

OPERATOR SPACES WITH COMPLETE BASES, LACKING COMPLETELY UNCONDITIONAL BASES

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ABSTRACT. We construct a Hilbertian operator space X such that the set of completely bounded operators on X consists of Hilbert-Schmidt perturbations of a certain representation of the second dual to the James space. This space possesses an orthonormal basis (e_i) such that all basis projections are completely contractive, yet any n -dimensional block subspace has complete unconditionality constant of at least $c\sqrt{n}$ (c is a constant).

1. INTRODUCTION AND THE MAIN RESULT

The main purpose of this paper is to give an example of an operator space with a completely bimonotone basis such that every block subsequence thereof has “the worst possible” constant of complete unconditionality (the definitions are given below). We start by recalling some useful definitions and results related to Banach and operator spaces. For more information, the reader is referred to [LT] (Banach spaces), [ER], [Pa], or [Pi2] (operator spaces).

Suppose $(e_i)_{i=1}^\infty$ is a basis in an operator space X . Henceforth we shall assume all our bases to be *normalized*. That is, if $(e_i)_{i=1}^\infty$ is a basis, we shall assume that $\|e_i\| = 1$ for every i . For $n \in \mathbb{N}$ denote by P_n the “natural” projection onto $\text{span}\{e_i \mid 1 \leq i \leq n\}$ (that is, $P_n e_i = e_i$ if $i \leq n$, and $P_n e_i = 0$ for $i > n$). We say that the basis (e_i) is *complete* if $\sup_n \|P_n\|_{cb} < \infty$. The basis (e_i) is *completely bimonotone* if $\|P_n\|_{cb} = \|I_X - P_n\|_{cb} = 1$ for every $n \in \mathbb{N}$ (I_X stands for the identity on X). (e_i) is called *C -completely unconditional* if, for any sequence $c_1, c_2, \dots \in \mathbb{C}$, and for any $a_1, a_2, \dots \in B(H)$, $\|\sum_i e_i \otimes a_i\| \leq C(\sup_i |c_i|)^{-1} \|\sum_i c_i e_i \otimes a_i\|$ (here and below, \otimes stands for the injective tensor product of operator spaces). For an operator space X , its *complete unconditional constant* $\text{cunc}(X)$ is defined as the infimum of all the C 's for which X has a C -completely unconditional basis. Note that, if X is n -dimensional, then $\text{cunc}(X) \leq \sqrt{n}$. Indeed, by [Pi1], $d_{cb}(X, OH_n) = \sqrt{n}$, and $\text{cunc}(OH_n) = 1$.

If $(e_i)_{i=1}^\infty$ is a basis in a Banach space X , then a (finite or infinite) sequence (f_j) is called a *block sequence* if $f_j = \sum_{n_{j-1}}^{n_j-1} \alpha_i e_i$, with $n_0 \leq n_1 \leq n_2 \leq \dots$. The celebrated theorem of Krivine (see e.g. [Kr], [Le], or Chapters 11-12 of [MS]) states that, for any basis $(e_i)_{i=1}^\infty$ in a Banach space, there exists $p \in [1, \infty]$ such that, for any $\varepsilon > 0$

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and $n \in \mathbb{N}$, there exists a block sequence $(f_j)_{j=1}^n$ so that the inequality

$$(1 - \varepsilon) \left(\sum_{j=1}^n |\alpha_j|^p \right)^{1/p} \leq \left\| \sum_{j=1}^n \alpha_j e_j \right\| \leq (1 + \varepsilon) \left(\sum_{j=1}^n |\alpha_j|^p \right)^{1/p}$$

is satisfied for any n -tuple of scalars $\alpha_1, \dots, \alpha_n$. We shall see that this result fails spectacularly in the operator space setting. To this end, we follow [AG] to describe the ‘‘Banach algebraic’’ version of the James space.

For $x = (x_i)_{i=1}^\infty \in \ell_\infty$ define

$$\|x\|_{J_1} = \frac{1}{\sqrt{2}} \sup \left\{ \left(\sum_{j=1}^n |x_{i_{j+1}} - x_{i_j}|^2 \right)^{1/2} \mid i_1 < i_2 < \dots < i_n \right\}$$

(here, we identify i_{n+1} with i_1). The *James space* J is the completion of c_{00} (the space of all eventually null sequences) with respect to the norm $\|\cdot\|_{J_1}$. For the sake of convenience, we sometimes use an equivalent norm:

$$\|x\|_{J_0} = \sup_{i_1 < i_2 < \dots < i_{2n}} \max \left\{ \left(\sum_{j=1}^n |x_{i_{2j+1}} - x_{i_{2j}}|^2 \right)^{1/2}, \left(\sum_{j=1}^n |x_{i_{2j-1}} - x_{i_{2j}}|^2 \right)^{1/2} \right\}$$

(as before, we identify i_{2n+1} with i_1).

