Operator spaces with few completely bounded maps

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Abstract

We construct several examples of Hilbertian operator spaces with few completely bounded maps. In particular, we give an example of a separable 1-Hilbertian operator space X_0 such that, whenever X' is an infinite dimensional quotient of X_0, X is a subspace of X', and $T : X \to X'$ is a completely bounded map, then $T = \lambda I_X + S$, where S is compact Hilbert-Schmidt and $||S||_2/16 \leq ||S||_{cb} \leq ||S||_2$. Moreover, every infinite dimensional quotient of a subspace of X_0 fails the operator approximation property.

We also show that every Banach space can be equipped with an operator space structure without the operator approximation property.

1 Introduction and main results

Recall that a Banach space (or an operator space) X is said to have the *bounded* (resp. *completely bounded*) *approximation property* (*BAP*, *CBAP*) if there exists a net of finite rank operators T_i which converges to the identity

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on X point-norm and $\sup_i ||T_i|| < \infty$ (resp. $\sup_i ||T_i||_{cb} < \infty$). It is known that, while an infinite dimensional Banach space may fail the approximation property, it must have an infinite dimensional subspace possessing a basis (and, consequently, having a bounded approximation property).

Following [7], we say that a net of operators $(u_i) \subset CB(X)$ (X is an operator space) converges to $u \in CB(X)$ in the *point-norm topology* if $\lim(u_i \otimes I_{\mathcal{K}})(x) = (u \otimes I_{\mathcal{K}})(x)$ for any $x \in X \otimes \mathcal{K}$. X is said to have the *operator approximation property* (resp. *compact operator approximation property*) (abbreviated as *OAP* and *COAP*) if there exists a net of finite rank maps (resp. completely bounded compact maps) on X converging to I_X (the identity on X) in the point-norm topology.

The main result of this paper is the construction of an operator space X_0 which is isometric to ℓ_2, and such that every c.b. operator on any $X \hookrightarrow X_0/W$ is a linear combination of the identity on X (denoted by I_X) and a Hilbert-Schmidt operator. The construction is based on an original unpublished idea of the first named author [25]. Moreover, every infinite dimensional subspace X of a quotient of X_0 fails the COAP.

Note that we do not know of any Banach space X such that every $T \in B(X,X)$ is a sum of a scalar and a compact. However, [12] provides an example of X such that every $T \in B(X,X)$ is a sum of a scalar and a strictly singular operator.

Below we use standard Banach space and operator space terminology and results. For these, we refer the reader to [22] (Banach spaces) and [7], [30] (operator spaces). If H and K are Hilbert spaces and $T : H \to K$ is a linear map, denote by $s_n(T)$ the n-th singular value of T (we assume that $s_1(T) \geq s_2(T) \geq \ldots \geq 0$). For $1 \leq p \leq \infty$, set $||T||_p = (\sum_n s_n(T)^p)^{1/p}$. We say that $A \prec B$ or $A \succ B$ (here, A and B are functions of several parameters) if there exists an absolute constant c s.t. $A \leq cB$ (resp. $B \leq cA$). We say that $A \sim B$ if $A \prec B$ and $A \succ B$. Recall also that two operator spaces Y and Z are said to be totally completely incomparable if no infinite dimensional subspace of Y is completely isomorphic to any subspace of Z.

We now state the main result of this paper.

Theorem 1.1 *There exists an operator space X_0 which is isometric to ℓ_2, and such that for any $W \hookrightarrow X_0$, and any subspace X of the quotient $X' = X_0/W$ we have:*

1. *If $T : X \to X'$ is a completely bounded bounded map, then $T = \lambda I_X + S$, where $\lambda \in \mathbb{C}$ and S is a compact operator and $||S||_2/16 \leq ||S||_{cb} \leq$*

$||S||_2$. Consequently, if X is completely isomorphic to $Y \oplus_\infty Z$, then either Y or Z is finite dimensional.

In addition, if X is infinite dimensional then we have

2. *Any finite n-dimensional subspace E of X or X^* satisfies $ex(E) \geq \sqrt{n}/16$.*

3. *The space X fails the compact operator approximation property.*

4. *The space X contains uncountably many totally completely incomparable subspaces Y such that $CB(Y) = \mathbb{C}Id + S_2$ isomorphically.*

5. *No proper subspace of X is completely isomorphic to X.*

Remark 1 In [2], A. Arias constructed an example of a separable 1-Hilbertian operator space failing the OAP. However, the space constructed by Arias contains infinite dimensional subspaces possessing the CBAP.

Remark 2 The Banach space analogue of Part *4* is due to I. Gasparis in [8].

Remark 3 Theorem 1.1 is optimal in the following way ; S_2 can't be replaced by any smaller space (see Theorem 1.7). In fact, any completely bounded endomorphisms of X_0 (or X) is the sum of a multiple of the identity and a nuclear operator in the operator space sense. Moreover, the space X_0 does not have any completely basic sequence.

In [13] (see also [11]), T. Gowers and B. Maurey constructed a Banach space with an unconditional basis, every operator on which is a sum of a strictly singular and a diagonal operators. A similar result can be proved in the non-commutative setting. We call a basis (e_i) in an operator space E *complete* if the basis projections are uniformly completely bounded. The basis (e_i) is called *M-complete* if the basis projections have c.b. norms not exceeding M, and *M-completely unconditional* if, for any sequences $a_i \in \mathcal{K}$ and $\omega_i \in \mathbb{C}$ with $\sup_i |\omega_i| \leq 1$, we have $||\sum e_i \otimes a_i|| \geq ||\sum e_i \otimes \omega_i a_i||$. E is said to have a *local complete basis structure* if there exists $M > 0$ and a sequence of finite dimensional spaces $E_1 \hookrightarrow E_2 \hookrightarrow \ldots \hookrightarrow E$ s.t. E_k has a M-complete basis, and $E = \overline{\cup_i E_i}$.

Theorem 1.2 *There exists an operator space X which is isometric to ℓ_2, has a 1-completely unconditional basis, and such that every completely bounded* *map on X is a sum of a diagonal and Hilbert-Schmidt operators. Moreover, if X' is a block subspace of X, then every completely bounded map on X' is a sum of a diagonal and Hilbert-Schmidt operators.*

In [38], S. Szarek used some ideas of E. Gluskin to construct a Banach space without a basis, but with the Bounded Approximation Property. Once again, this result has a non-commutative analogue for an Hilbertian operator space structure (otherwise, just take the minimal operator space structure on Szarek's space). We shall say that an operator space X has *M-completely unconditional finite dimensional decomposition* (*FDD*) if there exist finite dimensional subspaces $E_i \hookrightarrow X$ and projections P_i from X onto E_i s.t. $||\sum_i \omega_i P_i||_{cb} \leq M \sup_i |\omega_i|$ for any finite sequence (ω_i), $P_i P_j = 0$ for $i \neq j$, and $X = \overline{\text{span}}[E_1, E_2, \ldots]$.

Theorem 1.3 *There exists an operator space X, isometric to ℓ_2, and having 1-completely unconditional FDD, but no local complete basis structure.*

In [23], P. Mankiewicz constructed a superreflexive Banach space with infinitely many multiplicative functionals. A non-commutative analogue of this result can be obtained.

Theorem 1.4

- *For any $n \in \mathbb{N}$ there exists an operator space X isometric to ℓ_2 such that there are exactly n non-zero multiplicative functionals on $CB(X)$.*
- *There exists an operator space X isometric to ℓ_2 and such that there is an uncountable family I of multiplicative functionals $(\phi_i)_{i \in I}$ on $CB(X)$ for which $||\phi_i|| = 1$ and $||\phi_i - \phi_j|| \geq 1$ whenever $i \neq j$.*

A Banach space X is said to have the *Schroeder-Bernstein property* if, whenever Y is isomorphic to a complemented subspace of X and vice versa, then X is isomorphic Y. The first example of a Banach space failing this property was constructed by Gowers and Maurey in [13]. Below we present a similar result for Hilbertian operator spaces.

