and (a_i) is a sequence of elements of N satisfying the *canonical anti-commutation relations* (*CAR*): $a_i a_j + a_j a_i = 0$, $a_i^* a_j + a_j a_i^* = \delta_{ij} I$, where I is the identity in N. Examples of such sequences in the hyperfinite II_1 factor can be found, in particular, in Section 5.2 of [2], Example III.5.4 of [3], or Section 9.3 of [31]. It is easy to see that $xy + yx = 0$ and $xy^* + y^*x = \tau(xy^*)I$ for any $x, y \in X = \mathrm{span}[a_i \,|\, i \in \mathbb{N}]$. Example III.5.4 of [3] shows that $\|x\|_2 = \|x\|/\sqrt{2}$ for any $x \in X$ (here $\|\cdot\|$ is the norm of N and $\|\cdot\|_p$ is the norm of $L_p(\tau)$). Therefore, the norms $\|\cdot\|_p$ and $\|\cdot\|_2$ are equivalent on X for $1 \leq p \leq \infty$. Set $X_p = \mathrm{span}[a_i \,|\, i \in \mathbb{N}] \hookrightarrow L_p(\tau)$.

The C^*-algebra generated by X is called *the CAR algebra*. It turns out (see [3]) that this algebra is unique, and is, in fact, a 2^∞ UHF algebra. The CAR algebra is also called *the fermion algebra*: $x^2 = 0$ for any $x \in X$, which corresponds to the Pauli principle of the impossibility of two fermions sharing one quantum state, see [2] for more on this.

Proposition 3.6. *For $1 \leq p < \infty$, X_p is completely isomorphic to RC_p.*

Remark. It is known (see e.g. Section 9.3 of [31]) that $X = X_\infty$ is not completely isomorphic to $R \cap C = RC_\infty$.

Proof. We have shown already that the norms $\|\cdot\|_p$ and $\|\cdot\|_2$ are equivalent on X. It therefore suffices to prove that, if $u : X_p \to X_p$ is a unitary (with respect to the norm $\|\cdot\|_2$), then it is a complete isometry, i.e. that

$$\|\sum a_i \otimes b_i\|_{S_p[L_p(\tau)]} = \|\sum ua_i \otimes b_i\|_{S_p[L_p(\tau)]}. \tag{3.7}$$

for every $b_1, \dots, b_n \in S_p$. In this setting, the sequence (ua_i) also satisfies the CAR, and therefore, by Theorem 5.2.5 of [2], u extends to a $*$-isomorphism α of the CAR algebra A, generated by X. Moreover, by the proof of this theorem (or by the construction given in [3]), α preserves the trace τ. Thus, $Id_{B(\ell_2)} \otimes \alpha$ is a $*$-isomorphism on $B(\ell_2) \otimes A$, preserving the trace $tr \otimes \tau$. Approximating the function $t \mapsto |t|^p$ by polynomials, we see that

$$tr \otimes \tau(|\sum a_i \otimes b_i|^p) = tr \otimes \tau(|\sum ua_i \otimes b_i|^p),$$

which implies (3.7). $\blacksquare$

As a consequence of this proposition, we can estimate the projection constant of $X_\infty^n = \mathrm{span}[a_1, \dots, a_n] \hookrightarrow N$.

Corollary 3.7. *There exists a constant $c > 0$ such that any projection P from $B(H)$ onto its subspace X_∞^n satisfies $\|P\|_{cb} \geq c\sqrt{n}$.*

Remark. (1) It is not hard to see that there exists a bounded projection from $B(H)$ onto X_∞ (this follows from the equivalence of the $\|\cdot\|_2$ and $\|\cdot\|_\infty$ norms).

(2) By [29], for every n-dimensional subspace $E \hookrightarrow B(H)$ there exists a projection P from $B(H)$ onto E s.t. $\|P\|_{cb} \leq \sqrt{n}$.

Proof. It is known that X_∞^n embeds completely isometrically into M_N (with $N = 2^n$). Suppose P is a projection from M_N onto X_∞^n. Denote by I the identity map on X_∞^n, and by u the formal identity from X_∞^n to X_1^n (the span of $a_1, \ldots, a_n$ in S_1^N, equipped with the normalized trace). Clearly, u is a contraction. The reasoning preceding Proposition 3.6, together with Hölder's inequality, show that $\|u^{-1}\| \leq 2$. By the definition of the completely 1-summing norm (see Chapter 13 of [6]), $\pi_1(u) = 1$. By Proposition 3.6, X_1^n is c_1-completely isomorphic to $R_n + C_n$, where c_1 is an absolute constant. Therefore, by Chapter 10 of [31], $\|u^{-1}\|_{cb} \leq c_2\sqrt{n}$, with a constant c_2. This implies

$$\pi_1(I) = \pi_1(u^{-1} \circ u) \leq \|u^{-1}\|_{cb}\pi_1(u) \leq c_2\sqrt{n}.$$

Hence, by Proposition 13.2.1 of [6], $\nu(P) \leq \|P\|_{cb}\pi_1(I) \leq c_2\sqrt{n}\|P\|_{cb}$ (here $\nu(\cdot)$ is the complete nuclear norm of an operator, described in detail in Chapter 12 of [6]). On the other hand, by duality, $n = \operatorname{rank} P \geq \nu(P)$. ■

Finally, we show that the spaces R_q, C_q, $R_q \cap C_q$, and $R_q + C_q$ do not embed into $L_p(\tau)$ when $1 \leq q < p \leq 2$. More precisely, we have:

Theorem 3.8. *Suppose $1 \leq q < p \leq 2$, N is a von Neumann algebra with the faithful normal semi-finite trace τ, E is an n-dimensional subspace of $L_p(\tau)$, and G is one of the following four spaces: RC_q^n, R_q^n, C_q^n, and $RC_{q'}^n$ (here, as usual, $1/ + 1/q' = 1$). Then $d_{cb}(E, G) \geq n^{1/2q - 1/2p}$.*

In fact, the proof of theorem also works for non-semifinite von Neumann algebras, and for the L_p spaces associated with them (such spaces are induced not by traces but by faithful states; a brief introduction can be found, for instance, in [35]).