J can be renormed to become a Banach algebra: for $x \in \ell_\infty$, set

$$\|x\|_J = \sup \{ \|xy\|_{J_1} \mid y \in c_{00}, \|y\|_{J_1} \leq 1 \}$$

(here xy refers to the pointwise product of the two sequences). It follows from [AG] that the norms $\|\cdot\|_{J_1}$, $\|\cdot\|_{J_0}$, and $\|\cdot\|_J$ are equivalent on J . Moreover, we have:

$$(1.1) \quad \begin{aligned} \|x\|_{J_1} &\leq \|x\|_{J_0} \leq \sqrt{2} \|x\|_{J_1} \leq \sqrt{2} \|x\|_J \text{ for any } x \in \ell_\infty, \\ \|x\|_J &\leq 2 \|x\|_{J_1} \text{ for any } x \in J. \end{aligned}$$

We denote the space of all $x \in \ell_\infty$ for which $\|x\|_J$ is finite by \mathcal{A} . It is shown in [AG] that $\mathcal{A} = J + \mathbf{1} = \{x \in \ell_\infty \mid \|x\|_{J_1} < \infty\}$. Here, $\mathbf{1} = (1, 1, \dots) \in \ell_\infty$. \mathcal{A} can be identified with the second dual of $(J, \|\cdot\|_J)$. Most importantly, \mathcal{A} is a Banach algebra, with the identity $\mathbf{1}$. Moreover, $\|x\|_J \geq \|x\|_\infty$ for any x , where $\|\cdot\|_\infty$ is the ℓ_∞ norm.

The N -dimensional James space J_N (or \mathcal{A}_N) is defined as follows: for a finite sequence $a = (a_i)_{i=1}^N$, consider $a' = (a_1, \dots, a_{N-1}, a_N, a_N, a_N, \dots)$. Abusing notation somewhat, we write $\|a\|_J$ ($\|a\|_{J_0}$, $\|a\|_{J_1}$) instead of $\|a'\|_J$ (respectively, $(\|a'\|_{J_0}$, $\|a'\|_{J_1})$). We sometimes write J_∞ and \mathcal{A}_∞ instead of J and \mathcal{A} .

We need several more definitions to formulate the main result. Let $(\delta_i)_{i=1}^\infty$ be the canonical basis in ℓ_2 . Denote by D the ‘‘diagonal projection’’: for $T \in B(\ell_2)$, define $DT \in B(\ell_2)$ by letting $DT\delta_i = \langle T\delta_i, \delta_i \rangle \delta_i$.

Let $\pi : \mathcal{A} \rightarrow B(\ell_2)$ be the representation, defined by setting $\pi(x) = \text{diag } x$. Here, $\text{diag } x$ is the diagonal operator whose entries are elements of the sequence x . In other

words, for the sequence $x = (x_i)$, $(\text{diag } x)\delta_i = x_i\delta_i$. By the reasoning above, π is unital and contractive. The main result of this paper states:

Theorem 1.1. (a) *There exists an operator space X , isometric to ℓ_2 , such that $T \in CB(X)$ if and only if $T = \pi(a) + S$, with $a \in \mathcal{A}$, $S \in S_2$, and $DS = 0$ (D is the diagonal projection defined above). Then*

$$\frac{\|a\|_J}{32\sqrt{2}} \leq \|\pi(a)\|_{cb} \leq \|a\|_J \quad \text{and} \quad \|S\|_{cb} \leq \|S\|_2 \leq 16 \min\{\|S\|_{cb}, \|T\|_{cb}\}.$$

Consequently,

$$\max\left\{\frac{\|a\|_J}{544\sqrt{2}}, \frac{\|S\|_2}{16}\right\} \leq \|T\|_{cb} \leq \|a\|_J + \|S\|_2.$$

The basis $(\delta_i)_{i=1}^\infty$ is completely bimonotone in X . Moreover, X is completely hereditarily indecomposable – that is, no infinite dimensional subspace of X is completely isomorphic to a direct sum of two infinite dimensional operator spaces.

(b) Suppose $\mathbf{f} = (f_j)_{j=1}^N$ is a normalized block sequence of (δ_i) ($N \in \mathbb{N} \cup \{\infty\}$). Consider the block subspace $X_{\mathbf{f}} = \text{span}[f_j \mid 1 \leq j \leq N] \hookrightarrow X$, and define the representation $\pi_{\mathbf{f}} : \mathcal{A}_N \rightarrow CB(X_{\mathbf{f}})$ by setting $\pi_{\mathbf{f}}(a)f_j = a_j f_j$ for any $a = (a_j)_{j=1}^N \in \mathcal{A}_N$. Then:

- (1) $T \in CB(X_{\mathbf{f}})$ if and only if $T = \pi_{\mathbf{f}}(a) + S$, with $a \in \mathcal{A}_N$, $S \in S_2$, and $D_{\mathbf{f}}S = 0$ ($D_{\mathbf{f}}$ is the diagonal projection on $X_{\mathbf{f}}$, defined by setting $D_{\mathbf{f}}Tf_j = \langle Tf_j, f_j \rangle f_j$).

