FINITE REPRESENTABILITY OF HOMOGENEOUS HILBERTIAN OPERATOR SPACES IN SPACES WITH FEW COMPLETELY BOUNDED MAPS

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Abstract. For every homogeneous Hilbertian operator space H, we construct a Hilbertian operator space X such that every infinite dimensional subquotient Y of X is completely indecomposable, and fails the Operator Approximation Property, yet H is completely finitely representable in Y. If H satisfies certain conditions, we also prove that every completely bounded map on such Y is a compact perturbation of a scalar.

1. Introduction and the main result

In [GM], T. Gowers and B. Maurey gave the first example of a hereditarily indecomposable Banach space Z (recall that an infinite dimensional space Z is called *hereditarily indecomposable* if it is not isomorphic to a direct sum of two infinite dimensional Banach spaces). Since then, a variety of hereditarily indecomposable Banach spaces were constructed. An overview of the current state of affairs is given in [M].

A non-commutative counterpart of this space was obtained by E. Ricard and the author in [OR]. There, we gave an example of an operator space X, isometric to ℓ_2 (as a Banach space), such that an operator $T : Y \to X$ (Y being a subspace of X) is completely bounded if and only if $T = \lambda J_Y + S$, where J_Y is the natural embedding, $\lambda \in \mathbb{C}$, and S is a Hilbert-Schmidt map. In particular, X is *completely hereditarily indecomposable* – that is, no infinite dimensional subspace $Y \hookrightarrow X$ is completely isomorphic to an ℓ_∞ sum of two infinite dimensional operator spaces. Moreover, X fails the Operator Approximation Property (see below for the definition). For any n-dimensional subspace $Y \hookrightarrow X$, there exists a unitary $U : Y \to Y$ s.t. $\|U\|_{cb} \geqslant \sqrt{n}/16$.

Our present goal is to construct completely hereditarily indecomposable operator spaces with "some structure" – that is, spaces which are saturated with "nice" finite dimensional subspaces. More precisely, for any homogeneous Hilbertian operator space H, we construct a Hilbertian operator space X such that:

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- For any infinite dimensional subspace Y of a quotient of X, $n \in \mathbb{N}$, and $\varepsilon > 0$, there exists a subspace $F \hookrightarrow Y$ which is $(1+\varepsilon)$-completely isomorphic to an n-dimensional subspace of H.
- Any Y as above is completely hereditarily indecomposable, and fails the Operator Approximation Property.

If H satisfies certain conditions, then, in addition, any c.b. map on Y is a compact perturbation of a scalar.

Below we recall some facts and definitions concerning operator spaces. For more information, the reader is referred to [ER], [Pa], or [Pi].

We say that an operator space is *c-Hilbertian* if its underlying Banach space is c-isomorphic to a Hilbert space. X is *c-homogeneous* if $\|T\|_{cb} \leqslant c\|T\|$ for any $T \in B(X)$. An infinite dimensional operator space X is called *completely indecomposable* if it is not completely isomorphic to an ℓ_∞ direct sum of two infinite dimensional operator spaces (equivalently, any c.b. projection on X has finite dimensional kernel, or finite dimensional range).

We use the term *subquotient* to mean a subspace of a quotient.

An operator space X is said to have the *Operator Approximation Property* (*OAP*, for short) if, for any $x \in \mathcal{K} \otimes X$ and $\varepsilon > 0$, there exists a finite rank map $T : X \to X$ s.t. $\|(I_{\mathcal{K}} \otimes T)x - x\| < \varepsilon$ (here $\mathcal{K}$ is the space of compact operators on ℓ_2, and $\otimes$ denotes the minimal (injective) tensor product). X has the *Compact Operator Approximation Property* (*COAP*) if, for any $x \in \mathcal{K} \otimes X$ and $\varepsilon > 0$, there exists a compact map $T : X \to X$ s.t. $\|(I_{\mathcal{K}} \otimes T)x - x\| < \varepsilon$. More details about the OAP, as well as several equivalent reformulations of this property, can be found in Chapter 11 of [ER].

The *complete Banach-Mazur distance* between the operator spaces X and Y is defined as

$$d_{cb}(X, Y) = \inf\{\|T\|_{cb}\|T^{-1}\|_{cb} \,|\, T \in CB(X, Y)\}.$$

We say that an operator space Y is *c-completely finitely representable* in X if for any finite dimensional subspace $Z \hookrightarrow Y$ there exists $W \hookrightarrow X$ s.t. $d_{cb}(W, Z) \leqslant c$. Y is called *c-completely complementably finitely representable* in X if for any finite dimensional subspace $Z \hookrightarrow Y$ there exists a projection $P \in CB(X)$ s.t. $\|P\|_{cb} \leqslant c$, and $d_{cb}(P(X), Z) \leqslant c$.

If H is a 1-homogeneous 1-Hilbertian operator space, we denote by H_n the n-dimensional operator space, completely isometric to (any) n-dimensional subspace of H. We say that H has property $(\mathcal{P})$ if there exists a sequence $(m(n)) \subset \mathbb{N}$ s.t.

$$\lim_{n\to\infty} \frac{1}{n}\|id : \mathrm{MIN}_{m(n)}(R_n + C_n) \to H_n\|_{cb} = 0.$$

Here, id is the formal identity map between n-dimensional Hilbert spaces, and the space $\mathrm{MIN}_k(X)$ (X being an operator space) is such that

$$\|x\|_{\mathcal{K}\otimes\mathrm{MIN}_k(X)} = \sup\{\|I_{\mathcal{K}} \otimes u(x)\|_{\mathcal{K}\otimes M_k} \,|\, u \in CB(X, M_k),\ \|u\|_{cb} \leqslant 1\},$$

where, as usual, M_k stands for the pace of $k \times k$ matrices. The reader is referred to [OR] for more information about MIN_k. For future reference, we need to consider a special case of the functor MIN_k – namely, MIN_1 (denoted by MIN for the sake of brevity). If X is a Banach or operator space, and $x \in \mathcal{K} \otimes X$, then

$$\|x\|_{\mathcal{K}\otimes\mathrm{MIN}(X)} = \sup\{\|I_{\mathcal{K}} \otimes f(x)\|_{\mathcal{K}} \,|\, f \in X^*,\ \|f\|_{cb} \leqslant 1\}.$$

In other words, if $a_1, \ldots, a_n \in \mathcal{K}$, and $x_1, \ldots, x_n \in X$, then

$$\|\sum a_i \otimes x_i\|_{\mathcal{K}\otimes\mathrm{MIN}(X)} = \sup\{\|\sum f(x_i)a_i\|_{\mathcal{K}} \,|\, f \in X^*,\ \|f\|_{cb} \leqslant 1\}.$$