We start by proving two lemmas concerning tensor products. Here we identify $(x_1, \ldots, x_n) \in \ell_2^n(X)$ (X being a Banach space) with $\sum_i \delta_i \otimes x_i \in \ell_2^n \otimes X$, where $(\delta_i)_{i=1}^n$ is the canonical basis of ℓ_2^n. As before, $\overset{\smile}{\otimes}$ and $\otimes_h$ stand for the injective (minimal) and Haagerup tensor products of operator spaces, respectively.

Lemma 3.9. *Suppose $1 \leq q < p \leq 2$. Then*

1. $\|Id : (R_n \cap C_n)\overset{\smile}{\otimes}RC_q^n \to \ell_2^n(\ell_2^n)\| \geq n^{1/2q}$.
2. $\|Id : (R_n \cap C_n)\overset{\smile}{\otimes}L_p(\tau) \to \ell_2^n(L_p(\tau))\| \leq n^{1/2p}$.

Proof. (1) When $q = 1$, we can identify (on the Banach space level) $\ell_2^n(\ell_2^n)$ with S_2^n, and $(R_n \cap C_n)\overset{\smile}{\otimes}RC_1^n$ with $CB(R_n \cap C_n) = M_n$. However, by interpolation and by [29],

$$\|Id : RC_1^n \to RC_q^n\|_{cb} \leq \|Id : RC_1^n \to OH_n\|_{cb}^{1/q'} = n^{1/2q'}.$$

Therefore,

$$\|Id : (R_n \cap C_n)\breve{\otimes} RC_q^n \to \ell_2^n(\ell_2^n)\| \geq \frac{\|Id : (R_n \cap C_n)\breve{\otimes} RC_1^n \to \ell_2^n(\ell_2^n)\|}{\|Id : (R_n \cap C_n)\breve{\otimes} RC_1^n \to (R_n \cap C_n)\breve{\otimes} RC_q^n\|}$$
$$= n^{1/2q}.$$

(2) It is easy to see that $\|Id : R_n \breve{\otimes} L_1(\tau) \to \ell_2^n(L_1(\tau))\| \leq n^{1/2}$. Indeed, for any element $\sum_i e_i \otimes x_i \in R_n \breve{\otimes} L_1(\tau)$ (where (e_i) is an orthonormal basis in R_n) we have $\|\sum_i e_i \otimes x_i\| \geq \max_i \|x_i\|$. Moreover, by [29], $R_n \breve{\otimes} L_2(\tau)$ can be identified with $S_4(\ell_2^n, L_2(\tau))$, while $\ell_2^n(L_2(\tau))$ is canonically isometric to $S_2(\ell_2^n, L_2(\tau))$. Therefore, $\|Id : R_n \breve{\otimes} L_2(\tau) \to \ell_2^n(L_1(\tau))\| \leq n^{1/4}$.

By definition of complex interpolation of operator spaces (see Chapter 2.7 of [31]), $(R_n \breve{\otimes} L_1(\tau), R_n \breve{\otimes} L_2(\tau))_{2/p'} = R_n \breve{\otimes} L_p(\tau)$. Moreover, by Theorem 5.6.3 of [1], $(\ell_2^n(L_1(\tau)), \ell_2^n(L_2(\tau)))_{2/p'} = \ell_2^n(L_p(\tau))$. Thus, $\|Id : R_n \breve{\otimes} L_p(\tau) \to \ell_2^n(L_p(\tau))\| \leq n^{1/2p}$. Similarly we establish that $\|Id : C_n \breve{\otimes} L_p(\tau) \to \ell_2^n(L_p(\tau))\| \leq n^{1/2p}$. The definition of $R_n \cap C_n$ the implies the second half of the lemma. ■

Lemma 3.10. *Suppose* $1 \leq p \leq 2 \leq q \leq \infty$, p' *satisfies the equality* $1/p + 1/p' = 1$, *and* G *is either* R_q^n, C_q^n, *or* RC_q^n. *Then*

(1) $\|Id : \ell_2^n(L_p(\tau)) \to RC_{p'}^n \otimes_h L_p(\tau)\| \leq n^{1/2 - 1/p'}$.

(2) $\|Id : \ell_2^n(\ell_2^n) \to RC_{p'}^n \otimes_h G\| = n^{1/2p - 1/2q}$.

Proof. To prove (1), note first that

$$\|\sum_{i=1}^n \delta_i \otimes x_i\|_{(R_n \cap C_n) \otimes_h L_1(\tau)} \leq \sum_{i=1}^n \|x_i\| \leq \sqrt{n}\Big(\sum_{i=1}^n \|x_i\|^2\Big)^{1/2},$$

hence $\|Id : \ell_2^n(L_1(\tau)) \to (R_n \cap C_n) \otimes_h L_1(\tau)\| \leq \sqrt{n}$. Moreover, $\ell_2^n(L_2(\tau))$ is isometric to $OH_n \otimes_h L_2(\tau)$ by [29]. Since the Haagerup tensor product is stable under complex interpolation (see [29] or Theorem 5.22 of [31]), we are done.