Then

$$\frac{\|a\|_J}{32\sqrt{2}} \leq \|\pi(a)\|_{cb} \leq (\sqrt{2} + 1)\|a\|_J \quad \text{and} \quad \|S\|_{cb} \leq \|S\|_2 \leq 16 \min\{\|S\|_{cb}, \|T\|_{cb}\}.$$

Consequently,

$$\max\left\{\frac{\|a\|_J}{544\sqrt{2}}, \frac{\|S\|_2}{16}\right\} \leq \|T\|_{cb} \leq (\sqrt{2} + 1)\|a\|_J + \|S\|_2.$$

- (2) If P_1, P_2, \dots, P_M are non-zero projections in $B(X_{\mathbf{f}})$ such that $P_i P_j = 0$ whenever $i \neq j$, then there exist $\omega_1, \dots, \omega_M \in \mathbb{C}$ for which $|\omega_1| = \dots = |\omega_M| = 1$ and $\|\sum_{j=1}^M \omega_j P_j\|_{cb} \geq \sqrt{M}/700$. In particular, $\text{cunc}(X_{\mathbf{f}}) \geq \sqrt{\dim X_{\mathbf{f}}}/700$.
- (3) If Y is a proper subspace of $X_{\mathbf{f}}$, or Y is a subspace of X which contains $X_{\mathbf{f}}$ as a proper subspace, then Y is not completely isomorphic to $X_{\mathbf{f}}$.

Remark 1.2. By 1.1, $\|a\|_J \leq 2\|a\|_{J_1} \leq 2\sqrt{2}\|a\|_2$ for any $a \in \ell_2$. Therefore, Theorem 1.1 and (2.2) imply the existence of a constant $C > 1$ such that, for any $T \in B(X_{\mathbf{f}})$, we have

$$C^{-1}\|T\|_{cb} \leq \inf\{\|a\|_J + \|S\|_2 \mid T = \pi_{\mathbf{f}}(a) + S, a \in \mathcal{A}, S \in S_2\} \leq C\|T\|_{cb}$$

($T \in CB(X)$ iff the term in the middle is bounded).

The proof of Theorem 1.1 is presented in the next section.

2. PROOF OF THE MAIN RESULT

Here and below, we shall denote by \mathcal{K} the space of compact operators on ℓ_2 . $B_{\mathcal{A}}$ shall denote the closed unit ball of \mathcal{A} (equipped with the norm $\|\cdot\|_J$). I_X stands for the identity map on X .

By [OR], there exists a family of operator spaces $(E_i)_{i=1}^{\infty}$ such that:

- (1) E_i is isometric to $\ell_2^{n_i}$ for some $n_i \in \mathbb{N}$, and $\{i \mid n_i = \dim E_i = j\}$ is infinite for any $j \in \mathbb{N}$.
- (2) For any operator $u : E_i^* \rightarrow E_j$, we have $\|u\|_1/(4 + 2^{-i}) \leq \|u\|_{cb} \leq \|u\|_1$ if $i = j$, $\|u\|_{cb} = \|u\|_2$ if $i \neq j$ (here, $\|v\|_p$ denotes the norm of v in the Schatten class S_p).

Following [OR], find a sequence of operators $u_i : \ell_2 \rightarrow \ell_2^{n_i}$ such that $\|u_i\|_2 = 1$ and, for any $\varepsilon > 0$, $n \in \mathbb{N}$, and $u : \ell_2 \rightarrow \ell_2^n$, there exists $i \in \mathbb{N}$ for which $n_i = n$ and $\|u_i - u\|_1 < \varepsilon$. On the Banach space level, we identify the range of u_i with E_i described above.

We define the operator space X as follows: for $x \in \ell_2 \otimes \mathcal{K}$, let

$$(2.1) \quad \|x\|_{X \otimes \mathcal{K}} = \sup \{ \|(u_i \pi(a) \otimes I_{\mathcal{K}})x\|_{E_i \otimes \mathcal{K}} \mid i \in \mathbb{N}, a \in B_{\mathcal{A}} \}.$$

Clearly, X is an operator space (Ruan's axioms are satisfied), and X is isometric to ℓ_2 as a Banach space. For any $a \in \mathcal{A}$, the definition implies that $\|\pi(a)\|_{cb} \leq \|a\|$. If Y is an operator space isometric to ℓ_2 and $T : Y \rightarrow X$ is a Hilbert-Schmidt operator, then

$$(2.2) \quad \|T\|_{cb} \leq \|T\|_2$$

Indeed,

$$\|T\|_{cb} = \sup \{ \|u_i \pi(a) T\|_{cb} \mid i \in \mathbb{N}, a \in B_{\mathcal{A}} \} \leq \sup \{ \|u_i \pi(a) T\|_1 \mid i \in \mathbb{N}, a \in B_{\mathcal{A}} \}.$$