Note that, for any 1-homogeneous 1-Hilbertian space H, $\|id : \mathrm{MIN}(\ell_2^n) \to H_n\|_{cb} \geqslant \|id : \mathrm{MIN}_{m(n)}(R_n + C_n) \to H_n\|_{cb}$, hence H has property $(\mathcal{P})$ whenever $\limsup_n \|id : \mathrm{MIN}(\ell_2^n) \to H_n\|_{cb}/n = 0$. In particular (by Chapter 10 of [Pi]), the spaces OH, $R + C$, and $R \cap C$ have $(\mathcal{P})$. To describe another large class of spaces possessing $(\mathcal{P})$, recall that an operator space X is *exact* if there exists $C > 0$ such that for any finite dimensional subspace $E \hookrightarrow X$ there exists $F \hookrightarrow M_N$ s.t. $d_{cb}(E, F) \leqslant C$. The infimum of all such constants C is called *the exactness constant* of X, and denoted by $\mathrm{ex}\,(X)$. Observe that H has property $(\mathcal{P})$ if $\lim_{n\to\infty} \mathrm{ex}\,(H_n)/\sqrt{n} = 0$. Indeed, by Smith's Lemma (Proposition 8.11 of [Pa]), there exists a sequence of positive integers $r(1) < r(2) < \ldots$ s.t., for every operator space X, and every $v \in CB(X, H_n)$,

$$\|v : X \to H_n\|_{cb} \leqslant 2\mathrm{ex}\,(H_n)\|I_{M_{r(n)}} \otimes v : M_{r(n)} \otimes X \to M_{r(n)} \otimes H_n\|$$

(we could have used $1 + \varepsilon$ instead of 2). Then, by [OR],

$$\begin{aligned}
&(2\mathrm{ex}\,(H_n))^{-1}\|id : \mathrm{MIN}_{r(n)}(R_n + C_n) \to H_n\|_{cb} \\
&\leqslant \|I_{M_{r(n)}} \otimes id : M_{r(n)} \otimes \mathrm{MIN}_{r(n)}(R_n + C_n) \to M_{r(n)} \otimes H_n\| \\
&= \|I_{M_{r(n)}} \otimes id : M_{r(n)} \otimes (R_n + C_n) \to M_{r(n)} \otimes H_n\| \leqslant \|id : R_n + C_n \to H_n\|_{cb}.
\end{aligned}$$

However, by Theorem 10.6 of [Pi],

$$\|id : R_n + C_n \to H_n\|_{cb} \leqslant \|id : R_n + C_n \to \mathrm{MAX}(\ell_2^n)\|_{cb} = \sqrt{n}.$$

This establishes property $(\mathcal{P})$.

The main result of this paper is

Theorem 1.1. *Suppose H is a separable 1-homogeneous 1-Hilbertian operator space. Then there exists a separable 1-Hilbertian operator space X such that for every infinite dimensional subquotient Y of X we have:*

1. *For any $\varepsilon > 0$, H is $(1 + \varepsilon)$-completely complementably finitely representable in Y.*
2. *Y is completely indecomposable.*

(3) *Y fails the Compact Operator Approximation Property.*
(4) *If H has property $(\mathcal{P})$, then every completely bounded map on Y is a compact perturbation of a scalar.*

Clearly, the COAP implies the OAP. By Chapter 11 of [ER], the OAP passes from an operator space to its predual. Therefore, dualizing the space X constructed in Theorem 1.1, we conclude:

Corollary 1.2. *Suppose H is a separable 1-homogeneous 1-Hilbertian operator space, whose dual H^* has property $(\mathcal{P})$. Then there exists a separable 1-Hilbertian operator space X such that for every infinite dimensional subquotient Y of X we have:*

(1) *For any $\varepsilon > 0$, H is $(1+\varepsilon)$-completely complementably finitely representable in Y.*
(2) *Y is completely indecomposable.*
(3) *Y fails the Operator Approximation Property.*
(4) *Every completely bounded map on Y is a compact perturbation of a scalar.*

In Section 2, we present a modification of the construction of asymptotic sets on the unit sphere of ℓ_2 (initially due to E. Odell and T. Schlumprecht [OS1]). In Section 3, we use these asymptotic sets to construct the space X from Theorem 1.1. Furthermore, we establish that all infinite dimensional subquotients of X are completely indecomposable, and H is completely complementably finitely representable in all such subquotients. In Section 4 we prove that all infinite-dimensional subquotients of X fail the OAP. Finally, in Section 5 we show that any c.b. map on an infinite dimensional subquotient of X is a compact perturbation of a scalar multiple of the identity, provided H has property $(\mathcal{P})$.

2. ASYMPTOTIC SETS IN ℓ_2

First we recall some Banach space notions, to be used in this and subsequent sections. All spaces are presumed to be infinite dimensional, unless stated otherwise. For a space X, $\mathbf{B}_X = \{x \in X \mid \|x\| \leqslant 1\}$ and $\mathbf{S}_X = \{x \in X \mid \|x\| = 1\}$ stand for the unit ball and the unit sphere of X, respectively.

We say that a sequence $(\delta_i)_{i=1}^\infty$ is a *basis* in a Banach space X if for every $x \in X$ there exists a unique sequence of scalars (a_i) s.t. $x = \sum_{i=1}^\infty a_i \delta_i$. Equivalently (see e.g. Proposition 1.a.3 of [LT]), the projections $P_n \in B(X)$, defined via $P_n(\sum_{i=1}^\infty a_i\delta_i) = \sum_{i=1}^n a_i\delta_i$, are well defined, and $\sup_n \|P_n\| < \infty$. If E is a finite subset of $\mathbb{N}$, we write $E(\sum_{i=1}^\infty a_i\delta_i) = \sum_{i\in E} a_i\delta_i$. The *support* of $a = \sum_{i=1}^\infty a_i\delta_i$ (denoted by $\operatorname{supp} a$) is the set of $i \in \mathbb{N}$ for which $a_i \neq 0$.

If E and F are finite subsets of $\mathbb{N}$, we write $E < F$ if $\max E < \min F$. If a Banach space X has a basis $(\delta_i)_{i\in\mathbb{N}}$, we write $a < b$ $(a, b \in X)$ if $\operatorname{supp} a < \operatorname{supp} b$.

The basis $(\delta_i)_{i=1}^\infty$ is called 1-*subsymmetric* if $\| \sum_i a_i \delta_i \| = \| \sum_i \omega_i a_i \delta_{n_i} \|$ for any finite sequence (a_i), any (ω_i) with $|\omega_i| = 1$, and any increasing sequence $n_1 < n_2 < \dots$ (sometimes, the term "1-unconditional 1-subsymmetric" is used to describe bases with this property).

For $\mathcal{S}_1, \mathcal{S}_2 \subset X$, we set $\mathrm{dist}(\mathcal{S}_1, \mathcal{S}_2) = \inf\{\|x_1 - x_2\| \,|\, x_1 \in \mathcal{S}_1, x_2 \in \mathcal{S}_2\}$.

A set $A \subset X$ is called *asymptotic* if, for every infinite dimensional $Y \subset X$, $\mathrm{dist}(A, Y) = 0$. If $(\delta_i)_{i\in\mathbb{N}}$ is a 1-subsymmetric basis for X, we say that $A \subset X$ is *spreading* (*unconditional*) if, for any $\sum_{i=1}^\infty a_i \delta_i \in A$, we have $\sum_{i=1}^\infty a_i \delta_{n_i} \in A$ for any $n_1 < n_2 < \dots$ (resp. $\sum_{i=1}^\infty \omega_i a_i \delta_i \in A$ for any $|\omega_i| = 1$).