We are going to establish (2) for $G = C_q^n$. The other cases are handled in a similar fashion. Recall that, by Proposition 5.16 of [31], for $x \in E \otimes_h F$ (E and F are operator spaces) we have

$$\|x\|_{E \otimes_h F} = \inf\{\|u\|_{cb}\|v\|_{cb} \,|\, u \in CB(E^*, R),\ v \in CB(R, F),\ \tilde{x} = v \circ u\} \qquad (3.8)$$

(here $\tilde{x} \in CB(E^*, F)$ is the operator corresponding to x). Observe that

$$\|u\|_{CB(RC_p^n, R_n)} = \|u\|_{2p} \qquad (3.9)$$

for any $u \in B(\ell_2^n, \ell_2^n)$. Indeed, the equality holds for $p = 1, 2$ (see [29] or Chapter 10 of [31]), hence, by interpolation, $\|u\|_{CB(RC_p^n, R_n)} \leq \|u\|_{2p}$. Similar interpolation techniques show that $\|w\|_{CB(R_n + C_n, RC_p^n)} \leq \|w\|_{2p'}$. By Hölder's Inequality,

$$\begin{aligned}\|u \circ w\|_2 &= \|u \circ w\|_{CB(R_n + C_n, R_n)} \\ &\leq \|u\|_{CB(RC_p^n, R_n)}\|w\|_{CB(R_n + C_n, RC_p^n)} \leq \|u\|_{2p}\|w\|_{2p'}\end{aligned}$$

for any u and w. However, for every $u \in CB(\ell_2^n, \ell_2^n)$ there exists $w \in CB(\ell_2^n, \ell_2^n)$ s.t. $\|w\|_{2p'} = 1$ and $\|u \circ w\|_2 = \|u\|_{2p}$. This implies (3.9).

Similarly, we show that any $u \in CB(R_n, C_q^n)$ satisfies

$$\|u\|_{CB(R_n, C_q^n)} = \|u\|_{2q'} \tag{3.10}$$

(here, q' satisfies $1/q + 1/q' = 1$).

By (3.8), (3.9) and (3.10),

$$\|x\|_{RC_{p'}^n \otimes_h RC_q^n} = \inf\{\|u\|_{2p}\|v\|_{2q'} \,|\, \tilde{x} = v \circ u\} = \|\tilde{x}\|_{1/2p+1/2q'}$$

(here $1/q + 1/q' = 1$). Moreover, $\ell_2^n(\ell_2^n)$ can be naturally identified with S_2^n, and $\|Id : S_2^n \to S_{1/2p+1/2q'}^n\| = n^{1/2p-1/2q}$. ■

Proof of Theorem 3.8. First suppose $G = RC_q^n$ (which we identify with ℓ_2^n), and consider an invertible operator $u : RC_q^n \to E$. By Lemma 3.9,

$$\begin{aligned} n^{1/2q} &\leq \|Id : (R_n \cap C_n) \breve{\otimes} RC_q^n \to \ell_2^n(\ell_2^n)\| \\ &\leq \|I_{R_n \cap C_n} \otimes u : (R_n \cap C_n) \breve{\otimes} RC_q^n \to (R_n \cap C_n) \breve{\otimes} E\| \\ &\cdot \|Id : (R_n \cap C_n) \breve{\otimes} E \to \ell_2^n(E)\| \cdot \|I_{\ell_2^n} \otimes u^{-1} : \ell_2^n(E) \to \ell_2^n(\ell_2^n)\| \\ &\leq n^{1/2p}\|u\|_{cb}\|u^{-1}\|. \end{aligned}$$

Thus, $\|u\|_{cb}\|u^{-1}\| \geq n^{1/2q-1/2p}$.

Now suppose $G = C_q^n$, R_q^n, or $RC_{q'}^n$. For $u \in CB(G, E)$, Lemma 3.10 implies:

$$\begin{aligned} n^{1/2p-1/2q} &= \|Id : \ell_2^n(\ell_2^n) \to RC_{p'}^n \otimes_h G\| \\ &\leq \|I_{\ell_2} \otimes u : \ell_2^n(\ell_2^n) \to \ell_2^n(E)\| \cdot \|Id : \ell_2^n(E) \to RC_{p'}^n \otimes_h E\| \\ &\cdot \|I_{RC_{p'}^n} \otimes u^{-1} : RC_{p'}^n \otimes_h E \to RC_{p'}^n \otimes_h G\| \\ &\leq n^{1/2-1/p'}\|u^{-1}\|_{cb}\|u\|. \end{aligned}$$

Thus,

$$\|u^{-1}\|_{cb}\|u\| \geq \frac{n^{1/2p-1/2q}}{n^{1/2-1/p'}} = n^{1/2p+1/p'-1/2-1/2q'} = n^{1/2q-1/2p}. \quad ■$$

4. COMMUTATIVE L_p SPACES

In this section, we assume that μ is a σ-finite measure, and $1 \leq p < \infty$. Then $L_\infty(\mu)$ is a commutative von Neumann algebra, which we equip with its "natural" (minimal) operator space structure, and $L_p(\mu) = (\mathrm{MIN}(L_\infty(\mu)), \mathrm{MAX}(L_1(\mu)))_{1/p}$. Our main result is:

Theorem 4.1. *Any homogeneous Hilbertian subspace of $L_p(\mu)$ is completely isomorphic to RC_p. More precisely:*

1. *Suppose $e_1, e_2, \ldots$ is an M-completely unconditional basis of $E \hookrightarrow L_p$ ($1 \leq p \leq 2$) such that $\|e_i\| \geq 1$, and $\|\sum \alpha_i e_i\| \leq L(\sum |\alpha_i|^2)^{1/2}$ for all scalars*

$\alpha_1, \alpha_2, \dots$. Then (e_i) is C^2ML-completely equivalent to an orthonormal basis of RC_p, with

$$\frac{1}{CM}|||(a_i)|||_p \le \|\sum_i e_i \otimes a_i\|_{L_p(S_p)} \le CL|||(a_i)|||_p$$

for any $a_1, a_2, \dots \in S_p$. Here C is an absolute constant.