However,

$$\|u_i \pi(a) T\|_1 \leq \|u_i\|_2 \|\pi(a)\| \|T\|_2 = \|T\|_2,$$

and we are done. Therefore, $\pi(a) + S \in CB(X)$ whenever $a \in \mathcal{A}$ and $S \in S_2$, with $\|\pi(a) + S\|_{cb} \leq \|a\|_J + \|S\|_2$.

Suppose now $a = (1, \dots, 1, 0, 0, \dots)$, or $a = (0, \dots, 0, 1, 1, \dots)$. In either case, the definition of $\|\cdot\|_J$ implies that $\|a\|_J = 1$. Therefore, $\|P_n\|_{cb} = \|I_X - P_n\|_{cb} = 1$ for any n (here, as before, P_n is the orthogonal projection onto $\text{span}[\delta_i \mid 1 \leq i \leq n]$). Thus, $(\delta_i)_{i=1}^{\infty}$ is a completely bimonotone basis in X .

Now consider a normalized block sequence $\mathbf{f} = (f_j)_{j=1}^N$, where $f_j = \sum_{i=n_{j-1}}^{n_j-1} \alpha_i \delta_i$, and the space $X_{\mathbf{f}}$ spanned by this sequence (here $1 = n_0 < n_1 < \dots$). Begin by obtaining upper estimates on c.b. norms of operators taking $X_{\mathbf{f}}$ to X . By (2.2), any Hilbert-Schmidt operator $S : X_{\mathbf{f}} \rightarrow X$ satisfies $\|S\|_{cb} \leq \|S\|_2$. We shall show that $\|\pi_{\mathbf{f}}(a)\|_{cb} \leq (1 + \sqrt{2})\|a\|_J$ for any $a \in \mathcal{A}$. Indeed, take $a = (a_1, a_2, \dots) \in \mathcal{A}$, and

define $b = (b_i)$ by setting $b_i = a_j$ for $i \in I_j = [n_{j-1}, n_j - 1]$ (if N is finite, then we set $n_j = n_N - N + j$ for $j \geq N$).

Lemma 2.1. *In the above notation, $(1 + \sqrt{2})\|a\|_J \geq \|b\|_J \geq \|a\|_J$.*

Proof. First show that $\|b\|_J \geq \|a\|_J$. Indeed, fix $\varepsilon > 0$, and find a sequence $x = (x_j) \in c_{00}$ s.t. $\|x\|_{J_1} = 1$, $\|ax\|_{J_1} > \|a\|_J - \varepsilon$. Define $y = (y_i)$ by setting $y_i = x_j$ for $i \in I_j$. Clearly, $\|y\|_{J_1} = \|x\|_{J_1}$, and $\|by\|_{J_1} = \|ax\|_{J_1}$. Thus, $\|b\|_J > \|a\|_J - \varepsilon$ for any $\varepsilon > 0$.

Now suppose $\|b\|_J = 1$, and $b = (b_i)$. Show that $\|a\|_J \geq 1/(1 + \sqrt{2})$. To this end, let $\beta = \lim_i b_i$. If $|\beta| \geq 1/(1 + \sqrt{2})$, we arrive to the desired conclusion via the inequality

$$\|a\|_J \geq |\lim_j a_j| = |\beta| \geq \frac{1}{1 + \sqrt{2}}.$$

Otherwise, let $b' = b - \beta \mathbf{1}$. By (1.1),

$$\|a\|_J \geq \|a\|_{J_1} = \|b'\|_{J_1} \geq \frac{1}{\sqrt{2}}\|b'\|_J \geq \frac{1}{\sqrt{2}}(\|b\|_J - |\beta|) \geq \frac{1}{1 + \sqrt{2}}.$$

■

As a corollary, we get (in the above notation):

$$(2.3) \quad \|\pi_{\mathbf{f}}(a)\|_{cb} \leq \|\pi(b)\|_{cb} \leq \|b\|_J \leq (\sqrt{2} + 1)\|a\|_J.$$

We now pass to estimating c.b. norms of operators on $X_{\mathbf{f}}$ from below. The following lemma will be one of our main tools.