The idea of constructing a sequence of asymptotic sets, satisfying certain conditions, was used by E. Odell and T. Schlumprecht in [OS1] in order to prove that ℓ_p is distortable for $1 < p < \infty$. Below we prove a sharper version of one of their results.

Theorem 2.1. *Suppose $\varepsilon_1 > \varepsilon_2 > \dots$ is a sequence of positive numbers, and $(K_i)_{i=1}^\infty$ is a sequence of positive integers. Then there exists a sequence of asymptotic spreading unconditional sets $A_1, A_2, \dots$, consisting of unit vectors in ℓ_2 with finite support, such that*

$$\sum_{k=1}^{K_n} \big|\langle a, b_k\rangle\big|^2 < \varepsilon_m^2 \tag{2.1}$$

whenever $m < n$, $a \in A_m$, $b_1, \dots, b_{K_n} \in A_n$, and $b_1 < \dots < b_{K_n}$.

The Schlumprecht space S is essential for proving this theorem. Recall (see [GM, OS1, OS2, S]) that S has a 1-subsymmetric basis $(\delta_i)_{i=1}^\infty$, and

$$\| \sum_i a_i \delta_i \| = \sup \big\{ \sup_i |a_i|, \sup_{n \geqslant 2,\, E_1 < \dots < E_n} \frac{1}{\phi(n)} \sum_{j=1}^n \| \sum_{i \in E_j} a_i \delta_i \| \big\} \tag{2.2}$$

(here $\phi(t) = \log(t+1)$). Using the ideas of [OS1], we first present "nice" sets in S and its dual.

Lemma 2.2. *Suppose $\sigma_1 > \sigma_2 > \dots$ is a sequence of positive numbers, and $(K_i)_{i=1}^\infty$ is a sequence of positive integers. Then there exist spreading unconditional sets $B_1, B_2, \dots \subset \mathbf{S}_S$ and $B_1^*, B_2^*, \dots \subset \mathbf{B}_{S^*}$, consisting of vectors with finite support, such that:*

1. *B_n is asymptotic for every n.*
2. *$\big|\langle a, Eb\rangle\big| < \sigma_{\min\{m,n\}}$ if $a \in B_n$, $b \in B_m^*$, and $E \subset \mathbb{N}$.*
3. *For every $a \in B_m$ there exists $b \in B_m^*$ satisfying $|\langle a, b\rangle| > 1 - \sigma_m$.*
4. *Suppose $m < n$, $a \in B_m$, $b_1, \dots, b_{K_n} \in B_n^*$, $b_1 < \dots < b_{K_n}$, and $E_1 < \dots < E_{K_n}$. Then $\sum_{k=1}^{K_n} \big|\langle a, E_k b_k\rangle\big| < 2\sigma_m$.*

Sketch of the proof. We rely on the construction from Section 2 of [GM] (summarized in [OS1] as Lemma 3.3). There, T. Gowers and B. Maurey show the existence of a rapidly increasing sequence $p_k \nearrow \infty$, and a rapidly decreasing sequence $\sigma'_k \searrow 0$, with the following property: for $n \in \mathbb{N}$, define

$$B^*_n = \Big\{ \frac{1}{\phi(p_n)} \sum_{j=1}^{p_n} b_j \Big| b_j \in S^*,\ \|b_j\| = 1,\ b_1 < \ldots < b_{p_n} \Big\} \subset \mathbf{B}_{S^*},$$

and let B_n be the set of all $(\sum_{i=1}^{p_n} x_i)/\|\sum_{i=1}^{p_n} x_i\| \in \mathbf{S}_S$, where $(x_i)_{i=1}^{p_n}$ is a RIS sequence of length p_n, with constant $1+\sigma'_n$ (we do not reproduce the definition of RIS, as it is quite cumbersome, and is not really necessary here; suffices to say that above, $x_1 < x_2 < \ldots < x_{p_n}$). Then the sets B_n and B^*_n are unconditional and spreading, and the statements (1), (2), and (3) of the lemma hold. It remains to prove (4).

By passing to a subsequence, we can assume that $\phi(K_n p_n) < 2\phi(p_n)$ for every n (recall that $\phi(t) = \log(t+1)$). Suppose m, n, a, and $(b_k)_{k=1}^{K_n}$ are as in (4). The sets B_m and B^*_n are unconditional, hence it suffices to prove (4) when all the entries of a and (b_k) are non-negative, and $E_k = \operatorname{supp} b_k$ for each k. In this situation, we have to show that $\langle a, \sum_{k=1}^{K_n} b_k \rangle < 2\sigma_m$. By construction,

$$b_k = \frac{1}{\phi(p_n)} \sum_{j=1}^{p_n} b_{jk},$$

where $b_{jk} \in \mathbf{B}_{S^*}$ $(1 \leqslant j \leqslant p_n)$ are such that $b_{1k} < \ldots < b_{p_n k}$. By passing from b_{jk} to $E_k b_{jk}$ if necessary, we can assume that $\operatorname{supp} b_{jk} \subset \operatorname{supp} b_k$ for each j, hence

$$b_{11} < b_{21} < \ldots < b_{p_n 1} < b_{12} < \ldots < b_{p_n K_n}.$$

Let

$$\tilde{b} = \frac{1}{\phi(p_n K_n)} \sum_{k=1}^{K_n} \sum_{j=1}^{p_n} b_{jk} = \frac{\phi(p_n)}{\phi(p_n K_n)} \sum_{k=1}^{K_n} b_k.$$

By (2.2), $\|\tilde{b}\| \leqslant 1$, hence $\|\sum_{k=1}^{K_n} b_k\| \leqslant \phi(p_n K_n)/\phi(p_n) < 2$. Moreover, $a = \alpha \sum_{s=1}^{p_m} a_s$, where $\|a_s\| = 1$ for each s, $a_1 < a_2 < \ldots < a_{p_m}$, and $\alpha = \|\sum_{s=1}^{p_m} a_s\|$. By (2.2), $\alpha \leqslant \phi(p_m)/p_m$. By Lemma 5 of [GM] (and by the choice of sequences (p_n) and (σ'_n)), $\langle a, \tilde{b} \rangle \leqslant 2\alpha < \sigma_m$. Thus, $\langle a, \sum_{k=1}^{K_n} b_k \rangle < 2\sigma_m$, as desired. ■

Proof of Theorem 2.1. Below we view elements of S, S^*, and ℓ_2 as sequences (via the expansions with respect to the canonical bases of these spaces). Operations of multiplication etc. are defined pointwise.