(2) *Suppose $e_1, e_2, \dots$ is a M-completely unconditional basis of $E \hookrightarrow L_p$ ($2 \le p < \infty$) such that $\|e_i\| \le 1$ and $L\|\sum \alpha_i e_i\| \ge (\sum |\alpha_i|^2)^{1/2}$ for all scalars $\alpha_1, \alpha_2, \dots$. Then there exists an absolute constant C such that, for any $a_1, a_2, \dots \in S_p$,*

$$\frac{1}{CL\sqrt{p}}|||(a_i)|||_p \le \|\sum_i e_i \otimes a_i\|_{L_p(S_p)} \le CM\sqrt{p}|||(a_i)|||_p.$$

(3) *Suppose E is a λ-Hilbertian H-homogeneous subspace of L_p ($1 \le p < \infty$). Then E is $\alpha pH\lambda^2$-completely isomorphic to RC_p, where α is a constant.*

For the proof we need several lemmas.

Lemma 4.2. *Suppose $1 \le p \le 2$, and $u : \ell_2 \to L_p$ is a linear map. Then $\|u : RC_p \to L_p\|_{cb} \le C_1\|u\|$, where C_1 is a constant.*

The following lemma was established in [9]. We provide a proof for the benefit of the reader.

Lemma 4.3. *If $u : \mathrm{MIN}(L_\infty) \to R \cap C$ is a linear map, then u is completely bounded, and $\|u\|_{cb} \le \big(2/\sqrt{\pi}\big)\|u\|$.*

Proof. An operator $v : X \to Y$ (here X is an operator space and Y is a Banach space) is called $(2, c)$-*summing* if there exists a constant C such that for every n-tuple $x_1, \dots, x_n \in X$ the inequality $\sum_i \|vx_i\|^2 \le C^2\|\sum x_i^* x_i\|$ holds. The infimum of all C's as above is called the $(2, c)$-*summing norm* of v and is denoted by $\pi_{2,c}(v)$. The class of $(2, r)$-summing maps and the $(2, r)$-summing norms are defined in a similar fashion, with $\|\sum x_i x_i^*\|$ instead of $\|\sum x_i^* x_i\|$. If $X = \mathrm{MIN}(E)$, then

$$\|\sum_i x_i^* x_i\| = \|\sum_i x_i x_i^*\| = \sup_{f \in E^*, \|f\| \le 1} \sum_i |f(x_i)|^2,$$

hence $\pi_2(v) = \pi_{2,r}(v) = \pi_{2,c}(v)$ for every $v : \mathrm{MIN}(E) \to Y$.

Note that

$$\begin{aligned}\|u &: \mathrm{MIN}(L_\infty) \to R \cap C\|_{cb} \\ &= \max\{\|u : \mathrm{MIN}(L_\infty) \to R\|_{cb}, \|u : \mathrm{MIN}(L_\infty) \to C\|_{cb}\}.\end{aligned}$$

By [29], $\|u : \mathrm{MIN}(L_\infty) \to R\|_{cb} = \pi_{2,r}(u)$ and $\|u : \mathrm{MIN}(L_\infty) \to C\|_{cb} = \pi_{2,c}(u)$. Therefore, $\|u : \mathrm{MIN}(L_\infty) \to R \cap C\|_{cb} = \pi_2(u)$. However, by Theorem 5.4 of [25] or Corollary 10.10 of [38], $\pi_2(u) \le (2/\sqrt{\pi})\|u\|$. ■

Proof of Lemma 4.2. By (1.1), it suffices to show that

$$\|I_{S_p} \otimes u : S_p[RC_p] \to S_p[L_p]\| \le C_1\|u\|.$$

Since $S_p[RC_p]$ coincides (as a Banach space) with $S_p[R_p] + S_p[C_p]$, it suffices to show that for any $x_1, \dots, x_k \in S_p$

$$\|\sum_{i=1}^{k} u\delta_i \otimes x_i\|_{L_p(S_p)} \leq C_1 \|u\| \cdot \|\sum \delta_i \otimes x_i\|_{C_p[S_p]} \tag{4.1}$$

and

$$\|\sum_{i=1}^{k} u\delta_i \otimes x_i\|_{L_p(S_p)} \leq C_1 \|u\| \cdot \|\sum \delta_i \otimes x_i\|_{R_p[S_p]}, \tag{4.2}$$

where $(\delta_i)_{i=1}^k$ is an orthonormal basis in ℓ_2^k. Consider a bilinear operator $T : B(\ell_2^k, L_p(\mu)) \times S_p[C_p] \to L_p(\mu, S_p)$, defined by

$$T\big(u \times \sum_i \delta_i \otimes x_i\big) = \sum_{i=1}^{k} u\delta_i \otimes x_i.$$

Clearly,

$$\|T : B(\ell_2^k, L_2) \times S_2[C_2] \to L_2(S_2)\| \leq 1,$$

and by Lemma 4.3 (and (1.1)),

$$\|T : B(\ell_2^k, L_1) \times S_1[C_1] \to L_1(S_1)\| \leq \frac{2}{\sqrt{\pi}}.$$