Lemma 2.2. *Consider the operators $T : Y \rightarrow X$ (Y is a subspace of X), $u : X \rightarrow \ell_2^n$, and $v : \ell_2^n \rightarrow X$, such that $\|u\|_2 = \|v\| = 1$. Let $C = \sup\{\|u\pi(a)v\|_1 \mid a \in B_{\mathcal{A}}\}$. Then $\|T\|_{cb} \geq \|uTv\|_1/(4 \max\{C, 1\})$.*

Proof. Fix $\varepsilon > 0$. By a small perturbation method, we may assume that $n = n_i$ for i so large that $2^{-i} < \varepsilon$, and $u = u_i$ (here we identify E_i with $\ell_2^{n_i}$). Then v can be viewed as a map from E_i^* into X . By (2.1),

$$\|v\|_{cb} = \sup \{ \|u_j \pi(a)v\|_{cb} \mid j \in \mathbb{N}, a \in B_{\mathcal{A}} \}.$$

However, $\|\pi(a)\| \leq \|a\|_J$, and therefore,

$$\begin{aligned} \|u_j \pi(a)v\|_{cb} &\leq \|u_j \pi(a)v\|_2 \leq \|u_j\|_2 \|\pi(a)\| \|v\| \leq 1 \text{ for } j \neq i, \\ \|u_i \pi(a)v\|_{cb} &\leq \|u_i \pi(a)v\|_1 \leq C. \end{aligned}$$

Thus, $\|v\|_{cb} \leq \max\{1, C\}$.

In a similar fashion, we conclude that

$$\|Tv\|_{cb} \geq \|u_i Tv\|_{cb} \geq \frac{1}{4 + \varepsilon} \|u_i Tv\|_1.$$

Thus,

$$\|T\|_{cb} \geq \frac{\|Tv\|_{cb}}{\|v\|_{cb}} \geq \frac{\|uTv\|_1}{(4 + \varepsilon) \max\{C, 1\}}.$$

We complete the proof by observing that ε can be chosen arbitrarily small. \blacksquare

We use this lemma to estimate the Hilbert-Schmidt norms of off-diagonal parts of c.b. maps on $X_{\mathbf{f}}$. As in the statement of Theorem 1.1, let $D_{\mathbf{f}}$ be the diagonal projection on $X_{\mathbf{f}}$. The following lemma yields estimates on $\|(I - D_{\mathbf{f}})T\|_2$ and $\|D_{\mathbf{f}}T\|_{cb}$.

Corollary 2.3. *Suppose $\mathbf{f} = (f_j)_{j=1}^N$ is a normalized block sequence in X (N is either infinite or finite). Then $\|(I - D_{\mathbf{f}})T\|_2 \leq 16\|T\|_{cb}$, and $\|D_{\mathbf{f}}T\|_{cb} \leq 17\|T\|_{cb}$.*

Proof. We can assume that N is finite. Denote by P_j ($1 \leq j \leq N$) the orthogonal projection onto $\text{span}[\delta_i \mid n_{j-1} \leq i \leq n_j - 1]$. For any $S \subset \{1, 2, \dots, N\}$, let $P_S = \sum_{j \in S} P_j$. By Lemma 2.2, $4\|T\|_{cb} \geq \|uP_S T P_S\|_1$ for any operator u satisfying $\|u\|_2 \leq 1$. Thus, $4\|T\|_{cb} \geq \|P_S T P_S\|_2$. Note that

$$\text{Ave}_{S \subset \{1, 2, \dots, N\}} \langle P_S T P_S f_j, f_i \rangle = \begin{cases} \langle T f_j, f_i \rangle / 4 & i \neq j \\ 0 & i = j \end{cases}$$

(the average is taken over all subsets of $\{1, 2, \dots, N\}$). Therefore,

$$\|T\|_{cb} \geq \frac{1}{4} \text{Ave}_S \|P_S T P_S\|_2 \geq \frac{1}{4} \|\text{Ave}_S P_S T P_S\|_2 = \frac{1}{16} \|(I - D_{\mathbf{f}})T\|_2.$$

By (2.2), $\|D_{\mathbf{f}}T\|_{cb} \leq \|T\|_{cb} + \|(I - D_{\mathbf{f}})T\|_{cb} \leq \|T\|_{cb} + \|(I - D_{\mathbf{f}})T\|_2 \leq 17\|T\|_{cb}$. \blacksquare

Next we estimate the norms of ‘‘diagonal’’ operators on $X_{\mathbf{f}}$ (we say that an operator T on $X_{\mathbf{f}} = \text{span}[f_j]$ is diagonal if $\langle T f_j, f_i \rangle = 0$ whenever $i \neq j$).

Proposition 2.4. *Suppose $\mathbf{f} = (f_j)_{j=1}^N$ ($N \in \mathbb{N} \cup \{\infty\}$) is a normalized block sequence of $(\delta_i)_{i=1}^{\infty}$, $X_{\mathbf{f}} = \text{span}[(f_j)_{j=1}^N]$, and the operator $T : X_{\mathbf{f}} \rightarrow X_{\mathbf{f}}$ is defined by setting $T f_j = t_j f_j$ ($1 \leq j \leq N$). Then*

$$\max \left\{ \frac{1}{32\sqrt{2}} \|t\|_J, \frac{1}{8\sqrt{2}} \|t\|_{J_0} \right\} \leq \|T\|_{cb} \leq (\sqrt{2} + 1) \|t\|_J$$

where $t = (t_j)_{j=1}^N$.