Suppose $B_1, B^*_1, B_2, B^*_2, \ldots$ are as in the previous lemma, with $2\sigma_k/(1-\sigma_k) < \varepsilon_k$. Define A_k as the set of vectors $x \in \ell_2$ for which $|x|^2 = ab/\langle a, b \rangle$, with $a \in B_k$, $b \in B^*_k$, $a, b \geqslant 0$, and $\langle a, b \rangle > 1 - \sigma_k$. It follows from [OS1] that the sets A_k are asymptotic, spreading, and unconditional. To show (2.1), suppose $m < n$, and consider non-negative $x, y_1, \ldots, y_{K_n} \in \ell_2$ s.t. $x^2 = ab$ and $y_k^2 = a_k b_k$ with $a \in B_m$, $b \in B^*_m$, $a_k \in B_n$, $b_k \in B^*_n$ (for $1 \leqslant k \leqslant K_n$), and $y_1 < y_2 < \ldots < y_{K_n}$. Let $E_k = \operatorname{supp} y_k$. By

Cauchy-Schwartz Inequality,

$$\sum_k \langle x, y_k \rangle^2 = \sum_k \langle \sqrt{a}\sqrt{b}, \sqrt{E_k a_k}\sqrt{E_k b_k}\,\rangle^2 \leqslant \sum_k \langle a, E_k b_k \rangle \langle a_k, E_k b \rangle.$$

By the previous lemma, $\sum_k \langle a, E_k b_k \rangle < 2\sigma_m$, and $\langle a_k, E_k b \rangle < \sigma_m$. Therefore,

$$\sum_k \Big\langle \frac{x}{\|x\|}, \frac{y_k}{\|y_k\|} \Big\rangle^2 \leqslant \frac{2\sigma_m^2}{(1-\sigma_m)^2}.$$

This establishes (2.1). ■

3. Construction and basic properties of X

Construct a sequence of sets A_n as in Theorem 2.1, with $\varepsilon_n = 239^{-n}$ and $K_n = 10^n$. Let $(\delta_i)_{i=1}^N$ and $(\delta_i)_{i=1}^\infty$ be the canonical bases in ℓ_2^N and ℓ_2, respectively.

Denote by $\mathcal{U}$ the set of operators $U : \ell_2 \to \ell_2^{K_n}$ (n even) of the form

$$U\xi = \sum_{j=1}^{K_n} \langle \xi, f_j \rangle \delta_j \ \text{ with } f_1, \ldots, f_{K_n} \in A_n,\ f_1 < \ldots < f_{K_n},$$

or

$$U\xi = \frac{1}{\sqrt{2}} \sum_{j=1}^{K_n} \langle \xi, f_{j+K_n} + \varepsilon f_j \rangle \delta_j \ \text{ with } f_1 < \ldots < f_{2K_n},\ \varepsilon = \pm 1,$$

$$\text{and either } f_1, \ldots, f_{2K_n} \in A_n, \text{ or } f_1, \ldots, f_{K_n} \in A_n,\ f_{K_n+1}, \ldots, f_{2K_n} \in A_{n+2}$$

(in both cases, $\xi \in \ell_2$). Let (U_i) be a countable dense subset in $\mathcal{U}$ (that is, for every $U \in \mathcal{U}$ and every $\varepsilon > 0$ there exists $i \in \mathbb{N}$ s.t. the range spaces of U and U_i coincide, and $\|U - U_i\|_1 < \varepsilon$).

Denote by $\mathcal{W}$ the set of operators $W \in B(\ell_2)$ s.t. $W\xi = \sum_{j=1}^{K_n} \langle \xi, f_j \rangle \delta_j$ for $\xi \in \ell_2$, where n is odd, and $f_1 < \ldots < f_{K_n}$ belong to A_n.

Following [OR], fix a sequence $s_0 < s_1 < \ldots$ (increasing "sufficiently fast"), and define spaces $E_i = \mathrm{MIN}_{s_i}(\mathrm{MAX}_{s_{i-1}}(R_{n_i} \cap C_{n_i}))$, for which:

1. $n_i = 100^j$ for some $j = j(i) \in \mathbb{N}$, and moreover, for each $j \in \mathbb{N}$ the number 100^j occurs infinitely many times in the sequence (n_i).
2. For any operator $u : E_i^* \to E_j$, we have $\|u\|_1/5 \leqslant \|u\|_{cb} \leqslant \|u\|_1$ if $i = j$, $\|u\|_{cb} = \|u\|_2$ if $i \neq j$.
3. If, in addition, H has property $(\mathcal{P})$, then $\lim_{j\to\infty} \gamma_j / 100^j = 0$, where

   $$\gamma_j = \|id : \mathrm{MIN}_{s_{i-1}}(R_{100^j} + C_{100^j}) \to H_{100^j}\|_{cb},$$

   and i is the smallest integer satisfying $n_i = 100^j$ (or in other words, $i = \min\{k \,|\, j = j(k)\}$). Consequently, $\|id : E_i^* \to H_{100^{j(i)}}\|_{cb} \leqslant \gamma_j$ for any i.

Define the operator space X by setting, for $x \in \mathcal{K} \otimes \ell_2$,

$$\|x\|_{\mathcal{K}\otimes X} = \max\big\{ \|x\|_{\mathcal{K}\otimes \mathrm{MIN}(\ell_2)}, \sup_{i\in\mathbb{N}} \|(I_{\mathcal{K}} \otimes U_i)x\|_{\mathcal{K}\otimes E_i}, \sup_{W\in\mathcal{W}} \|(I_{\mathcal{K}} \otimes W)x\|_{\mathcal{K}\otimes H} \big\} \tag{3.1}$$

(recall that, for $x = \sum_i a_i \otimes \delta_i \in \mathcal{K} \otimes \mathrm{MIN}(\ell_2)$,

$$\|x\|_{\mathcal{K}\otimes\mathrm{MIN}(\ell_2)} = \sup\{\| \sum_i \alpha_i a_i \|_{\mathcal{K}} \mid \sum_i |\alpha_i|^2 \leqslant 1\}).$$

It is easy to check that X satisfies Ruan's axioms, hence it is an operator space. Also, X is isometric to ℓ_2. We shall show that it has all the desired properties. Start by showing that elements of $\mathcal{U}$ and $\mathcal{W}$ "ignore" each other.