Theorem 1.5 *There exist operator spaces X and Y, isometric to ℓ_2, such that X is completely isomorphic to a completely complemented subspace of Y, and Y is completely isomorphic to a completely complemented subspace of X, but X is not completely isomorphic to Y.*

It has been long known (see e.g. [22], Theorem 1.e.15) that any separable dual Banach space with the approximation property also possesses the *metric approximation property*. This is not true in the operator space case.

Theorem 1.6 *For $n = 2, 3, \dots$ there exists an operator space X_n such that*

1. *X_n is isometric to ℓ_2 and $\sqrt{2n}$-completely isomorphic to* $\mathrm{MIN}(\ell_2)$.

2. *If X is a subspace of a quotient of X_n and E and F are mutually orthogonal n-dimensional subspaces of X and $T : \mathrm{span}[E, F] \to \mathrm{span}[E, F]$ is such that $T|_E = I_E$ and $T|_F = 0$, then $||T||_{cb} \geq n^{1/2}/8$.*

   Consequently, if X is infinite dimensional, and if X' is an infinite dimensional subspace X and $T_i : X' \to X$ is a net of compact completely bounded maps and $\lim_i T_i = I_{X'}$ point-norm, then $\sup_i ||T_i||_{cb} \geq n^{1/2}/8$.

In particular, the space $\ell_2((X_n)_{n\geq 2})$ is an Hilbertian operator space with the OAP but failing the CBAP.

Above, $\ell_2((X_n)_{n\geq 2})$ denotes the space $\big((\oplus_{n\geq 2} X_n)_{c_0}, (\oplus_{n\geq 2} X_n)_{\ell_1}\big)_{1/2}$ (cf. [35]).

We give another variation of Theorem 1.1.

Consider a symmetric sequence space V. We say that V has property $\mathcal{P}$ if, for any $e \in \ell_\infty$, $\lim_n \|P_n e\|_E < \infty$ implies $e \in V$, and $\|e\|_V = \lim_n \|P_n e\|_V$ where P_n is the projection onto the n-first coordinates. We denote by S_V the space of operators on ℓ_2 whose sequences of singular values belong to V (with the obvious norm).

Theorem 1.7 *Suppose V is a symmetric sequence space. The following are equivalent:*

1. *There exists an operator space X_0 such that whenever X' is an infinite dimensional quotient of X_0 and X is an infinite dimensional subspace of X', then, for any $T : X \to X'$ completely bounded, we have a decomposition $T = f(T)Id_{X'} + S$, where S is compact, and $\|S\|_V \sim \|S\|_{cb}$*

2. *V has property $\mathcal{P}$, and the natural embedding of ℓ_2 into V is continuous.*

Above, we have dealt with a panopticum of "pathological" operator space structures on ℓ_2. It is possible to extend some of these bad properties on arbitrary Banach spaces.

Theorem 1.8 *Let X be an infinite dimensional Banach space, then there exists an operator space structure on X failing the OAP. Moreover, if X is a dual Banach space, this operator space structure can also be a dual one.*

In Section 2, we introduce the functors MIN_n and MAX_n and establish some basic facts about them. In Section 3, we construct many different operator space structures on ℓ_2^n using these functors. Section 4 is devoted to the construction of the space appearing in Theorem 1.1. In Section 5, we modify this example to prove Theorems 1.2 to 1.7. The last section deals with Theorem 1.8.

2 The functors MIN_n and MAX_n

Below we follow [7] and [30] in using standard operator space conventions and notations. We denote by M_n the space of $n \times n$ matrices, and by $E_{i,j}$ the standard matrix units. The latter is considered to be embedded into the space $\mathcal{K}$ of compact operators on ℓ_2 "in the natural way". Similarly, $\mathcal{K}_0$ denotes the set of compact operators with finitely many entries, and is viewed as a (non-closed) subspace of $\mathcal{K}$. Unless noted otherwise, $\otimes$ denotes the minimal (injective) tensor product of operator spaces.

If E is a Banach space, $e \in E$ and $f \in E^*$, we denote the action of f on e by $\langle f, e\rangle$. We identify a Hilbert space with its own dual. If $x = \sum e_i \otimes a_i \in E \otimes \mathcal{K}$ and $y = \sum f_j \otimes b_j \in E^* \otimes \mathcal{K}$, we set

$$\langle y, x\rangle \stackrel{\text{def}}{=} \sum_{ij} \langle f_j, e_i\rangle b_j \otimes a_i \in \mathcal{K} \otimes \mathcal{K} \simeq \mathcal{K}.$$

If E is an operator space and $n \in \mathbb{N}$, we denote by $\mathrm{MIN}_n(E)$ and $\mathrm{MAX}_n(E)$ the operator spaces which are isometric to E as Banach spaces, and have operator space structures such that, for $x \in E \otimes \mathcal{K}_0$,

$$\begin{aligned} ||x||_{\mathrm{MIN}_n(E)\otimes\mathcal{K}_0} &\stackrel{\text{def}}{=} \sup\{||(u \otimes I_{\mathcal{K}_0})x|| \,;\, u \in CB(E, M_n),\ ||u||_{cb} \le 1\}; \\ ||x||_{\mathrm{MAX}_n(E)\otimes\mathcal{K}_0} &\stackrel{\text{def}}{=} \sup\{||(u \otimes I_{\mathcal{K}_0})x|| \,;\, u \in B(E, B(H)),\ ||u \otimes I_{M_n}|| \le 1\}. \end{aligned}$$

Note that $\mathrm{MIN}_1(E) = \mathrm{MIN}(E)$ and $\mathrm{MAX}_1(E) = \mathrm{MAX}(E)$ (the minimal and maximal quantizations of E, discussed thoroughly in [28] and [29]) completely isometrically.

The spaces $\mathrm{MIN}_n(E)$ and $\mathrm{MAX}_n(E)$ were mentioned in [17] (defined in a slightly different way), and more generally studied in [21].

Before proceeding further, we need to collect some informations about these spaces. Note first that the formal identity maps $\mathrm{MAX}_n(E) \to E \to \mathrm{MIN}_n(E)$ are completely contractive.

The first basic property that we will need is that $\mathrm{MIN}_n(E)$ and $\mathrm{MAX}_n(E)$ approximate E in a weak sense :

Lemma 2.1 *Let E be an operator space, then the formal identity maps*

$$I_E \otimes I_{M_n} : \mathrm{MIN}_n(E) \otimes M_n \to E \otimes M_n \to \mathrm{MAX}_n(E) \otimes M_n$$

are isometries.

Thus for any $x \in \mathcal{K} \otimes E$,

$$\|x\|_{\mathcal{K}\otimes E} = \lim_{n\to\infty} \|x\|_{\mathcal{K}\otimes \mathrm{MIN}_n(E)} = \lim_{n\to\infty} \|x\|_{\mathcal{K}\otimes \mathrm{MAX}_n(E)}$$

And for any map $u : E \to F$,

$$\|u\|_{cb(E,F)} = \lim_{n\to\infty} \|u\|_{cb(\mathrm{MAX}_n(E),F)} = \lim_{n\to\infty} \|u\|_{cb(E,\mathrm{MIN}_n(F))}.$$

Proof : The contractiveness of the second identity map is clear by the definition of $\mathrm{MAX}_n(E)$. For the first map, it follows immediately from Proposition 2.2.2 and Lemma 2.3.4 of [7]. This is reminiscent of Smith's lemma (Proposition 2.2.2 of [7]): to compute norms of $x \in M_n(B(H))$, one only needs n vectors in H.

Noticing that $\|x\|_{\mathcal{K}\otimes \mathrm{MIN}_n(E)}$ is increasing whereas $\|x\|_{\mathcal{K}\otimes \mathrm{MAX}_n(E)}$ is decreasing, the second affirmation follows from the first and an obvious approximation argument.