Proof. The inequality $\|T\|_{cb} \leq (\sqrt{2} + 1) \|t\|_J$ has already been established in (2.3). To establish the reverse inequality, fix $i_1 < i_2 < \dots < i_{2n}$, and let

$$C^2 = \sum_{j=1}^n |t_{i_{2j+1}} - t_{i_{2j}}|^2.$$

Let $c_j = t_{i_{2j+1}} - t_{i_{2j}}$, and define operators $v : \ell_2^n \rightarrow X$ and $u : X \rightarrow \ell_2^n$ by setting $v e_j = (f_{i_{2j+1}} - f_{i_{2j}}) / \sqrt{2}$ for $1 \leq j \leq n$ (here $(e_j)_{j=1}^n$ is an orthonormal basis in ℓ_2^n), and

$$u \xi = \frac{1}{\sqrt{2}C} \sum_{j=1}^n \bar{c}_j \langle \xi, f_{i_{2j+1}} + f_{i_{2j}} \rangle e_j.$$

Then $\|v\| = \|u\|_2 = 1$. Moreover, $\|u\pi(a)v\|_1 \leq \sqrt{2}$ whenever $\|a\|_J \leq 1$. Indeed,

$$(2.4) \quad u\pi(a)ve_j = \frac{c_j}{C} \left\langle \pi(a) \frac{f_{i_{2j+1}} - f_{i_{2j}}}{\sqrt{2}}, \frac{f_{i_{2j+1}} + f_{i_{2j}}}{\sqrt{2}} \right\rangle e_j.$$

Write

$$\frac{f_{i_{2j+1}} - f_{i_{2j}}}{\sqrt{2}} = \sum_{i \in S_j} \beta_i \delta_i, \quad \frac{f_{i_{2j+1}} + f_{i_{2j}}}{\sqrt{2}} = \sum_{i \in S_j} \gamma_i \delta_i.$$

Here the sets S_j are ‘‘consecutive.’’ $S_j = [L_{j-1}, L_j - 1]$ for $1 \leq j \leq n - 1$, with $L_0 < L_1 < \dots < L_n$, while S_n is either $[L_{n-1}, L_n - 1]$ or $[L_{n-1}, L_n - 1] \cup [1, L_0 - 1]$.

We know that $\sum_{i \in S_j} \beta_i \bar{\gamma}_i = 0$ for any j . Let

$$A_j = \left\langle \pi(a) \frac{f_{i_{2j+1}} - f_{i_{2j}}}{\sqrt{2}}, \frac{f_{i_{2j+1}} + f_{i_{2j}}}{\sqrt{2}} \right\rangle = \sum_{i \in S_j} a_i \beta_i \bar{\gamma}_i = \sum_{i \in S_j} (a_i - a_{L_{j-1}}) \beta_i \bar{\gamma}_i.$$

Let $B_j = \max\{|a_i - a_{L_{j-1}}| \mid i \in S_j\}$. We obtain:

$$A_j \leq B_j \sum_{i \in S_j} |\beta_i \bar{\gamma}_i| \leq B_j \left(\sum_{i \in S_j} |\beta_i|^2 \right)^{1/2} \left(\sum_{i \in S_j} |\gamma_i|^2 \right)^{1/2} = B_j.$$

By (1.1), $\sum_j B_j^2 \leq \|a\|_{J_0}^2 \leq 2$. By (2.4),

$$\|u\pi(a)v\|_1 = \sum_{j=1}^n \frac{|c_j|}{C} A_j \leq \frac{1}{C} \left(\sum_{j=1}^n |c_j|^2 \right)^{1/2} \left(\sum_{j=1}^n B_j^2 \right)^{1/2} \leq \sqrt{2}.$$

Now observe that, by definition of u , $uTve_j = |c_j|^2/(2C)e_j$ for $1 \leq j \leq N$, and therefore, $\|uTv\|_1 = C/2$. By Lemma 2.2,

$$\|T\|_{cb} \geq \frac{C}{8\sqrt{2}} = \frac{1}{8\sqrt{2}} \left(\sum_{j=1}^n |t_{i_{2j+1}} - t_{i_{2j}}|^2 \right)^{1/2}.$$

Similarly, we show that

$$\|T\|_{cb} \geq \frac{1}{8\sqrt{2}} \left(\sum_{j=1}^n |t_{i_{2j-1}} - t_{i_{2j}}|^2 \right)^{1/2}.$$

Since the sequence (i_k) is arbitrary, we conclude that $\|T\|_{cb} \geq \|t\|_{J_0}/(8\sqrt{2})$.