Lemma 3.1. *If $U \in \mathcal{U}$ and $W \in \mathcal{W}$, then $\|UW^*\|_1 \leqslant 1$.*

Proof. It suffices to prove that $\|UV\|_1 \leqslant 1/2$ when $U \in B(\ell_2, \ell_2^{K_n})$ and $V \in B(\ell_2, \ell_2)$ are given by

$$U\xi = \sum_{j=1}^{K_m} \langle \xi, g_j \rangle \delta_j, \ \text{ and } \ V\delta_i = \left\{ \begin{array}{ll} f_i & i \leqslant K_n \\ 0 & i > K_n \end{array} \right. , \tag{3.2}$$

where $f_1 < \ldots < f_{K_n}$ belong to A_n, and $g_1 < \ldots < g_{K_m}$ belong to A_ℓ, for $\ell \geqslant m$, and $n \notin \{m, \ell\}$. Indeed, the adjoint of any element of $\mathcal{W}$ equals V as above, while any element of $\mathcal{U}$ either equals to a U of the above form, or can be represented as $(U_1 + U_2)/\sqrt{2}$, with U_1 and U_2 resembling U in (3.2). Note that, for U and V as in (3.2),

$$UV\delta_i = \left\{ \begin{array}{ll} \sum_{j=1}^{K_m} \langle f_i, g_j \rangle \delta_j & i \leqslant K_n \\ 0 & i > K_n \end{array} \right. ,$$

and therefore,

$$\|UV\|_2^2 = \sum_{i=1}^{K_n} \sum_{j=1}^{K_m} |\langle f_i, g_j \rangle|^2. \tag{3.3}$$

To estimate $\|UV\|_1$, suppose first that $n < \ell$. By construction of A_n and A_ℓ, $\sum_{j=1}^{K_m} |\langle f_i, g_j \rangle|^2 < \varepsilon_n^2$ for $1 \leqslant i \leqslant K_n$. Therefore, by (3.3), $\|UV\|_2^2 \leqslant K_n \varepsilon_n^2$. Moreover, $\operatorname{rank} UV \leqslant \operatorname{rank} U = K_n$, hence

$$\|UV\|_1 \leqslant \sqrt{\operatorname{rank} UV} \|UV\|_2 = K_n \varepsilon_n < \frac{1}{2},$$

by our choice of K_n and ε_n. If $n > \ell$, we similarly obtain $\|UV\|_1 < K_m \varepsilon_\ell \leqslant K_\ell \varepsilon_\ell < 1/2$ (we use the fact that $m \leqslant \ell$). ∎

We shall identify subquotients of X with subspaces of X (as linear spaces). More precisely, suppose $X'' \hookrightarrow X' \hookrightarrow X$. Then $Y = X/X''$ and $Y' = X'/X''$ are identified with $X \ominus X''$ and $X' \ominus X''$, respectively.

Proposition 3.2. *H is $(1 + \varepsilon)$-completely complementably finitely representable in any infinite dimensional subquotient of X.*

Proof. Fix an odd n, and consider $f_1, \ldots, f_{K_n} \in A_n$ such that $f_1 < \ldots < f_{K_n}$. Denote by $X_{\mathbf{f}}$ the span of $f_1, \ldots, f_{K_n}$ in X. We shall show that $X_{\mathbf{f}}$ is completely contractively complemented in X, and completely isometric to H_{K_n}. Indeed, there exists $W_0 \in \mathcal{W}$ s.t. $W_0 \xi = \sum_{j=1}^{K_n} \langle \xi, f_j \rangle \delta_j$ for $\xi \in X$. By (3.1), $\|W_0\|_{cb} = 1$.

Consider W_0^* as an operator $V : H \to X$. Then

$$\|V\|_{cb} = \max\big\{\|V\|_{CB(H, \mathrm{MIN}(\ell_2))}, \sup_{i \in \mathbb{N}} \|U_i V\|_{CB(H, E_i)}, \sup_{W \in \mathcal{W}} \|WV\|_{CB(H)}\big\}.$$

But $\|V\|_{CB(H, \mathrm{MIN}(\ell_2))} = \|V\| = 1$, $\|WV\|_{CB(H)} = \|WV\| \leqslant 1$, and $\|U_i V\|_{CB(H, E_i)} \leqslant \|U_i V\|_1 \leqslant 1$ by Lemma 3.1. Thus, both W_0 and V are complete contractions, hence $X_{\mathbf{f}}$ is completely isometric to H_{K_n}. Moreover, $P = V W_0$ is a completely contractive projection onto $X_{\mathbf{f}}$.

Now consider $Y' = X'/X''$ (with $X'' \hookrightarrow X' \hookrightarrow X$). By perturbing X' and X'' slightly, and identifying Y' with a subspace of X (as explained above), we can assume that $Y' \cap A_n$ contains $f_1 < \ldots < f_{K_n}$. Denote by Z the span of $f_1, \ldots, f_{K_n}$ in Y'. We claim that Z is completely isometric to H_{K_n}, and completely contractively complemented in Y'. Indeed, consider the orthogonal projection P from X onto Z. Above we have established that P is completely contractive as an operator on X. Therefore, for any $z \in \mathcal{K} \otimes Z$,

$$\begin{aligned} \|z\|_{\mathcal{K} \otimes X'} \geqslant \|z\|_{\mathcal{K} \otimes Y'} &= \inf\big\{\|z + x\|_{\mathcal{K} \otimes X'} \,|\, x \in \mathcal{K} \otimes X''\big\} \\ &\geqslant \inf\big\{\|(I_{\mathcal{K}} \otimes P)(z + x)\|_{\mathcal{K} \otimes X'} \,|\, x \in \mathcal{K} \otimes X''\big\} = \|z\|_{\mathcal{K} \otimes X'}, \end{aligned}$$

since $X'' \subset \ker P$. Thus, Z is completely isometric to the span of $f_1, f_2, \ldots, f_{K_n}$ in X', which, by the above, is completely isometric to H_{K_n}. Moreover, P (viewed as an operator on Y) is completely contractive. ■

The following result yields a useful lower estimate for c.b. norms of operators on X and its subquotients.

Proposition 3.3. *Suppose $X'' \hookrightarrow X' \hookrightarrow X$, and let Y and Y' are the quotient spaces X/X'' and X'/X'', respectively.*

(a) Consider the operators $T : Y' \to Y$, $U : Y \to \ell_2^{100^n}$, and $V : \ell_2^{100^n} \to Y'$, such that $U, V^* \in \mathcal{U}$. Then

$$\|T\|_{cb} \geqslant \frac{\|UTV\|_1}{5 \max\{10^n, \|UV\|_1\}}.$$

Consequently, $\|T\|_{cb} \geqslant \|UTV\|_1/(5 \cdot 10^n)$ whenever U and V as above satisfy $UV = 0$.

(b) Suppose H has property $(\mathcal{P})$, and consider the operators $T : Y' \to Y$, $U : Y \to \ell_2^{100^n}$, and $V : \ell_2^{100^n} \to Y'$, such that $U \in \mathcal{U}$. Then

$$\|T\|_{cb} \geqslant \frac{\|UTV\|_1}{5 \max\{10^n \|V\|, \gamma_n \|V\|, \|UV\|_1\}}.$$

For the proof, we need the following two lemmas. Below, X'', X', X'', Y', and Y are as in the statement of Proposition 3.3.