For the last assertion, since $\|u\|_{cb(E,\mathrm{MIN}_n(F))}$ is increasing, we have

$$\begin{aligned} \|u\|_{cb(E,F)} &= \sup\{\|u(x)\|_{\mathcal{K}\otimes F} \; ; \; \|x\|_{\mathcal{K}\otimes E} \leq 1\} \\ &= \sup\{\|u(x)\|_{\mathcal{K}\otimes \mathrm{MIN}_n(F)} \; ; \; n \geq 1, \|x\|_{\mathcal{K}\otimes E} \leq 1\} \\ &= \sup\{\|u\|_{cb(E,\mathrm{MIN}_n(F))} \; ; \; n \geq 1\}. \end{aligned}$$

And, in the same way, using the fact that $\|x\|_{\mathcal{K}\otimes \mathrm{MAX}_n(E)}$ decreases to $\|x\|_{\mathcal{K}\otimes E}$

$$\begin{aligned} \|u\|_{cb(E,F)} &= \sup\{\|u(x)\|_{\mathcal{K}\otimes F} \; ; \; \|x\|_{\mathcal{K}\otimes E} \leq 1\} \\ &= \sup\{\|u(x)\|_{\mathcal{K}\otimes F} \; ; \; n \geq 1, \|x\|_{\mathcal{K}\otimes \mathrm{MAX}_n(E)} \leq 1\} \\ &= \sup\{\|u\|_{cb(\mathrm{MAX}_n(E),F)} \; ; \; n \geq 1\}. \end{aligned}$$

■

The functors MIN_n and MAX_n have the following universal properties

Proposition 2.2 *For any operator spaces E and F and any linear map $u : E \to F$, we have*

$$\begin{aligned} ||u : E \to \mathrm{MIN}_n(F)||_{cb} &= ||u \otimes I_{M_n} : M_n(E) \to M_n(F)|| \\ \text{and} \quad ||u : \mathrm{MAX}_n(E) \to F||_{cb} &= ||u \otimes I_{M_n} : M_n(E) \to M_n(F)||. \end{aligned}$$

Proof : By its definition, $\mathrm{MIN}_n(F)$ is embedded in a direct sum of some M_n. Hence according to Smith's lemma, the completely bounded norm of $u : E \to \mathrm{MIN}_n(F)$ is exactly the bounded norm of $u \otimes I_{M_n} : M_n(E) \to M_n(\mathrm{MIN}_n(F))$. Then, the first equality follows from the previous lemma since $M_n(\mathrm{MIN}_n(F))$ and $M_n(F)$ are isometric.

For the second equality, notice that $v = u/\|u \otimes I_{M_n}\|_{M_n(E) \to M_n(F)}$ is a map from E to some $B(H)$ containing F with $\|v \otimes Id_n\| \leq 1$ and the definition of MAX_n, we have $\|v\|_{cb(\mathrm{MAX}_n(E),F)} \leq 1$. The other inequality is clear since the identity $\mathrm{MAX}_n(E) \to E$ is completely contractive. ■

To summarize these observations, we state a characterization of $\mathrm{MIN}_n(E)$ and $\mathrm{MAX}_n(E)$ via "extremal" properties of these spaces.

Lemma 2.3 *Let E be an operator space, $\mathrm{MIN}_n(E)$ (resp. $\mathrm{MAX}_n(E)$) is the only operator space structure on E (say $\tilde{E}$) such that $M_n(\tilde{E}) = M_n(E)$ isometrically, and, $||u||_{cb(X,\tilde{E})} = ||u \otimes I_{M_n}||_{B(M_n(X),M_n(E))}$ (resp. $||v||_{cb(\tilde{E},X)} = ||v \otimes I_{M_n}||_{B(M_n(E),M_n(X))}$) for any $u : X \to E$ (resp. $v : E \to Y$).*

Below, we prove a useful duality relation.

Lemma 2.4 *Suppose E is an operator space and $n \in \mathbb{N}$. Then*

(a) $(\mathrm{MAX}_n(E))^* = \mathrm{MIN}_n(E^*)$;

(b) $(\mathrm{MIN}_n(E))^* = \mathrm{MAX}_n(E^*)$.

Proof : We will use the following result application of local reflexivity for Banach spaces from [9] : for any $n \geq 1$, for any finite dimensional operator space F, any operator space X and any bounded map $T : F \to X^{**}$, there exist a net of maps $T_i : F \to X$, converging to T in point-weak* topology, and such that $||T_i \otimes I_{M_n}|| \leq ||T \otimes I_{M_n}||$.

(a) Pick $u \in M_k((\mathrm{MAX}_n(E))^*)$. By the previous lemma and by definition $M_k((\mathrm{MAX}_n(E))^*) = CB(\mathrm{MAX}_n(E), M_k)$, so

$$||u||_{M_k((\mathrm{MAX}_n(E))^*)} = \sup\{||\langle u, x\rangle|| \,;\, x \in M_n(E),\ ||x|| \leq 1\}$$

The preceding remark for $F = M_n^*$ and $X = E$ gives

$$\begin{aligned} ||u||_{M_k((\mathrm{MAX}_n(E))^*)} &= \sup\{||\langle u, y\rangle|| \,;\, y \in M_n(E^{**}),\ ||y|| \leq 1\} \\ &= \sup\{||(v \otimes I_{M_k})u|| \,;\, v \in CB(E^*, M_n),\ ||v||_{cb} \leq 1\} \\ &= ||u||_{M_k(\mathrm{MIN}_n(E^*))}. \end{aligned}$$

(b) We start by showing that $\mathrm{MIN}_n(E)^{**} = \mathrm{MIN}_n(E^{**})$ completely isometrically.

First recall that for any operator space X, we have $M_n(X^{**}) = (M_n(X))^{**}$ completely isometrically and by Lemma 2.1 $M_n(\mathrm{MIN}_n(X)) = M_n(X)$ isometrically. We deduce the following chain of isometries:

$$\begin{aligned} M_n(\mathrm{MIN}_n(E)^{**}) &= (M_n(\mathrm{MIN}_n(E)))^{**} = (M_n(E))^{**} \\ &= M_n(E^{**}) = M_n(\mathrm{MIN}_n(E^{**})). \end{aligned}$$

Moreover, the previously mentioned result of [9] (with $X = \mathrm{MIN}_n(E)$) implies that for any finite dimensional F and $T : F \to \mathrm{MIN}_n(E)^{**}$

$$||T||_{cb} \leq \limsup_i ||T_i||_{cb} = \limsup_i ||T_i \otimes I_{M_n}|| \leq ||T \otimes I_{M_n}||.$$

Since the c.b. norm of an operator is the supremum of the c.b. norms of its restrictions to finite dimensional subspaces, $||T||_{cb} = ||T \otimes I_{M_n}||$ for any $T : X \to \mathrm{MIN}_n(E)^{**}$. Thus, the assumptions of Lemma 2.3 are satisfied by $\mathrm{MIN}_n(E)^{**}$ for the operator space E^{**}, so $\mathrm{MIN}_n(E)^{**} = \mathrm{MIN}_n(E^{**})$ completely isometrically.

Finally, consider the formal identity map $id : \mathrm{MAX}_n(E^*) \to (\mathrm{MIN}_n(E))^*$. By part (a) of the lemma, the map $id^* : (\mathrm{MIN}_n(E))^{**} \to (\mathrm{MAX}_n(E^*))^* = \mathrm{MIN}_n(E^{**})$ is a complete isometry, hence so is id. ∎

Recall that an operator space E is C-homogeneous if for any bounded linear map $u : E \to E$, we have $\|u\|_{cb} \leq C\|u\|$.

We shall make use of the following observation:

Lemma 2.5 *Suppose E is a C-homogeneous operator space and $n \in \mathbb{N}$, then so are* $\mathrm{MIN}_n(E)$ *and* $\mathrm{MAX}_n(E)$.

In view of the discussion above, we can regard MIN_n and MAX_n as functors on the category of operator spaces. We shall also note that, for any operator space E, $\mathrm{MIN}_n(E)$ embeds completely isometrically into $\ell_\infty(I, M_n)$ for a sufficiently large set I. Hence, $\mathrm{MIN}_n(E)$ is 1-exact in the terminology of [32].