Now let $t_0 = \lim_{i \rightarrow \infty} t_i$, and $t' = t - t_0 \mathbf{1}$. By (1.1), $\|t'\|_J \leq 2\|t'\|_{J_0} = 2\|t\|_{J_0}$. Moreover, $\|t\|_J \leq 2 \max\{\|t'\|_J, |t_0|\}$. Finally, $\|T\|_{cb} \geq \|T\| \geq |t_0|$, and therefore,

$$\|T\|_{cb} \geq \max \left\{ |t_0|, \frac{\|t'\|_J}{16\sqrt{2}} \right\} \geq \frac{\|t\|_J}{32\sqrt{2}}.$$

■

This lemma shows that X (and, consequently, any space $X_{\mathbf{f}}$ described above) is completely hereditarily indecomposable. Indeed, otherwise there exists a block sequence $\mathbf{f} = (f_j)_{j=1}^{\infty}$, and a completely bounded projection P on $X_{\mathbf{f}} = \text{span}[\mathbf{f}]$ s.t.

$Pf_j = f_j$ if j is even, and $Pf_j = 0$ if j is odd. Proposition 2.4 rules out such a possibility.

This completes the proof of parts (a) and (b)(1) of Theorem 1.1. Next we show part (b)(2).

Proposition 2.5. *Suppose $\mathbf{f} = (f_j)_{j=1}^N$ is a block subsequence of (δ_i) (here, N may be either finite or infinite), and P_1, P_2, \dots, P_M are non-zero projections in $B(X_{\mathbf{f}})$ (where $X_{\mathbf{f}} = \text{span}[\mathbf{f}]$), satisfying $P_i P_j = 0$ whenever $i \neq j$. Then*

$$\sup \left\{ \left\| \sum_{j=1}^M \omega_j P_j \right\|_{cb} \mid |\omega_1| = \dots = |\omega_M| = 1 \right\} \geq \frac{\sqrt{M}}{700}.$$

Proof. Suppose, for the sake of contradiction, that $\left\| \sum_{j=1}^M \omega_j P_j \right\|_{cb} < \sqrt{M}/700$ whenever $|\omega_1| = \dots = |\omega_M| = 1$. As before, denote by $D_{\mathbf{f}}$ the operator of projection onto the diagonal in $B(X_{\mathbf{f}})$. Let $A_j = (I - D_{\mathbf{f}})P_j$, and $B_j = D_{\mathbf{f}}P_j$. We identify B_j with the sequence $b_j = (b_{ji})_{i=1}^N$. By Corollary 2.3, $\left\| \sum_{j=1}^M \omega_j A_j \right\|_2 < 16\sqrt{M}/700$ whenever $|\omega_1| = \dots = |\omega_M| = 1$. In particular,

$$(2.5) \quad \sum_{j=1}^M \|A_j\|_2^2 = \text{Ave} \left\| \sum_{j=1}^M \omega_j A_j \right\|_2^2 < \frac{M}{1000}$$

(we average over the torus \mathbb{T}^M).

If S is a subset of $\{1, 2, \dots, M\}$, define $P_S = \sum_{j \in S} P_j$, $A_S = \sum_{j \in S} A_j$, and $B_S = \sum_{j \in S} B_j$. Then $P_S = P_S^2$, and $D_{\mathbf{f}}(A_S B_S) = D_{\mathbf{f}}(B_S A_S) = 0$. Therefore,

$$B_S = D_{\mathbf{f}}(P_S) = D_{\mathbf{f}}(P_S^2) = D_{\mathbf{f}}\left(\left(\sum_{j \in S} (A_j + B_j)\right)^2\right) = D_{\mathbf{f}}(A_S^2) + B_S^2.$$

Thus,

$$(2.6) \quad \|B_S - B_S^2\| \leq \|A_S^2\|$$

In particular, $\|B_j - B_j^2\| \leq \|A_j^2\|$.

Denote by \mathcal{I} the set of all $j \in [1, M]$ for which $\|A_j\| \leq 1/5$ (by (2.5), $|\mathcal{I}| \geq 4M/5$). For $j \in \mathcal{I}$, the previous paragraph implies that $|b_{ji} - b_{ji}^2| < 1/5$ for any i . Thus, $\min\{|b_{ji}|, |1 - b_{ji}|\} < 1/3$. Let $\mathcal{I}_j = \{i \mid |b_{ji} - 1| < 1/3\}$. Note that $\mathcal{I}_j \neq \emptyset$, since otherwise, $\|P_j\| \leq \|A_j\| + \|B_j\| < 1/5 + 1/3$, which contradicts the fact $\|P_j\| \geq 1$.