By [4] or [26] (cf. [23]), for $1/p = (1-\gamma)/2 + \gamma/1$

$$B(\ell_2^n, L_p) = \big(B(\ell_2^n, L_2), B(\ell_2^n, L_1)\big)_\gamma$$

with equivalent norms, and the equivalence constant is independent of n and p. Therefore,

$$\|T : B(\ell_2^k, L_p) \times S_p[C_p] \to L_p(S_p)\| \leq C_1$$

for some absolute constant C_1. This establishes (4.1). (4.2) is proved in the same manner. ■

Proof of Theorem 4.1. (1) $1 \leq p \leq 2$. Consider an operator $u : RC_p \to E$, defined by $u\delta_i = e_i$ (here (δ_i) is an orthonormal basis in RC_p). By Lemma 4.2, $\|u\|_{cb} \leq C_1\|u\| \leq C_1 L$. By Lemma 3.2 and a simple averaging argument, $\|u^{-1}\|_{cb} \leq C_2 M$.

(2) $2 \leq p < \infty$. Consider an operator $u : E \to RC_p$, defined by $ue_i = \delta_i$, where (δ_i) is an orthonormal basis in ℓ_2. Then $\|u\| \leq L$ and, by Theorem 76 of [23], u has an extension $\tilde{u} : L_p(\mu) \to RC_p$ such that $\|\tilde{u}\| \leq C_2\sqrt{p}L$, where C_2 is an absolute constant. By Lemma 4.2, $\|u\|_{cb} \leq \|\tilde{u}\|_{cb} \leq C_1\|\tilde{u}\| \leq C_1 C_2 \sqrt{p} L$. On the other hand, by an averaging argument and Theorem 2.1, $\|u^{-1}\|_{cb} \leq C\sqrt{p}M$.

(3) Consider the case $1 \leq p \leq 2$. There exists an operator $u : RC_p \to E$ such that $\|u\| \leq \lambda$ and $\|u^{-1}\| = 1$. Pick an orthonormal basis (δ_i) in RC_p, and let $e_i = u\delta_i$. The basis (e_i) is $H\lambda$-completely unconditional. Indeed, pick a set $S \subset \mathbb{N}$. Denote the basis projections from E onto $\mathrm{span}[e_i | i \in S]$ and from ℓ_2 onto $\mathrm{span}[\delta_i | i \in S]$ by P_S and Q_S, respectively. Then

$$\begin{aligned}\|I - 2P_S\|_{cb} \leq& H\|I - 2P_S\| = H\|u(I - 2Q_S)u^{-1}\| \\ \leq& H\|u\| \cdot \|I - 2Q_S\| \cdot \|u^{-1}\| \leq H\lambda.\end{aligned}$$

We complete the proof by applying part (1). The case of $p \geq 2$ follows from (2) in a similar fashion. ■

Remark. A statement similar to Theorem 4.1 can be obtained from Theorems A and C by a change of density argument (relying on Theorem 8 and Proposition 43 of [23] if $1 \leq p \leq 2$, Theorem 76 if $2 \leq p < \infty$). However, this approach yields worse estimates on complete equivalence constants than those of our theorem.

In [24] we proved the case of $p = 1$ of Theorem 4.1, but the technique employed was different from the one described below.

Suppose X is a Hilbertian subspace of $L_p(\mu)$, where μ is a finite measure, and $p \in (2, \infty)$. By [18], the norms $\|\cdot\|_2$ and $\|\cdot\|_p$ are equivalent on X. Then the norms $\|\cdot\|_2$ and $\|\cdot\|_q$ are equivalent on X for $1 \leq q \leq p$. Below we prove an operator space analogue of this statement.

Proposition 4.4. *Suppose μ is a finite measure, $2 \leq p < \infty$, and X is a subspace of $L_p(\mu)$. Let $\lambda = d_{cb}(X, RC_p)$. If $1 \leq q \leq p$, denote by X_q the image of X in $L_q(\mu)$. Then:*

1. *For $2 \leq q \leq p$, $d_{cb}(X_q, RC_q) \leq (C_3\sqrt{p}\lambda^2)^\theta$, where C_3 is a constant and $1/q = (1-\theta)/2 + \theta/p$.*
2. *For $p' = p/(p-1)$, $d_{cb}(X_{p'}, RC_{p'}) \leq C_3\sqrt{p}\lambda^2$.*
3. *For $p' \leq q \leq 2$, $d_{cb}(X_q, RC_q) \leq (C_4 p\lambda^3)^\theta$, where $1/q = (1-\theta)/2 + \theta/p'$, and C_4 is a constant.*

The following lemma may be of independent interest.

Lemma 4.5. *Consider an interpolation pair (H_0, H_1) of 1-homogeneous 1-Hilbertian operator spaces, and assume they share the underlying Hilbert space H. Let $H_\theta = (H_0, H_1)_\theta$ be the complex interpolation space. Consider an invertible positive operator $A : H \to H$ and define an operator space H_1' by setting, for $x \in M_n(H)$,*

$$\|x\|_{M_n(H_1')} = \|(I_{M_n} \otimes A)(x)\|_{M_n(H_1)}.$$

Then for $0 \leq \theta \leq 1$ $H_\theta' = (H_0, H_1')_\theta$ is completely isometric to H_θ and, moreover,

$$\|x\|_{M_n(H_\theta')} = \|(I_{M_n} \otimes A^\theta)(x)\|_{M_n(H_\theta)}.$$

Proof. Set $S = \{z \in \mathbb{C} \mid 0 \leq \Re z \leq 1\}$. Fix $k \in \mathbb{N}$ and let $\mathcal{F}$ be the space of continuous functions $f : S \to M_k(H)$ which are analytic in the interior of S, equipped with the norm

$$\|f\|_{\mathcal{F}} = \max\{\sup_t \|f(it)\|_{M_k(H_0)}, \sup_t \|f(1+it)\|_{M_k(H_1)}\}.$$