To conclude this section, we turn to some examples. Recall that $R \cap C$ is the Hilbertian 1-homogeneous operator space generated in $B(\ell_2)$ by the matrices $\{\delta_i = E_{0,i} + E_{i,0}\,;\, i \geq 1\}$. We denote by $R_n \cap C_n$ its n-dimensional version. Thus for any $x = \sum m_i \otimes \delta_i \in M_k(R \cap C)$, we have

$$\|x\|_{M_k(R\cap C)} = \max\Big\{\big\|\sum_{i\geq 1} m_i m_i^*\big\|^{1/2}\,;\ \big\|\sum_{i\geq 1} m_i^* m_i\big\|^{1/2}\Big\}.$$

Its dual space is $R + C$, we denote by (δ_i^*) the dual basis of (δ_i). Its norm is defined by

$$\|x\|_{M_k(R+C)} = \inf\Big\{\big\|\sum_{i\geq 1} a_i a_i^*\big\|^{1/2} + \big\|\sum_{i\geq 1} b_i^* b_i\big\|^{1/2} \;\Big|\; x = \sum (a_i + b_i) \otimes \delta_i^* \in M_k(R + C)\Big\}.$$

We refer to [30] for more discussion on classical homogeneous Hilbertian operator spaces.

Lemma 2.6 *Suppose $n \in \mathbb{N}$ and $m \geq n$. Then, for any linear map $u \in B(\ell_2^n)$:*

$$||u||_1/4 \leq ||u : \mathrm{MIN}_m(R_n + C_n) \to \mathrm{MAX}_m(R_n \cap C_n)||_{cb} \leq ||u||_1,$$

where $\|u\|_p$ is the norm of u in the Schatten class S_p^n.

Proof : This is simple application of corollary 1.5 in [18] (see also Theorem 19.1 in [30]).

Recall that a map $a : E \to Y$ from an operator space E to a Banach space Y is $(2, RC)$-summing if there is a constant C such that for all n and for all $(x_1, \ldots, x_n)$ in E we have

$$\sum \|a(x_i)\|^2 \leq C^2 \big\|\sum x_i \otimes \delta_i\big\|_{E\otimes(R\cap C)}.$$

The best constant in this inequality is $\pi_{2,RC}(a)$.

Theorem 2.7 *[18] Any completely bounded map u from an 1-exact operator space E into the dual of a 1-exact operator space F factorizes as $u = a^*b$*

$$E \xrightarrow{b} H \xrightarrow{a^*} F^*,$$

where H is a Hilbert space and a and b are such that

$$\pi_{2,RC}(a)\pi_{2,RC}(b) \leq 4\|u\|_{cb}.$$

Conversely, if there is such a factorization we have

$$\|u\|_{cb} \leq \pi_{2,RC}(a)\pi_{2,RC}(b).$$

As mentioned earlier, $\mathrm{MIN}_n(E)$ is 1-exact for all E. Moreover $R_n \cap C_n$ sits in $M_n \oplus M_n$ and is also 1-exact. We will need that for any map $v : R_n + C_n \to \ell_2$, we have

$$\pi_{2,RC}(v) = \|v\|_2.$$

If $(x_1, \ldots, x_d)$ are in ℓ_2^n (identified with $R_n + C_n$), denote by $T : \ell_2^d \to \ell_2^n$ the map sending an orthonormal basis (e_i) to (x_i), $T(e_i) = x_i$. With this notation,

$$\pi_{2,RC}(v) = \sup\left\{\|vT\|_2 \ ; \ \left\|\sum x_i \otimes \delta_i\right\|_{(R_n+C_n)\otimes(R_d\cap C_d)} \leq 1 \right\}$$

But since $(R_n + C_n) \otimes (R_d \cap C_d) = CB(R_n \cap C_n, R_d \cap C_d)$, by homogeneity, we have $\left\|\sum x_i \otimes \delta_i\right\|_{(R_n+C_n)\otimes(R_d\cap C_d)} = \|T\|_\infty$ from what the estimate follows.

For the inequalities, by the previous theorem, any c.b $u : \mathrm{MIN}_m(R_n + C_n) \to \mathrm{MAX}_m(R_n \cap C_n)$ factorizes as $u = a^*b$ with $b : \mathrm{MIN}_m(R_n + C_n) \to \ell_2$ and $a : \mathrm{MIN}_m(R_n + C_n) \to \ell_2$. If $(x_1, \ldots, x_d) \in (\mathrm{MIN}_m(R_n + C_n))^d$ are as in the definition of $(2, RC)$-summing maps, the tensor $\sum x_i \otimes \delta_i$ has rank at most n so to compute its norm, we can assume that it sits in $\mathrm{MIN}_n(R_n + C_n) \otimes (R_n \cap C_n)$. Since $m \geq n$, $\mathrm{MIN}_m(R_n + C_n) \otimes R_n \cap C_n$ is isometric to $(R_n + C_n) \otimes R_n \cap C_n$ by Lemma 2.1, we deduce that the $(2, RC)$-summing norm of $a : \mathrm{MIN}_m(R_n + C_n) \to \ell_2$ is the same as $a : R_n + C_n \to \ell_2$, that is $\|a\|_2$. The situation is the same for b, so this ends the proof since the product of two maps in S_2 is in S_1 and conversely. ∎

Remark 4 Instead of [18], we could have used the stronger results about the non commutative Grothendieck inequality of [36]. But we don't need the full strength of it, so we prefer this approach which is, maybe, less elegant.

3 Well separated operator spaces

In this section, we use the MAX_n and MIN_n functors to produce many different operator space structure on ℓ_2. To be precise let (n_i) be a sequence of integers greater than 1, they may be thought of as dimensions.

Theorem 3.1 *There exist 1-exact 1-homogeneous operator spaces structures E_i on $\ell_2^{n_i}$, such that for any map $u : E_j^* \to E_i$, we have*

$$\begin{cases} (\mathbf{1}) \quad \|u\|_{cb(E_j^*,E_i)} = \|u\|_2 & \text{if } j \neq i \\ (\mathbf{2}) \quad \|u\|_1/(4+2^{-i}) \leq \|u\|_{cb(E_i^*,E_i)} \leq \|u\|_1 & \end{cases}$$

Proof : We construct inductively the spaces (E_i) and an increasing sequence of integers (s_i), with $s_i \geq n_{i-1}$, so that $E_i = \mathrm{MIN}_{s_i}(\mathrm{MAX}_{s_{i-1}}(R_{n_i} \cap C_{n_i}))$.

Put $s_{-1} = n_0$ and $E_0 = \mathrm{MIN}_{s_0}(\mathrm{MAX}_{s_{-1}}(R_{n_0} \cap C_{n_0}))$, we claim that if s_0 is big enough then $(\mathbf{2})$ is satisfied. Indeed, using Proposition 2.2 twice, we have

$$CB(E_0^*, E_0) = CB(\mathrm{MIN}_{s_{-1}}(R_{n_0} + C_{n_0}), E_0)$$

Then for each $u : E_0^* \to E_0$, we deduce, using Lemma 2.1, that

$$\lim_{s_0 \to \infty} \|u\|_{CB(E_0^*,E_0)} = \|u\|_{CB(\mathrm{MIN}_{s_{-1}}(R_{n_0}+C_{n_0}),\mathrm{MAX}_{s_{-1}}(R_{n_0}\cap C_{n_0}))}.$$

Then $(\mathbf{2})$ follows from Lemma 2.6 and an easy compactness argument (using ϵ-nets). Moreover, note that the majoration is trivial since it is true for any map between Hilbertian operator spaces.