Lemma 3.4. *Suppose P is the orthogonal projection from X onto Y', and $U_i : X \to E_i$ is as in the definition of X. Then $\|U_i|_{Y'}\|_{CB(Y',E_i)} \leqslant 1 + 2\|U_i - U_iP\|_1$.*

Proof. Observe first that

$$\|U_iP\|_{CB(X,E_i)} \leqslant 1 + \|U_i - U_iP\|_{CB(X,E_i)} \leqslant 1 + \|U_i - U_iP\|_1.$$

Moreover, $\|U_iP\|_{CB(X,E_i)} \geqslant \|U_iP|_{Y'}\|_{CB(Y',E_i)}$. Indeed, suppose $y \in M_n \otimes Y'$ satisfies $\|y\|_{M_n\otimes Y'} < 1$. Then there exists $x \in M_n \otimes X$ such that $\|x\|_{M_n\otimes X} < 1$, and $I_{M_n} \otimes P(x) = y$. We conclude that

$$\|I_{M_n} \otimes U_iP(y)\|_{M_n\otimes E_i} = \|I_{M_n} \otimes U_iP(x)\|_{M_n\otimes E_i} < \|U_iP\|_{CB(X,E_i)}.$$

To finish the proof, note that $\|U_i|_{Y'}\|_{CB(Y',E_i)} \leqslant \|U_iP|_{Y'}\|_{CB(Y',E_i)} + \|U_i - U_iP\|_1$. ■

Lemma 3.5. *Suppose V as an operator from E_i^* to Y'. Then*

$$\|V\|_{CB(E_i^*,Y')} \leqslant \max\big\{\|U_iV\|_1, \|V\|_2, \sup_{W\in\mathcal{W}} \|WV\|_{cb}\big\}.$$

Consequently:

1. *If $V^* \in \mathcal{U}$, then $\|V\|_{CB(E_i^*,Y')} \leqslant \max\{\|U_iV\|_1, \|V\|_2\}$.*
2. *If H has property $(\mathcal{P})$ and $n_i = 100^k$, then*

$$\|V\|_{CB(E_i^*,Y')} \leqslant \max\big\{\|U_iV\|_1, \max\{\sqrt{n_i}, \gamma_k\}\|V\|\big\}.$$

Proof. Let $q : X' \to Y'$ is the complete quotient map. By (3.1),

$$\begin{aligned}\|V\|_{CB(E_i^*,Y')} &= \|qV\|_{CB(E_i^*,Y')} \leqslant \|V\|_{CB(E_i^*,X)} \\ &= \max\big\{\|V\|_{CB(E_i^*,\mathrm{MIN}(\ell_2))}, \sup_{j\in\mathbb{N}} \|U_jV\|_{CB(E_i^*,E_j)}, \sup_{W\in\mathcal{W}} \|WV\|_{CB(E_i^*,H)}\big\}.\end{aligned}$$

However, $\|V\|_{CB(E_i^*,\mathrm{MIN}(\ell_2))} = \|V\|$, $\|U_iV\|_{cb} \leqslant \|U_iV\|_1$, while $\|U_jV\|_{cb} = \|U_jV\|_2 \leqslant \|V\|_2$ for $j \neq i$. If $V^* \in \mathcal{U}$, then, by Lemma 3.1, $\|WV\|_{cb} \leqslant \|WV\|_1 \leqslant 1$. If H has property $(\mathcal{P})$ and $n_i = 100^k$, then $\|WV\|_{cb} \leqslant \gamma_k\|V\|$. ■

Proof of Proposition 3.3. We observe that, for any $i \in \mathbb{N}$,

$$\|T\|_{cb} \geqslant \frac{\|U_iTV\|_{CB(E_i^*,E_i)}}{\|U_i|_Y\|_{CB(Y,E_i)}\|V\|_{CB(E_i^*,Y')}} \geqslant \frac{\|U_iTV\|_1}{5\|U_i|_Y\|_{CB(Y,E_i)}\|V\|_{CB(E_i^*,Y')}}.$$

Approximating U with operators U_i, and using estimates for $\|U_i\|_{cb}$ and $\|V\|_{cb}$ obtained in Lemmas 3.4 and 3.5, we achieve the result. ■

Corollary 3.6. *Any infinite dimensional subquotient of X is completely indecomposable.*

Proof. Suppose P is a projection on $Y' = X'/X''$ (here, $X'' \hookrightarrow X' \hookrightarrow X$), and both the range and the kernel of P are infinite dimensional. The sets A_n involved in the construction of X are asymptotic, and therefore, by a small perturbation argument, we can assume that for any even n there exist $f_1, \ldots, f_{2K_n} \in A_n \cap Y'$ s.t. $f_1 < \ldots < f_{2K_n}$, and

$$Pf_j = \begin{cases} f_j & j \leqslant K_n \\ 0 & j > K_n \end{cases}.$$

Consider the operators $U, V \in B(X, \ell_2^{K_n})$, defined by

$$U\xi = \frac{1}{\sqrt{2}} \sum_{s=1}^{K_n} \langle \eta, f_{s+K_n} - f_s \rangle \delta_s, \ V\xi = \frac{1}{\sqrt{2}} \sum_{s=1}^{K_n} \langle \eta, f_{s+K_n} + f_s \rangle \delta_s \ (\xi \in \ell_2).$$

Then $U, V \in \mathcal{U}$, and $UV^* = 0$. Therefore, by Proposition 3.3,

$$\|P\|_{cb} \geqslant \frac{\|UPV^*\|_1}{5 \cdot 10^{n/2}} = \frac{10^n/2}{5 \cdot 10^{n/2}} = 10^{n/2-1}.$$

The even integer n can be arbitrarily large, hence P is not completely bounded. ■

4. Subquotients of X fail the OAP

As in the previous section, we assume that $X'' \hookrightarrow X' \hookrightarrow X$, and $Y' = X'/X''$ is infinite dimensional. We establish

Theorem 4.1. *Y' fails the Compact Operator Approximation Property.*

Our main tool is

Lemma 4.2. *Suppose Z is an operator space with the Compact Operator Approximation Property, $(Z_i)_{i=0}^{\infty}$ a sequence of finite dimensional subspaces of Z, $(F_i)_{i=1}^{\infty}$ a sequence of 1-exact operator spaces, and the function $f : \mathbb{N} \to (2, \infty)$ is such that $\lim_{n\to\infty} f(n) = \infty$. Then there exists a compact operator $\psi : Z \to Z$ such that $\psi|_{Z_0} = I_{Z_0}$, and $\|u_i \psi|_{Z_i}\|_{cb} \leqslant f(i)\|u_i\|_{cb}$ for any $i \in \mathbb{N}$ and $u_i : Z \to F_i$.*

We omit the proof, as it is identical to the proof of Lemma 6.1 of [OR].

Proof of Theorem 4.1. By a small perturbation argument, we may assume that Y' contains vectors f_{ij} ($j \in \mathbb{N}$, $1 \leqslant i \leqslant K_{2j}$) with finite support such that $f_{ij} \in A_{2j}$, and $f_{ij} < f_{k\ell}$ if $j < \ell$, or $j = \ell$ and $i < k$. For every $j \in \mathbb{N}$, $1 \leqslant m \leqslant 100$, and $\varepsilon = \pm 1$, define operators $A_{j,m,\varepsilon} : Y' \to \ell_2^{K_{2j}}$ and $B_{j,m,\varepsilon} : \ell_2^{K_{2j}} \to Y'$ by setting $m' = K_{2j}(m-1)$,

$$B_{j,m,\varepsilon}\delta_{ij} = \frac{1}{\sqrt{2}}\big(f_{ij} - \varepsilon f_{m'+i+1,j+1}\big) \text{ for } 1 \leqslant i \leqslant 100^j$$

($(\delta_{ij})_{i=1}^{K_{2j}}$ is the canonical basis of $\ell_2^{K_{2j}}$), and

$$A_{j,m,\varepsilon}\xi = \frac{1}{\sqrt{2}} \sum_{i=1}^{100^j} \langle \xi, f_{ij} + \varepsilon f_{m'+i+1,j+1} \rangle \delta_i \text{ for } \xi \in Y'.$$

We can assume that, for every triple (j, m, ε) as above, there exists $s = s(j, m, \varepsilon) \in \mathbb{N}$ for which $\dim E_s = K_{2j}$, and $U_s = A_{j,m,\varepsilon}$ (here, we identify E_s with $\ell_2^{K_{2j}}$).