$\mathcal{F}'$ is defined in a similar manner, with H_1' instead of H_1. Then, by definition of complex interpolation method,

$$\|x\|_{M_k(H_\theta)} = \inf\{\|f\|_{\mathcal{F}} \mid f(\theta) = x\} \quad \text{and} \quad \|x\|_{M_k(H_\theta')} = \inf\{\|f\|_{\mathcal{F}'} \mid f(\theta) = x\}$$

for every $x \in M_n(H)$. Consider an isometry $U : \mathcal{F}' \to \mathcal{F}$, defined by

$$\big(U(f)\big)(z) = \big(I_{M_n} \otimes e^{z \ln A}\big)(f(z)).$$

Clearly, U maps the unit ball of $M_n(H'_\theta)$ onto the unit ball of $M_n(H_\theta)$. ■

Proof of Proposition 4.4. Without loss of generality, assume that μ is a probability measure. Consider $T : X_p \to RC_p$ such that $\|T\|_{cb} = 1$ and $\|T^{-1}\|_{cb} \leq \lambda$. By Theorem 76 of [23], T extends to $S \circ \mathcal{M}_g$, where $\|S : L_2(\mu) \to RC_p\| \leq C_2\sqrt{p}$ (C_2 is a constant), and $\mathcal{M}_g : L_p(\mu) \to L_2(\mu)$ is an operator of multiplication by g. Here $g \geq 0$ is such a function that $\int g^r d\mu = 1$, with $r = 2p/(p-2)$ (i.e. $1/2 = 1/p + 1/r$). Note that $\|(\mathcal{M}_g|_{X_p})^{-1}\| \leq \|T^{-1}\|\|S\| \leq C_2\sqrt{p}\lambda$, hence $\mathcal{M}_g(X_2)$ is a closed subspace of $L_2(\mu)$.

Let P be the orthogonal projection on X_2. Note that $\|f\|_p \geq \|f\|_2$ for any $f \in L_p(\mu)$, and moreover, $\|f\|_p \leq \|(\mathcal{M}_g|_{X_p})^{-1}\|\|f\|_2 \leq C_2\sqrt{p}\lambda$ for any $f \in X_p$. Thus, by Lemma 4.2,

$$\|P : L_p(\mu) \to L_p(\mu)\|_{cb} \leq C_1\|P : L_p(\mu) \to L_p(\mu)\| \leq C_1C_2\sqrt{p}\lambda.$$

Thus, the operator $\tilde{T} = TP$ is well-defined both on L_p, and on on L_2.

(1) The operator $T : X_2 \to \ell_2$ is invertible and, by identifying X_2 with ℓ_2 in a suitable way, we can assume that T is positive. Consider two operator space structures on ℓ_2: let $H_1 = RC_p$ and let H_0 be such that $T : X_2 \to H_0$ is a complete isometry, i.e. for $x \in M_n(\ell_2)$,

$$\|x\|_{M_n(H_0)} = \|(I \otimes T^{-1})(x)\|_{M_n(X_2)}.$$

X_2 is isometric to OH, hence, by Lemma 4.5, $H_\theta = (H_0, H_1)_\theta$ is completely isometric to RC_q. Therefore,

$$\begin{aligned}\|T : X_q \to H_\theta\|_{cb} \leq & \|\tilde{T} : L_q(\mu) \to H_\theta\|_{cb} \\ \leq & \|\tilde{T} : L_2(\mu) \to H_0\|_{cb}^{1-\theta} \cdot \|\tilde{T} : L_p(\mu) \to H_1\|_{cb}^{\theta}.\end{aligned}$$

However,

$$\begin{aligned}\|\tilde{T} : L_2(\mu) \to H_0\|_{cb} &= \|T : X_2 \to H_0\|_{cb}\|P\| = 1, \\ \|\tilde{T} : L_p(\mu) \to H_1\|_{cb} &\leq \|P\|_{cb}\|T\|_{cb} \leq C_1C_2\sqrt{p}\lambda.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|T^{-1} : H_\theta \to X'_q\|_{cb} = & \|T^{-1} : H_\theta \to L_q(\mu)\|_{cb} \\ \leq & \|T^{-1} : H_0 \to L_2(\mu)\|_{cb}^{1-\theta} \cdot \|T^{-1} : H_1 \to L_p(\mu)\|_{cb}^{\theta} = \lambda^\theta.\end{aligned}$$

Thus, $d_{cb}(X_q, H_\theta) \leq (C_1C_2\sqrt{p}\lambda)^{2\theta}$.

(2) If Z is an operator space, we denote by $\overline{Z}$ its *complex conjugate*. The reader is referred to Section 2.9 of [31] for more information about this construction. By the results of Chapter 1 of [30],

$$\|\sum_i z_i \otimes \overline{a_i}\|_{S_p[Z]} = \|\sum_i \overline{z_i} \otimes a_i\|_{S_p[\overline{Z}]}$$

for any $z_1, z_2, \ldots \in Z$ and $a_1, a_2, \ldots \in S_p$. Therefore, if Z is a subspace of $L_s(\nu)$, then $\overline{Z}$ can be identified (completely isometrically) with $\{\overline{f} \mid f \in Z\}$.