Assume $i \geq 1$, and that $(E_j)_{j<i}$ and $(s_j)_{j<i}$ have been constructed. The same reasoning as before ensures that $(\mathbf{2})$ is satisfied if s_i is big enough. To deal with $(\mathbf{1})$, note that by the properties of MAX_n, for any $u : \ell_2^{n_j} \to \ell_2^{n_i}$ we have

$$\|u\|_{CB(E_j^*,E_i)} = \|u \otimes Id_{M_{s_j}}\|_{M_{s_j}(\mathrm{MIN}_{s_{j-1}}(R_{n_j}+C_{n_j})) \to M_{s_j}(E_i)}.$$

Since $s_i \geq s_{i-1} \geq s_j$, we have using Lemma 2.1 that

$$M_{s_j}(E_i) = M_{s_j}(\mathrm{MIN}_{s_i}(\mathrm{MAX}_{s_{i-1}}(R_{n_i} \cap C_{n_i}))) = M_{s_j}(R_{n_i} \cap C_{n_i})$$

isometrically, so that we get

$$\|u\|_{CB(E_j^*,E_i)} = \|u \otimes Id_{M_{s_j}}\|_{M_{s_j}(\mathrm{MIN}_{s_{j-1}}(R_{n_j}+C_{n_j})) \to M_{s_j}(R_{n_i}\cap C_{n_i})}.$$

Since the map u is of rank at most $n_j \leq s_j$, using Smith's lemma and the comparison between homogeneous Hilbertian operator spaces, we get

$$\|u\|_{CB(E_j^*,E_i)} \geq \|u\|_{CB(R_{n_j}+C_{n_j},R_{n_i}\cap C_{n_i})} = \|u\|_2.$$

For the majoration,

$$\|u\|_{CB(E_j^*,E_i)} \leq \|u\|_{CB(\mathrm{MIN}(\ell_2^{n_j}),R_{n_i}\cap C_{n_i})} = \|u\|_2.$$

It remains to deal with $CB(E_i^*, E_j)$. But notice that

$$\|u\|_{CB(E_j^*,E_i)} = \|u^t\|_{CB(E_i^*,E_j)}.$$

Hence the result follows from the previous estimate since $\|u\|_2 = \|u^t\|_2$.

As a conclusion, we prove the lemma by induction. ■

Remark 5 The same kind of arguments leads to the fact that for any map $u : \ell_2^{n_i} \to \ell_2^{n_i}$, we have

$$\|u\|_2/4 \leq \|u\|_{CB(R_{n_i}\cap C_{n_i},E_i)} \leq \|u\|_2 \quad \text{and} \quad \|u\|_{CB(E_i,R_{n_i}\cap C_{n_i})} = \|u\|_\infty,$$
$$\|u\|_1/4 \leq \|u\|_{CB(\mathrm{MIN}(\ell_2^{n_i}),E_i)} \leq \|u\|_1 \quad \text{and} \quad \|u\|_{CB(E_i,\mathrm{MIN}(\ell_2^{n_i}))} = \|u\|_\infty.$$

Consequently $\sqrt{n}/4 \leq d_{cb}(E_i, R_{n_i} \cap C_{n_i}) \leq \sqrt{n}$, and $n/4 \leq d_{cb}(E_i, \mathrm{MIN}(\ell_2^{n_i}))$ $\leq n$.

Remark 6 We don't state it but, in addition, the construction can be made so that if $u : E_j \to E_i$, we have

$$\|u\|_2/(4 + 2^{-\min(i,j)}) \leq \|u\|_{CB(E_j,E_i)} \leq \|u\|_2$$

This is related to Corollary 3.5.

If we do not take care of the 1-homogeneity, there is another approach to construct the spaces E_i using random matrices. Recall the following remarkable result of Haagerup and Thorbjørnsen (stated in a less precise way)

Theorem 3.2 *[14] Let $n \geq 1$ and let $G_1^N, \dots, G_n^N$ be a collection of standard independent Gaussian random $N \times N$ matrices, for all $N \geq 1$, on some probability space Ω (it means that the entries of the G_k^N's are standard complex Gaussian variables with mean 0 and $\mathbb{E}|(G_k^N)_{i,j}|^2 = 1/N$). Then, for any integer $m \geq 1$ and $(a_k)_{k\leq n} \in M_m^n$, we have almost surely for $\omega \in \Omega$*

$$\begin{aligned}\Big\|\sum_{k=1}^n a_k \otimes \delta_k\Big\|_{M_m(R\cap C)} &\leq \limsup_{N\to\infty} \Big\|\sum_{k=1}^n a_k \otimes G_k^N(\omega)\Big\|_{M_m(M_N)} \\ &\leq 2\Big\|\sum_{k=1}^n a_k \otimes \delta_k\Big\|_{M_m(R\cap C)}.\end{aligned}$$

Recall that if E is an operator space, $\overline{E}$ denotes its complex conjugate as an operator space ($M_n(\overline{E}) = \overline{M_n(E)}$). The following is an easy adaptation of chapter 20 in [30].

Corollary 3.3 *There exist 1-exact operator space structures E_i on $\ell_2^{n_i}$, such that for any map $u : E_j^* \to E_i$, we have*

$$\begin{cases} (\mathbf{1'}) & \|u\|_2/(2+2^{-i}+2^{-j}) \le \|u\|_{cb(\overline{E}_j^*,E_i)} \le \|u\|_2 \qquad \textit{if } j \neq i \\ (\mathbf{2'}) & \operatorname{Tr} u/(2+2^{-i}) \le \|u\|_{cb(\overline{E}_i^*,E_i)} \end{cases}$$

Proof : In addition to [14], we will use the following consequence of measure concentration (see [5]):

Lemma 3.4 *Let $G_1^N, \ldots, G_n^N$ be as above, then almost surely for $\omega \in \Omega$, we have :*

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr} G_i^N G_j^{N*} = \delta_{i,j}.$$

It follows from this lemma that for any $u = (u_{i,j}) : \ell_2^n \to \ell_2^n$, almost surely we have

$$\lim_{n\to\infty} \Big\| \sum_{i,j\le n} u_{i,j} G_i^N \otimes \overline{G}_j^N \Big\| \ge \operatorname{Tr} u.$$

Indeed, by testing the left hand side against $\frac{1}{N}\sum_{k,l\le N} E_{k,l} \otimes E_{k,l}$ (a norm one element of $S_{N^2}^1$), we conclude that it is greater than $\sum_{i,j\le n} u_{i,j} \frac{1}{N} \operatorname{Tr} G_i G_j^*$. A compactness argument (using ϵ-nets) implies that this result holds uniformly in u with $(1-\epsilon)\operatorname{Tr}$ instead of Tr.

The proof goes by induction. We construct operator spaces $(F_i)_{i\ge 0}$ such that

$$F_i = \operatorname{span}(G_1^{N_i}(\omega), \ldots, G_{n_i}^{N_i}(\omega))$$

for some increasing sequence of integers (N_i), $N_i \ge n_{i+1}$ and $\omega \in \Omega$. If we identify F_i with $\ell_2^{n_i}$ using the preceding basis, we have

$$(i) \quad \forall (a_k)_{k\le n_i} \in M_{N_{i-1}},$$

$$\frac{1}{1+4^{-i}} \Big\| \sum_{k=1}^{n_i} a_k \otimes \delta_k \Big\|_{M_{N_{i-1}}(R\cap C)} \le \Big\| \sum_{k=1}^{n_i} a_k \otimes G_k^N(\omega) \Big\|_{M_{N_{i-1}}(M_{N_i})}$$

$$\Big\| \sum_{k=1}^{n_i} a_k \otimes G_k^N(\omega) \Big\|_{M_{N_{i-1}}(M_{N_i})} \le (2+4^{-i}) \Big\| \sum_{k=1}^{n_i} a_k \otimes \delta_k \Big\|_{M_{N_{i-1}}(R\cap C)}$$

and

$$(ii) \quad \forall u : \ell_2^{n_i} \to \ell_2^{n_i}, \quad \frac{1}{1+4^{-i}} \operatorname{Tr} u \le \|u\|_{cb(\overline{F}_i^*, F_i)}$$

For $i = 0$, put $N_{-1} = n_0$. Then Theorem 3.2 (once again, with an ϵ-net argument) and the preceding lemma ensure that with N_i going to infinity, (i) and (ii) occur with probability 1. So it is possible to find F_0.

The construction of F_{i+1} is made in the same way.