Moreover, $\mathcal{I}_j \cap \mathcal{I}_k = \emptyset$ whenever $k, j \in \mathcal{I}$. Indeed, suppose $i \in \mathcal{I}_j \cap \mathcal{I}_k$. Let $S = \{j, k\}$. Then $\|B_S - B_S^2\| \geq |x - x^2|$, with $x = b_{ji} + b_{ki}$. However, $|x - 2| < 2/3$, and therefore,

$$|x - x^2| = |x| \cdot |x - 1| \geq \frac{4}{3} \cdot \frac{1}{3} = \frac{4}{9}.$$

On the other hand, by (2.6), $\|B_S - B_S^2\| \leq (\|A_j\| + \|A_k\|)^2 < 4/25$, a contradiction.

Changing the enumeration of our projections, we can assume that $\{1, 2, \dots, 2L\} \subset \mathcal{I}$ (with $L \geq 2M/5$), and there exists a sequence $i_1 < i_2 < \dots < i_{2L}$ s.t. $i_j \in \mathcal{I}_j$

for $1 \leq j \leq 2L$. By assumption, $\|\sum_{j=1}^L \omega_j P_{2j}\|_{cb} < \sqrt{M}/700$ whenever $|\omega_1| = \dots = |\omega_L| = 1$. By Lemma 2.3, $\|\sum_j \omega_j B_{2j}\|_{cb} < 17\sqrt{M}/700$. However, by Proposition 2.4,

$$\left\| \sum_j \omega_j B_{2j} \right\|_{cb}^2 \geq \frac{1}{128} \left\| \sum_j \omega_j b_{2j} \right\|_{J_0}^2 \geq \frac{1}{128} \sum_{k=1}^L \left| \sum_j \omega_j (b_{2j, i_{2k}} - b_{2j, i_{2k-1}}) \right|^2.$$

Averaging over the torus $\mathbb{T}^L = \{(\omega_1, \dots, \omega_L) \mid |\omega_1| = \dots = |\omega_L| = 1\}$, we obtain:

$$\text{Ave} \left| \sum_{j=1}^L \omega_j (b_{2j, i_{2k}} - b_{2j, i_{2k-1}}) \right|^2 = \sum_{j=1}^L |b_{2j, i_{2k}} - b_{2j, i_{2k-1}}|^2 \geq |b_{2k, i_{2k}} - b_{2k, i_{2k-1}}|^2 \geq \frac{1}{4},$$

and therefore,

$$\text{Ave} \left\| \sum_j \omega_j B_{2j} \right\|_{cb}^2 \geq \frac{1}{128} \sum_{k=1}^L \text{Ave} \left| \sum_{j=1}^L \omega_j (b_{2j, i_{2k}} - b_{2j, i_{2k-1}}) \right|^2 \geq \frac{L}{512} \geq \frac{M}{1280}.$$

By Lemma 2.3, this implies that, for some $(\omega_1, \dots, \omega_L) \in \mathbb{T}^L$,

$$\left\| \sum_j \omega_j P_{2j} \right\|_{cb} \geq \frac{1}{17} \cdot \frac{\sqrt{M}}{\sqrt{1280}} > \frac{\sqrt{M}}{610},$$

a contradiction. ■

The proof of Section (b)(3) of Theorem 1.1 follows from

Lemma 2.6. *Suppose X , $X_{\mathbf{f}}$, and Y are as in the statement of Theorem 1.1. Then no completely bounded map from $X_{\mathbf{f}}$ to Y can have a bounded inverse.*

Remark 2.7. Completely bounded operators with bounded inverse were called *semi-isomorphisms* in [ORo].

Proof. Suppose first that Y is a proper subspace of an infinite dimensional block subspace $X_{\mathbf{f}}$, and $T : X_{\mathbf{f}} \rightarrow Y$ is a completely bounded map. By part (b)(1) of Theorem 1.1, $JT = \pi_{\mathbf{f}}(a) + S$, where $J : Y \rightarrow X_{\mathbf{f}}$ is the natural embedding, $a \in \mathcal{A}$, and $S \in S_2$. Write $a = (a_1, a_2, \dots)$, and let $a' = \lim_k a_k$. If $a' = 0$, then T is compact. Otherwise, JT is a compact perturbation of $a' I_{X_{\mathbf{f}}}$. Hence, JT is a Fredholm operator with index 0 (see e.g. Section 2.c of [LT]). On the other hand, if $T : X_{\mathbf{f}} \rightarrow Y$ has a bounded inverse, then $0 = \dim \ker(JT) < \text{codim } \text{ran}(JT) = \dim X_{\mathbf{f}}/Y$, a contradiction.

The case of $X_{\mathbf{f}} \leftrightarrow Y \leftrightarrow X$ is dealt with in a similar fashion, except that now we compute the Fredholm index of TP (P is the orthogonal projection from Y onto $X_{\mathbf{f}}$). ■

This completes the proof of Theorem 1.1.

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