Consider $\overline{x} \in M_n(\overline{X_{p'}})$. By duality,

$$\begin{aligned}\|\overline{x}\| &= \sup\{\|\langle y, \overline{x}\rangle\| \mid y \in M_n(L_p(\mu)), \|y\| \leq 1\} \\ &= \sup\{\|\langle (I_{M_n} \otimes P)y, \overline{x}\rangle\| \mid y \in M_n(L_p(\mu)), \|y\| \leq 1\}\end{aligned}$$

(here, $\langle y, \overline{x}\rangle$ is an $n^2 \times n^2$ matrix). However, $\|P\|_{CB(L_p(\mu))} \leq C_1 C_2 \sqrt{p}\lambda$, hence

$$\begin{aligned}&\sup\{\|\langle y, \overline{x}\rangle\| \mid y \in M_n(X_p), \|y\| \leq 1\} \\ &\leq \|\overline{x}\| \leq C_1 C_2 \sqrt{p}\lambda \sup\{\|\langle y, \overline{x}\rangle\| \mid y \in M_n(X_p), \|y\| \leq 1\}.\end{aligned}$$

By duality,

$$d_{cb}(\overline{X_{p'}}, RC_{p'}) \leq d_{cb}(\overline{X_{p'}}^*, X_p) d_{cb}(X_p, RC_p) \leq C_1 C_2 \sqrt{p}\lambda^2.$$

To finish the proof, recall that RC_p is completely isometric to its complex conjugate.

(3) is proved along the lines of (2), except that we combine the inequality

$$\|P\|_{CB(L_{q'}(\mu))} \leq \|P\|^{\theta}_{CB(L_p(\mu))} \|P\|^{1-\theta}_{CB(L_2(\mu))} \leq (C_1 C_2 \sqrt{p}\lambda)^{\theta}$$

with the results of (1). ■

Finally, we discuss isomorphic copies of ℓ_p in L_p. We show first that any completely unconditional basic sequence in L_p ($1 \leq p < \infty$) which is equivalent to the canonical basis in ℓ_p, is in fact completely equivalent to it.

Theorem 4.6. *Suppose $f_1, f_2, \ldots$ is a normalized M-completely unconditional basic sequence in L_p which is equivalent to the canonical basis in ℓ_p, and $a_1, a_2, \ldots \in S_p$.*

(1) *If $1 \leq p \leq 2$, then there exists a constant $c > 0$ for which*

$$c\big(\sum \|a_i\|^p_{S_p}\big)^{1/p} \leq \|\sum f_i \otimes a_i\|_{L_p(S_p)} \leq M\big(\sum \|a_i\|^p_{S_p}\big)^{1/p}.$$

(2) *If $2 \leq p < \infty$, then there exists a constant $C > 0$ for which*

$$\frac{1}{M}\big(\sum \|a_i\|^p_{S_p}\big)^{1/p} \leq \|\sum f_i \otimes a_i\|_{L_p(S_p)} \leq C\big(\sum \|a_i\|^p_{S_p}\big)^{1/p}.$$

Proof. (1) The upper estimate follows directly from Lemma 3.2. To establish the lower estimate, recall that, by Proposition 3.4 of [5], there exist $c > 0$ and disjoint measurable sets S_i s.t. $\int_{S_i} |f_i(t)|^p d\mu \geq c^p$. By (1.3) and Khintchine-Kahane inequality, there exists a constant k s.t.

$$\begin{aligned}&kM^p \|\sum f_i \otimes a_i\|^p \geq \text{Ave}_{\pm} \|\sum \pm f_i \otimes a_i\|^p = \int \text{Ave}_{\pm} \|\sum \pm f_i(t) a_i\|^p \\ &\geq \sum_j \int_{S_j} \text{Ave}_{\pm} \|\sum \pm f_i(t) a_i\|^p \geq \sum_j \int_{S_j} \|f_j(t) a_j\|^p \geq c^p \sum \|a_j\|^p.\end{aligned}$$

(2) The left-hand side follows from Theorem 2.1. To prove the right-hand side, we once again use Proposition 3.4 of [5]. The said proposition implies that there exists $K < \infty$ s.t.

$$\sum_i \Big(\int |f_i(t)|^2 |\phi(t)|^{p-2} d\mu \Big)^s \leq K^s$$

whenever $\|\phi\|_{L_p} \leq 1$ (here $s = p/(p-2)$). Thus, the linear map $T : L_s(\mu) \to \ell_s$, defined by

$$T(\psi) = \Big(\int |f_i(t)|^2 \psi d\mu \Big)_{i=1}^{\infty},$$

satisfies $\|T\| \leq K$. T is a positive operator (that is, it maps positive functions into positive sequences), and therefore, by [28], $\|T\|_{cb} \leq K$. Dualizing, we conclude that, whenever $b_1, b_2, \ldots \in S_r$ (where $r = p/2$), $\int \| \sum |f_i(t)|^2 b_i \|^r \leq K^r \sum \|b_i\|^r$. Thus, for $a_1, a_2, \ldots \in S_p$,

$$\begin{aligned} \Big(\int \| \big(\sum |f_i(t)|^2 a_i^* a_i \big)^{1/2} \|_{S_p}^p d\mu \Big)^{1/p} &\leq K \Big(\sum \|a_i\|_{S_p}^p \Big)^{1/p}, \\ \Big(\int \| \big(\sum |f_i(t)|^2 a_i a_i^* \big)^{1/2} \|_{S_p}^p d\mu \Big)^{1/p} &\leq K \Big(\sum \|a_i\|_{S_p}^p \Big)^{1/p}. \end{aligned} \tag{4.3}$$

Using the complete unconditionality of (f_i) and (1.3), we conclude that

$$\begin{aligned} \| \sum f_i \otimes a_i \| \leq & M \mathrm{Ave}_{\pm} \| \sum \pm f_i \otimes a_i \| \\ \leq MB\sqrt{p} \max & \Big\{ \Big(\int \| \big(\sum |f_i(t)|^2 a_i^* a_i \big)^{1/2} \|_{S_p}^p d\mu \Big)^{1/p}, \\ & \Big(\int \| \big(\sum |f_i(t)|^2 a_i a_i^* \big)^{1/2} \|_{S_p}^p d\mu \Big)^{1/p} \Big\}. \end{aligned}$$

Together with (4.3), the last inequality concludes the proof. ■

The paper of the first author and N. Nielsen [14] shows that ℓ_p contains uncomplemented completely isomorphic copies of ℓ_p. However, in some case, we can guarantee complete complementability.