Finally to obtain the E_i's from the corollary, simply take $E_i = \frac{1}{2+4^{-i}} F_i \cap \mathrm{MIN}(\ell_2^{n_i})$, so as to make it Hilbertian. Notice that $(1')$ follows by applying (i) twice ; if $i > j$, first at rank i with $a_k = \sum_{l=0}^{n_j} u_{k,l} \overline{G}_l^{N_j}$ and second at rank j with $a_l = \sum_{k=0}^{n_i} u_{k,l} \delta_k$ (which sits in $M_{n_j} \oplus M_{n_j}$ by homogeneity). Finally, $(2')$ is exactly (ii). ■

Remark 7 The main drawback of this construction is the lack of information concerning the space E_i ; especially we don't know if it is possible to make it homogeneous (of course it is clear that it can be done up to some fixed matricial level but strictly less than N_i).

On the other hand, the spaces constructed in Theorem 3.1 (with $n_i = n$) can be used to show that the set of homogeneous Hilbertian operator space of dimension n is not separable if $n > 4$. We refer to [30] chapter 21 for further informations on this topic. For $n > 16$, we get easily the following stronger result :

Corollary 3.5 *There exists a family of homogeneous Hilbertian n-dimensional operator spaces $(E_t)_{t \in \mathbb{R}}$ such that $\|u\|_2/4 \le \|u\|_{cb} \le \|u\|_2$ for any $u \in CB(E_s, E_t)$ with $s \neq t$. Consequently, $d_{cb}(E_s, E_t) \ge n/16$.*

Proof : For any subset A of $\mathbb{N}$, define

$$X_A = \cap_{i \in A} E_i,$$

with E_i appearing in Theorem 3.1 all identified with ℓ_n^2. By Remark 3, $\|u\|_{cb} \le \|u\|_2$ for any $u \in CB(X_A, X_B)$. Moreover, suppose $i \in B \backslash A$. Then

$$\frac{\|u^* u\|_1}{4 + 2^{-i}} \le \|u^* \circ u \circ Id\|_{CB(E_i^*, E_i)} \le \|u^*\|_{CB(E_i^*, X_A)} \|u\|_{CB(X_A, X_B)} \|Id\|_{CB(X_B, E_i)}.$$

Since $\|u^* u\|_1 = \|u\|^2$, $Id \in CB(X_B, E_i)$ is a complete contraction, and $\|u^*\|_{CB(E_i^*, X_A)} \le \|u\|_2$, we conclude that $\|u\|_2/(4 + 2^{-i}) \le \|u\|_{CB(X_A, X_B)}$.

Now identify $\mathbb{N}$ with $\mathbb{N} \times \mathbb{N}$, and define the family $\mathcal{I}$ of subsets of $\mathbb{N} \times \mathbb{N}$ of the form $\cup_{i=1}^{\infty} \{i\} \times S_i$, where S_i is the set of even or odd positive integers. Then $\mathcal{I}$ has the cardinality of a continuum, and $|A \nsubseteq B| = \infty$ whenever A and B are distinct elements of $\mathcal{I}$. Thus, $\|u\|_2/4 \leq \|u\|_{CB(X_A,X_B)} \leq \|u\|_2$ for any such A and B.

Finally, observe that the spaces X_A and X_B are homogeneous. Thus, by [39], $d_{cb}(X_A, X_B) = \|Id\|_{CB(X_A,X_B)} \|Id\|_{CB(X_B,X_A)}$. ■

4 Construction of the main example

In this section, we present the construction of the space appearing in Theorem 1.1 and prove the main properties of this pathological operator space structure on ℓ_2.

Let $(u_i)_{i\geq 0}$ be a sequence of finite rank endomorphism of ℓ_2 with $\|u_i\|_2 \leq 1$ and dense in the unit ball of the Hilbert-Schmidt operators for the trace norm. Let $n_i = \operatorname{rank} u_i$, we will identify $\operatorname{ran} u_i$ with $\ell_2^{n_i}$. According to Theorem 3.1, we can equip the ranges $(\operatorname{ran} u_i)_{i\geq 1}$ with operator space structures $(E_i)_{i\geq 1}$ satisfying (**1**) and (**2**).

Definition 4.1 *We denote by X_0 the operator space structure on ℓ_2 given by the embedding :*

$$\Phi : \begin{cases} X_0 & \longrightarrow & \bigoplus_{i\geq 0} E_i \\ x & \longmapsto & \big(u_i(x)\big) \end{cases}$$

Thus for $x \in \ell_2 \otimes \mathcal{K}_0$, we have

$$\|x\|_{X_0 \otimes \mathcal{K}_0} = \sup_{i\geq 0} ||(u_i \otimes I_{\mathcal{K}})x||_{E_i \otimes \mathcal{K}}.$$

To be rigorous, we need to justify that Φ is an isometric embedding of ℓ_2. The boundedness of Φ by 1 follows from that of (u_i). Moreover since any rank one map of norm one has Hilbert-Schmidt norm one, we have that for any $x \in \ell_2$, $\|\Phi(x)\| \geq \|x\|_{\ell_2}$.

Fix a quotient X' of X_0. Since X_0 is Hilbertian we can make the identification $X' = X_0/W \approx W^{\perp}$. Let X be a subspace of X', so that it can be algebraically identified with a subspace of X_0.

Lemma 4.2 *Let E be an Hilbertian operator space and let T be any Hilbert-Schmidt map from E into X'. Then T is completely bounded and*

$$\|T\|_{cb} \leq \|T\|_2.$$

In particular, any Hilbert-Schmidt map $T : X \to X'$ is completely bounded.

Proof : Let $i : X' \to X_0$ be the formal inclusion map (which is simply a bounded map but not a completely bounded one) and $P : X_0 \to X'$ be the orthogonal projection. Since we have $T = PiT$ and $\|iT\|_2 = \|T\|_2$, it suffices to consider $X' = X_0$.

Now let $T : E \to X_0$ be a Hilbert-Schmidt map. From the definition of X_0, we have to check that the maps $u_iT : E \to E_i$ are uniformly completely bounded. But $\|u_iT\|_1 \leq \|u_i\|_2\|T\|_2 \leq \|T\|_2$, since any trace class operator V between Hilbertian operator spaces is completely bounded with $\|V\|_{cb} \leq \|V\|_1$ (we have already used it, see [28] for instance). ■

Now we turn to the converse of this lemma, we will prove the following

Theorem 4.3 *Suppose $T : X \to X'$ is a completely bounded map. Then $T = \lambda I_X + S$, where $\lambda \in \mathbb{C}$, I_X is the inclusion map and S is compact, with*

$$\max\{\, ||S||_2/16,\ |\lambda|\,\} \leq ||T||_{cb} \leq ||S||_2 + |\lambda|.$$

The proof will be divided in several lemmas.

Lemma 4.4 *With the above notations:*

$$\|T\|_{CB(X,X')} \geq$$
$$\frac{1}{4}\sup_{n\geq 0}\left\{|\mathrm{Tr}\,(yuT)|\,;\ y : \ell_2^n \to X,\ u : X' \to \ell_2^n, \textit{with } \|y\|_\infty\|u\|_2 \leq 1,\ uy = 0\right\}.$$

Proof : Take u and y as in the supremum. Without loss of generality, we can assume that $\|u\|_2 = 1$. We extend u to X_0 using the orthogonal projection onto X'. By the choice of the family (u_i), we can find arbitrarily good approximations of u by some u_i for the trace norm, so to avoid some ϵ's, we will assume that $u = u_i$ and u_i vanishes on W. Consider that $y : E_i^* \to X$, we compute its c.b. norm ; the same trick as in the previous lemma, $y = Piy$, ensures that its c.b norm from E_i^* to X is less than from E_i^* to X_0. By definition, we have

$$\|y\|_{CB(E_i^*,X_0)} = \sup_{j\geq 0}\|u_jy\|_{CB(E_i^*,E_j)}$$

Since by hypothesis $u_i y = 0$, we need to compute the norm for $j \neq i$. Using (1) from Theorem 3.1, recalling that the u_j's satisfy $\|u_j\|_2 \leq 1$, we obtain

$$\|y\|_{CB(E_i^*,X)} \leq \|y\|_{CB(E_i^*,X_0)} = \sup_{j\neq i} \|u_j y\|_{CB(E_i^*,E_j)} \leq \sup_{j\neq i} \|u_j y\|_2 \leq \|y\|_\infty \leq 1$$

In fact, this is an equality.