Suppose, for the sake of contradiction, that Y' has the COAP. By Lemma 4.2, there exists a compact operator $\psi : Y' \to Y'$ such that $\psi f_{i,3} = f_{i,3}$ for $1 \leqslant i \leqslant 100^3$, and

$$\|A_{j,m,\varepsilon}\psi B_{j,m,\varepsilon}\|_{cb} \leqslant j\|A_{j,m,\varepsilon}\|_{cb}\|B_{j,m,\varepsilon}\|_{cb} \text{ for } j \geqslant 3,\ 1 \leqslant m \leqslant 100,\ \varepsilon = \pm 1,$$

with $A_{j,m,\varepsilon}$ and $B_{j,m,\varepsilon}$ viewed as elements of $CB(Y', E_{s(j,m,\varepsilon)})$ and $CB(E^*_{s(j,m,\varepsilon)}, Y')$, respectively. However, $\|A_{j,m,\varepsilon}\|_{cb} \leqslant 1$, and $\|B_{j,m,\varepsilon}\|_{cb} \leqslant \sqrt{K_{2j}} = 10^j$ (by Lemma 3.4 and Lemma 3.5, respectively). Thus, we have

$$\|A_{j,m,\varepsilon}\psi B_{j,m,\varepsilon}\|_{CB(E^*_{s(j,m,\varepsilon)}, E_{s(j,m,\varepsilon)})} \leqslant j \cdot 10^j$$

for any appropriate triple (j, m, ε). By the basic properties of spaces E_i, we have

$$\mathrm{Re}\,\big(\mathrm{tr}(A_{j,m,\varepsilon}\psi B_{j,m,\varepsilon})\big) \leqslant \|A_{j,m,\varepsilon}\psi B_{j,m,\varepsilon}\|_1 \leqslant 5j \cdot 10^j.$$

An easy computation shows that

$$\mathrm{tr}(A_{j,m,\varepsilon}\psi B_{j,m,\varepsilon}) = \frac{1}{2}\sum_{i=1}^{K_{2j}} \langle \psi(f_{ij} - \varepsilon f_{m'+i+1,j+1}), f_{ij} + \varepsilon f_{m'+i+1,j+1}\rangle.$$

Therefore,

$$\begin{aligned}&\mathrm{Re}\,\big(\mathrm{tr}(A_{j,m,1}\psi B_{j,m,1} + A_{j,m,-1}\psi B_{j,m,-1})\big)\\ &= \mathrm{Re}\,\Big(\sum_{i=1}^{K_{2j}} \big(\langle \psi(f_{ij}), f_{ij}\rangle - \langle \psi(f_{m'+i+1,j+1}), f_{m'+i+1,j+1}\rangle\big)\Big) \leqslant 10^{j+1}j.\end{aligned}$$

Consequently,

$$\mathrm{Re}\,\Big(\sum_{i=1}^{K_{2j}} \langle \psi(f_{m'+i+1,j+1}), f_{m'+i+1,j+1}\rangle\Big) \geqslant \mathrm{Re}\,\Big(\sum_{i=1}^{K_{2j}} \langle \psi(f_{ij}), f_{ij}\rangle\Big) - 2 \cdot 10^{j+1}j.$$

Summing over all values of m ($1 \leqslant m \leqslant 100$), we obtain

$$S_{j+1} \geqslant 100(S_j - 2 \cdot 10^{j+1}j), \tag{4.1}$$

where $S_j = \mathrm{Re}\sum_{i=1}^{100^j} \langle \psi(f_{ij}), f_{ij}\rangle$. This allows us to show by induction that

$$S_j > \frac{j+1}{2j}100^j > \frac{100^j}{2} \tag{4.2}$$

whenever $j \geqslant 3$. Indeed, $\psi(f_{i,3}) = f_{i,3}$ for $1 \leqslant i \leqslant 100^3$, hence $S_3 = 100^3$. Assuming (4.2) holds for some $j \geqslant 3$, observe that

$$\frac{2 \cdot 10^{1+j}j}{S_j} < 10^{2-j}j < \frac{1}{j+2},$$

hence, by (4.1),

$$S_{j+1} \geqslant 100S_j\Big(1 - \frac{2 \cdot 10^{j+1}j}{S_j}\Big) > \frac{j+1}{2j}100^{j+1}\Big(1 - \frac{1}{j+2}\Big) = \frac{j+2}{2(j+1)}100^{j+1}.$$

This proves (4.2) for $j+1$.

On the other hand, ψ is compact, hence $\max_{1 \leqslant i \leqslant K_{2j}} \|\psi(f_{ij})\| < 1/2$ when j is sufficiently large. For such j, $S_j < 100^j/2$. This contradicts (4.2). ∎

As a corollary, we prove:

Corollary 4.3. *In the above notation, the spaces Y' and Y'^* are not exact.*

For the proof, we need a non-commutative analogue of the notion of a basis. We say that a sequence (x_i) in an operator space X is *C-completely basic* if it is a basis in $Y = \mathrm{span}[x_i \,|\, i \in \mathbb{N}]$, and moreover, the basis projections $P_n \in CB(Y)$ (defined by setting $P_n x_i = x_i$ if $i \leqslant n$, and $P_n x_i = 0$ if $i > n$) satisfy $\sup_n \|P_n\|_{cb} \leqslant C$. In this setting, $Y = \mathrm{span}[x_i \,|\, i \in \mathbb{N}]$ clearly has the OAP. Therefore, Corollary 4.3 is proved by combining Theorem 4.1 with

Lemma 4.4. *Suppose Z is an infinite-dimensional λ-exact operator space. Then Z contains a C-completely basic sequence for any $C > \lambda$.*

Proof. We select a C-completely basic sequence $(z_i) \subset Z$ inductively. More precisely, we select linearly independent vectors $z_1, z_2, \ldots \in Z$, finite codimensional subspaces $\ldots \hookrightarrow Z_2 \hookrightarrow Z_1 \hookrightarrow Z$, and finite rank projections $P_n \in CB(Z_n)$ such that, for any n, $z_1, \ldots, z_n \in Z_n$, $\mathrm{ran}\, P_n = \mathrm{span}[z_1, \ldots, z_n]$, $\|P_n\|_{cb} < C$, and $P_m z_n = 0$ whenever $m < n$ (then the operators $P_n|_{\mathrm{span}[z_k \,|\, k \in \mathbb{N}]}$ play the role of basis projections).