Proposition 4.7. *Suppose (f_i) is a sequence of independent mean zero random variables in $L_p(\mu)$ such that* $\mathrm{span}[(f_i)_{i \in \mathbb{N}}]$ *is isomorphic to ℓ_p, and $\|f_i\|_p = 1$ for every i ($1 \leq p < \infty$). Then the sequence (f_i) is completely equivalent to the canonical basis in ℓ_p, and* $\mathrm{span}[(f_i)_{i \in \mathbb{N}}]$ *is completely complemented.*

Proof. Consider a sequence (ε_i) of Bernoulli random variables in $L_p(\nu)$, independent of each other and of f_i's. Let $f_i' = f_i \varepsilon_i$ be elements of $L_p(\mu \times \nu)$. Clearly, the basic sequence (f_i') is completely unconditional. By a standard symmetrization trick (see e.g. Lemma 6.3 of [19]), the sequence (f_i) is also completely unconditional. By [5], the sequence (f_i) is equivalent to the canonical basis of ℓ_p. Therefore, by Theorem 4.6, (f_i) is completely equivalent to the canonical basis in ℓ_p.

To show that $\mathrm{span}[(f_i)_{i \in \mathbb{N}}]$ is completely complemented, we can assume that μ is a probability measure (other cases can be easily reduced to this one). After discarding finitely many f_i's if necessary, we obtain, by Theorem 3.5 of [5], a sequence of sets $E_i \in \mathcal{A}(f_i)$ (the smallest σ-algebra with respect to which f_i is measurable) s.t. $\sum_i \mu(E_i) < 1/2$, and $\int_{E_i} |f_i|^p > (4c)^p$, with $c > 0$ independent of i. Let $F_i = E_i \backslash (\cup_{j \neq i} E_j)$. By the independence of f_i's,

$$\int_{F_i} |f_i|^p = \frac{\mu(F_i)^p}{\mu(E_i)^p} \int_{E_i} |f_i|^p > (2c)^p. \tag{4.4}$$

Denote by (δ_i) the canonical basis of ℓ_p, and define $T \in B(\mathrm{span}[(f_i)_{i\in\mathbb{N}}], \ell_p)$ by setting $Tf_i = \delta_i$. We shall show that T is *regular*, that is, there exists a constant K s.t.

$$\| \sup_j |Tx_j| \| \leq K \| \sup_j |x_j| \|$$

for any finite sequence $(x_j) \subset \mathrm{span}[(f_i)_{i\in\mathbb{N}}]$. Indeed, write $x_j = \sum_i a_{ij} f_i$, with $a_{ij} \in \mathbb{C}$. Then $\| \sup_j |Tx_j| \|^p = \sum_i b_i^p$, where $b_i = \sup_j |a_{ij}|$. Moreover, for each i there exists $j(i)$ s.t. $|a_{i,j(i)}| \geq b_i/2$. By Jensen's inequality and (4.4),

$$\| \sup_j |x_j| \|^p \geq \sum_i \int_{F_i} | \sum_k a_{k,j(i)} f_k |^p \geq \sum_i \int_{F_i} |a_{k,j(i)}|^p |f_i|^p \geq c^p \sum_i b_i^p.$$

Therefore, T is regular. By [27], T extends to a regular operator $\tilde{T} : L_p \to \ell_p$. By [28], $\tilde{T}$ is completely bounded. Therefore, $T^{-1} \circ \tilde{T}$ is a completely bounded projection onto (a finite codimensional subspace of) $\mathrm{span}[(f_i)_{i\in\mathbb{N}}]$. ■

Corollary 4.8. *Suppose* (f_i) *is a sequence of independent random variables in* $L_p(\mu)$ *(*$1 \leq p < \infty$*), for which* $\mathrm{span}[(f_i)_{i\in\mathbb{N}}]$ *is isomorphic to* ℓ_p*. Then* $\mathrm{span}[(f_i)_{i\in\mathbb{N}}]$ *is completely isomorphic to* ℓ_p*, and completely complemented in* $L_p(\mu)$*.*

Proof. Once again, we can assume that μ is a probability measure. Random variables $g_i = f_i - \mathbb{E} f_i$ are independent, and $\mathbb{E} g_i = 0$. By the proof of Theorem A of [5], $\mathrm{span}[(g_i)_{i\in\mathbb{N}}]$ is isomorphic to ℓ_p. By Proposition 4.7, $\mathrm{span}[(g_i)_{i\in\mathbb{N}}]$ is completely isomorphic to ℓ_p, and completely complemented in $L_p(\mu)$. However, both $\mathrm{span}[(f_i)_{i\in\mathbb{N}}]$ and $\mathrm{span}[(g_i)_{i\in\mathbb{N}}]$ are 1-codimensional subspaces of $\mathrm{span}[1, (f_i)_{i\in\mathbb{N}}]$, hence these two subspaces are completely isomorphic, and both are completely complemented in $L_p(\mu)$. ■

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