Let us show that $u_i : X' \to E_i$ is completely contractive. First consider it as a map $u_i : X_0 \to E_i$. Then, by definition this map is completely contractive. Since it vanishes on W, it can be factorized completely contractively through X' ; the resulting map is then exactly $u_i : X' \to E_i$.

Since T is completely bounded, we have $\|u_i T y\|_{CB(E_i^*,E_i)} \leq \|T\|_{cb}$, and by Theorem 3.1

$$\|u_i T y\|_1 \geq \|u_i T y\|_{CB(E_i^*,E_i)} \geq \|u_i T y\|_1/(4+2^{-i}),$$

so this map is trace class and

$$\|T\|_{cb} \geq \|u_i T y\|_1/(4+2^{-i}) \geq \frac{1}{4+2^{-i}}|\mathrm{Tr}\,(u_i T y)| \geq \frac{1}{4+2^{-i}}|\mathrm{Tr}\,(y u_i T)|.$$

Since i can be arbitrarily large (recall that $u_i = u$), we are done. ∎

The next step is easy.

Lemma 4.5 *Let p and q be orthogonal projections with $qp = 0$. Then qTp is Hilbert-Schmidt and $\|qTp\|_2 \leq 4\|T\|_{cb}$.*

Proof : It suffices to consider the case of $n = \max\{\mathrm{rank}\, p, \mathrm{rank}\, q\} < \infty$. Just apply the preceding lemma with $y = pa$ where $a : \ell_2^n \to \mathrm{ran}\, p$, with $\|a\|_\infty = 1$ and $u = bq$ where $b : \mathrm{ran}\, q \to \ell_2^n$, with $\|b\|_2 = 1$, thus we get

$$|\mathrm{Tr}\,(abqTp)| \leq 4\|T\|_{cb}.$$

It is easy to see that ab describes the unit ball of Hilbert-Schmidt maps from $\mathrm{ran}\, q$ to $\mathrm{ran}\, p$. The duality between Schatten classes implies the result. ∎

We will make use of the following computation.

Lemma 4.6 *Let $n \geq 1$. We denote by $(G_{n,k}, dP)$ the Grassmanian manifold of k-dimensional subspaces of ℓ_2^n equipped with its Haar measure (relative to the unitary group), which we identify with the orthogonal projections of rank k. Then, for any $T \in M_n$,*

$$\int_{G_{n,k}} (1-P)TP\, dP = \frac{k(n-k)}{n^2-1}\Big(T - \frac{\mathrm{Tr}\, T}{n} Id\Big).$$

Proof : Consider the linear map ϕ given by the integral. Then, using invariance of dP under the inner automorphism induced by the unitary group (say U_n), we get

$$\forall U \in U_n, T \in M_n, \qquad \phi(UTU^*) = U\phi(T)U^*.$$

This means that ϕ is an intertwiner for the action of U_n on M_n induced by inner automorphism. Since this representation has two irreducible components (corresponding to span Id and its orthogonal for the Hilbert-Schmidt scalar product), by classical results (see [15] page 147) there are constants a and b such that

$$\phi(T) = aT + b\,\mathrm{Tr}\,(T)Id.$$

Since $\phi(Id) = 0$, we get $a + nb = 0$. The maps $T \mapsto (1-P)TP$ are all projections of rank $k(n-k)$. So computing the trace of ϕ gives $k(n-k) = an^2 + bn$. An easy calculation gives the result. ■

Proof of Theorem 4.3 : Let Q be the orthogonal projection onto $X' \ominus X$. Fix an increasing sequence (E_n) of subspaces of X of dimension $2n$ and call P_{E_n} the projection onto E_n. Consider a measure $\nu = \mu \times \epsilon$ on $G_{2n,n} \times \{0,1\}$, where μ is the uniform probability measure on $G_{2n,n}$, and $\epsilon(1) = 2n^2/(4n^2-1)$, $\epsilon(0) = (2n^2-1)/(4n^2-1)$. Combining the triangle inequality and the previous lemma, we get:

$$\begin{aligned} &\left\| \tfrac{n^2}{4n^2-1}(Q + P_{E_n})TP_{E_n} - \tfrac{\mathrm{Tr}\,P_{E_n}TP_{E_n}}{2n} Id_{E_n} \right\|_2 \\ \leq &\int \|(\epsilon Q + (1-P)P_{E_n})TP_{E_n}P\|_2 \, d\nu \leq 4\|T\|_{cb}. \end{aligned}$$

Take a free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$. The uniformly bounded functionals $T \mapsto \frac{\mathrm{Tr}\,P_{E_n}TP_{E_n}}{2n}$ converge weak-* to some f along $\mathfrak{U}$. In the same way $(Q + P_{E_n})TP_{E_n}$ and P_E converges to T and Id for the strong operator topology and are uniformly bounded. Since the Hilbert-Schmidt norm is lower semicontinuous for the strong operator topology, we deduce that

$$\|T - f(T)Id\|_2 \leq 16\|T\|_{cb}.$$

■

Remark 8 We did not try to reach the best constant in the proof, for instance computing the integral instead of using the triangular inequality yields a constant 2 instead of 4, and the use of Theorem 3.3 instead of 3.1 divides the constant by 2. Thus 16 can be replaced by 4.

Remark 9 We could have chosen a more spectral theoretic proof; the λ in the theorem constitutes the essential spectrum of T. Thus, assuming that 0 belongs to the essential spectrum of T, we can use Lemma 4.4 to construct u_n and y_n such that $\operatorname{Tr} y_n u_n T$ converges to $c\|T\|_2$ for some constant $c \geq (2\sqrt{2})^{-1}$.

The main motivation for this finite dimensional approach is

Corollary 4.7 *For any $n \geq 1$, there exists an operator space structure Y_n on ℓ_2^n such that any completely bounded map T on Y_n is of the form $T = \lambda Id + S$ with $\|S\|_2 \leq 16\|T\|_{cb}$. And conversely, any such map is completely bounded with $\|T\|_{cb} \leq |\lambda| + \|S\|_2$.*

It suffices to take as Y_n any n-dimensional subspace of X_0.

Remark 10 The space appearing in the previous corollary can be made exact, changing 16 to 17. It suffices to use Definition 4.1 with a finite number of u_i corresponding to some ϵ-net of the ball of Hilbert-Schmidt maps from X to X.

We conclude this section with some applications of Theorem 4.3.

Theorem 4.8 *Suppose X is infinite dimensional, then it fails the compact operator approximation property.*

Proof : First notice that by Theorem 4.3, any compact endomorphism of X is a limit of finite rank maps in the completely bounded norm, so that it suffices to show that X fails the OAP.

Let $T \in CB(X,X)$, by Theorem 4.3 and the closed graph theorem (or the proof of 4.3)

$$\|T\|_{cb} \sim |\lambda| + \|S\|_2. \qquad (*)$$

The classical criterion for the AP in Banach spaces has an analogue in the operator space setting (see [7]) : X has the OAP if and only if the natural map

$$\Phi \ : \ X^*\hat{\otimes}X \to CB(X,X)$$

is one to one, where $X^*\hat{\otimes}X$ is the projective operator space tensor product.