First pick an arbitrary non-zero $z_1 \in Z$. By Hahn-Banach Theorem, there exists a contractive projection P_1 onto $E_1 = \mathrm{span}[z_1]$. Moreover, P_1 has rank 1, hence it is completely contractive. Let $Z_1 = Z$.

Now suppose we have selected $z_1, \ldots, z_n$, $Z_1, \ldots, Z_n$, and $P_1, \ldots, P_n$, as above. Pick an arbitrary non-zero $z_{n+1} \in Z_n \cap (\cap_{m=1}^n \ker P_m)$. Let $E = \mathrm{span}[z_1, \ldots, z_{n+1}]$. Find $F \hookrightarrow M_N$ and $u : E \to F$ s.t. $\|u\|_{cb} = 1$, $\|u^{-1}\|_{cb} < C$. By Arveson-Wittstock-Stinespring-Paulsen extension theorem, there exists $\tilde{u} : Z_n \to M_N$ s.t. $\tilde{u}|_E = u$, and $\|\tilde{u}\|_{cb} = 1$. Let $Z_{n+1} = \mathrm{span}[E, \ker \tilde{u}] \hookrightarrow Z_n$, and note that $\dim Z_n / \ker \tilde{u} \leqslant \dim M_N < \infty$, hence $\dim Z_n / Z_{n+1} < \infty$. Furthermore, $\tilde{u}(Z_{n+1}) \subset F$. It is easy to see that $P_{n+1} = u^{-1} \tilde{u}|_{Z_{n+1}}$ is a projection from Z_{n+1} onto $\mathrm{span}[z_1, \ldots, z_{n+1}]$, with $\|P_{n+1}\|_{cb} < C$. Moreover, $P_m z_{n+1} = 0$ for $m \leqslant n$. ∎

5. Completely bounded maps on subquotients of X

In this section, we assume that H has property $(\mathcal{P})$, $X'' \hookrightarrow X' \hookrightarrow X$, $Y = X/X''$, and $Y' = X'/X''$ is infinite dimensional. We denote by $J_{Y'}$ the natural embedding of Y' into Y. We show:

Theorem 5.1. *Any completely bounded operator $S : Y' \to Y$ is of the form $S = cJ_{Y'} + S'$, where $c \in \mathbb{C}$ and S' is compact.*

For the proof, we need the following proposition (it may be known to specialists).

Proposition 5.2. *Suppose Z' is a subspace of a Hilbert space Z, and $T \in B(Z', Z)$. Then either T is a compact perturbation of a scalar multiple of J (the natural embedding of Z' into Z), or there exist mutually orthogonal projections of infinite rank $P \in B(Z')$, $Q \in B(Z)$ such that $QT|_{\operatorname{ran} P} \in B(\operatorname{ran} P, \operatorname{ran} Q)$ is invertible.*

Proof. First denote by Q_0 the orthogonal projection in $B(Z)$ whose kernel equals Z'. If there are no infinite rank projections P and Q s.t. $\operatorname{ran} Q \subset \operatorname{ran} Q_0$ and $QT|_{\operatorname{ran} P}$ is invertible, then $Q_0 T$ is compact. This reduces the problem to the case of $Z' = Z$.

We denote by $\mathcal{K}(H)$ the space of compact operators on H. We shall show that, if $c = \operatorname{dist}(T, \mathbb{C}I_Z + \mathcal{K}(Z)) > 0$, then there exist mutually orthogonal projections P and Q of infinite rank s.t. $QT|_{\operatorname{ran} P} \in B(\operatorname{ran} P, \operatorname{ran} Q)$ is invertible.

Note that $\operatorname{dist}(RTR, \mathbb{C}R + \mathcal{K}(\operatorname{ran} R)) = c$ for any orthogonal projection $R \in B(Z)$ with finite dimensional kernel. By Theorem 9.12 of [D], for such an R there exist mutually orthogonal norm 1 vectors $\xi(R), \eta(R) \in \operatorname{ran} R$ s.t. $\langle T\xi(R), \eta(R)\rangle > c/3$. This allows us to construct inductively vectors $(\xi_n)_{n\in\mathbb{N}}$ and $(\eta_n)_{n\in\mathbb{N}}$ in Z, such that, for any k, j,

$$\langle \xi_k, \eta_j\rangle = 0, \ \langle \xi_k, \xi_j\rangle = \langle \eta_k, \eta_j\rangle = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}, \ \langle T\xi_k, \eta_j\rangle \begin{cases} > c/3 & k = j \\ = 0 & k \neq j \end{cases}. \tag{5.1}$$

Indeed, let $R_1 = I_Z$, $\xi_1 = \xi(R_1)$, and $\eta_1 = \eta(R_1)$. Suppose $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ have already been selected in such a way that (5.1) holds whenever $j, k \leqslant n$. Let R_{n+1} be the orthogonal projection whose kernel is spanned by $(\xi_i)_{i=1}^n$, $(\eta_i)_{i=1}^n$, $(T\xi_i)_{i=1}^n$, and $(T^*\eta_i)_{i=1}^n$. Let $\xi_{n+1} = \xi(R_{n+1})$, $\eta_{n+1} = \eta(R_{n+1})$, and observe that now (5.1) holds for all $j, k \leqslant n+1$.

Denote by Q and P the orthogonal projections from Z onto $\operatorname{span}[\eta_n \mid n \in \mathbb{N}]$ and $\operatorname{span}[\xi_n \mid n \in \mathbb{N}]$, respectively. By the above, $QT|_{\operatorname{ran} P}$ is invertible. ■

Proof of Theorem 5.1. Suppose $T : Y' \to Y$ is not a compact perturbation of $J_{Y'}$. We shall show T is not completely bounded. By Proposition 5.2, there exist mutually orthogonal projections P and Q of infinite rank s.t. $\|QT\xi\| \geqslant \|\xi\|/C$ for any $\xi \in \operatorname{ran} P$ ($C > 0$). By a small perturbation argument, assume the existence of $f_1 < \ldots < f_{K_n}$ in $\operatorname{ran} Q \cap A_n$ (n even). Consider $U \in \mathcal{U}$ which sends f_j into δ_j ($1 \leqslant j \leqslant K_n$), and annihilates $\operatorname{span}[f_1, \ldots, f_{K_n}]^\perp$. Define $V : \ell_2^{K_n} \to \operatorname{ran} P \hookrightarrow Y'$ by setting $V\delta_j = (QT)^{-1} f_j$ (once again, $1 \leqslant j \leqslant K_n$). Then $\|V\| \leqslant C$, $UV = 0$, and UTV is the identity on $\ell_2^{K_n}$. Applying Lemma 3.3, we conclude that

$$\|T\|_{cb} \geqslant \frac{100^n}{5C \max\{\gamma_n, 10^n\}}.$$

n can be chosen to be arbitrarily large, hence T is not completely bounded. ■

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