By [7], $(X^*\hat{\otimes}X)^* = CB(X,X^{**})$ under the trace duality. But since X is Hilbertian, we have $X = X^{**}$, and dualizing $(*)$ gives for any $x \in X^*\hat{\otimes}X$, as $S_2^* = S_2$,

$$\|x\|_{X^*\hat{\otimes}X} \sim |\operatorname{Tr} x| + \|\Phi(x)\|_2.$$

Let (e_n) be an orthonormal sequence in X (identifying X and X^*) and put $x_n = \frac{1}{n}\sum_{k=1}^n e_k \otimes e_k$. Then

$$\begin{aligned}\|x_n - x_m\|_{X^*\hat{\otimes}X} &\le C(|\mathrm{Tr}\,(x_n - x_m)| + \|x_n\|_2 + \|x_m\|_2)\\ &\le C(0 + n^{-1/2} + m^{-1/2}),\end{aligned}$$

so that (x_m) is a Cauchy sequence in $X^*\hat{\otimes}X$ and converges to some x satisfying $\mathrm{Tr}\,x = 1$, hence $x \neq 0$. Moreover since $\|\Phi(x_n)\|_{CB(X,X)} \le \|\Phi(x_n)\|_2 = n^{-1/2}$, we have $\Phi(x) = 0$ and Φ is not one to one. ∎

As an easy consequence, we get the following surprising fact of which we don't know any analogue in the Banach space setting :

Corollary 4.9 *With the previous notation,* $\dim\ker\Phi = 1$.

Proof : Indeed, if $y \in \ker\Phi$, then $\tilde{y} = y - (\mathrm{Tr}\,y)\,x$ is still in $\ker\Phi$ and $\mathrm{Tr}\,\tilde{y} = 0$, then $\|\tilde{y}\| = 0$, thus $\ker\Phi = \mathrm{span}\,x$. ∎

Actually, the proof of the theorem also shows that

Corollary 4.10 *Any completely bounded map T on X is of the form $T = \lambda Id + S$, where S is nuclear in the operator space sense.*

The question of the existence of such an Hilbertian operator space was raised in [33].

Suppose X is an infinite dimensional subquotient of X_0. Then the exactness constants of finite dimensional subspaces of X are "as large as they can be" (up to a multiplicative constant).

Theorem 4.11 *Assume X is infinite dimensional. If E is an n-dimensional subspace of X or X^*, then $ex(E) \ge \sqrt{n}/16$.*

Proof : We only sketch it.

Suppose first $E \hookrightarrow X$. By [27], for any $\varepsilon > 0$ we can find a subspace $X' \hookrightarrow X$ s.t. $E \hookrightarrow X'$, $\dim X/X' < \infty$, and there exists a projection P from X' onto E with $||P||_{cb} \le ex(E) + \varepsilon$. The estimate of $ex(E)$ now follows from the lower bound in Theorem 4.3.

Now suppose $E \hookrightarrow X^*$. Then there exists a rank n projection P, acting on a finite codimensional $Y^* \hookrightarrow X^*$, and such that $||P||_{cb} \le ex(E) + \varepsilon$. Thus P^* is a rank n projection on Y, which is an infinite dimensional quotient of a subspace of X_0. Hence, $||P^*||_{cb} \ge n^{1/2}/16$. ∎

We end this section with the proof of Theorem 1.1.

Proof of Theorem 1.1 : It remains to prove (4) and (5).

To establish (4), let $(e_i)_{i=1}^{\infty}$ be an orthonormal basis in X. For $A \subset \mathbb{N}$, let $X_A = \mathrm{span}[e_i\,;\, i \in A]$. Choose uncountably many sets $\{A_i; i \in I\}$ such that $|A_i \cap A_j| < \infty$ whenever $i \neq j$ (see Lemma 5.4). We shall show that, if $i \neq j$, then if $Y \subset X_{A_i}$ and $Z \subset X_{A_j}$ are infinite dimensional, then they are not completely isomorphic. Let $T : Y \to Z$ be a c.b. map with bounded inverse. Then $T = cI_Y + S$, where $c \neq 0$ and S is compact. Let $W = T^{-1}(Z \cap Y^{\perp})$. Since $Z \cap Y^{\perp} \supset Z \cap X_{A_j \setminus A_i}$, and $A_j \setminus A_i$ is cofinite in A_j and as $Z \subset X_{A_j}$, $Z \cap X_{A_j \setminus A_i}$ is of finite codimension in Z. So W must be infinite dimensional. For $y \in W$, $Ty = cy + Sy \in Y^{\perp}$, and therefore, $Ty \perp y$. Let P be the orthogonal projection onto $Z \cap Y^{\perp}$. For any $y \in W$, we have $Ty = PTy = cPy + PSy = PSy$, and therefore, $||Sy|| \geq ||PSy|| \geq ||y||/||T^{-1}||$. Since W is infinite dimensional, this contradicts the compactness of S. The second assertion follows from (1).

Finally we deal with (5). Let X'' be a proper subspace of an infinite dimensional $X' \hookrightarrow X$. Suppose, for the sake of contradiction, that $T : X' \to X''$ is a completely bounded isomorphism. Denote by J the natural embedding of X'' into X'. Then $JT = cI_{X'} + S$, where S is compact and $c \neq 0$. By Proposition 2.c.10 of [22], $\dim(\ker JT) = \dim(X'/\mathrm{ran}\, JT)$, which is impossible. ■

It is easy to see that, if an operator space Z has the CBAP, then $Z \otimes_{\min} \mathcal{K}$ has the BAP. It may be tempting to conjecture that $X_0 \otimes_{\min} \mathcal{K}$ fails the BAP. However, surprisingly, $X_0 \otimes_{\min} \mathcal{K}$ has the MAP.

Proposition 4.12 *The Banach space $X_0 \otimes_{\min} \mathcal{K}$ has the MAP.*

Proof : We have to show that for any finite sequence $(x_i)_{i \leq n}$ in $X_0 \otimes_{\min} \mathcal{K}$ and for any $\epsilon > 0$, there is a finite rank map $\phi : X_0 \otimes_{\min} \mathcal{K} \to X_0 \otimes_{\min} \mathcal{K}$ of bounded norm one such that $\|\phi(x_i) - x_i\| \leq \epsilon$.

Let $P_N : \mathcal{K} \to M_N$ be the canonical projection. For N large enough, we have for all $i \in [1; n]$, $\|(Id_{X_0} \otimes P_N)x_i - x_i\| \leq \epsilon$. Fix such a N.

Let $k \in \mathbb{N}$, be such that $s_{k-1} > N$ (where s_k is the sequence appearing in the definition of the E_k). Let P be any orthogonal projection of X_0 onto a subspace containing $(\ker u_i)^{\perp}$ for all $i \leq k$ (so that $u_i P = u_i$), then $\|P \otimes Id_{M_N}\| = 1$. Indeed, we have

$$\|P \otimes Id_{M_N}\| = \sup_{j \geq 0} \|u_j P \otimes Id_{M_N} : M_N(X_0) \to M_N(E_j)\|.$$

If $j \leq k$, then since $u_j P = u_j$, the map on the right side appears in the definition of $M_N(X_0)$ so is completely contractive.

If $j > k$, then as $s_{k-1} > N$, using Lemma 2.1, $M_N(E_j) = M_N(R_{n_j} \cap C_{n_j})$. The map $u_j P$ goes from an Hilbertian operator space to $R_{n_j} \cap C_{n_j}$, so its c.b. norm is dominated by its Hilbert-Schmidt norm which is one by assumption on u_j.

Now let P be the orthogonal projection onto $\mathrm{span}((\mathrm{ran}\, u_i)_{i\leq k}, (x_i^{k,l})_{i\leq N})$, where $x_i^{k,l}$ are the entries of $(Id_{X_0} \otimes P_N)x_i$. Then $P \otimes P_N = (P \otimes Id_{M_N})(Id_{X_0} \otimes P_N)$ has norm bounded by 1 and

$$\begin{aligned} \|(P \otimes P_N)(x_i) - x_i\| &\leq \|(P \otimes P_N)(x_i) - (Id_{X_0} \otimes P_N)(x_i)\| \\ &+ \|(Id_{X_0} \otimes P_N)(x_i) - x_i\| \leq 0 + \epsilon. \end{aligned}$$

So the map $\phi = P \otimes P_N$ is finite rank and suits the assumptions. ■

Remark 11 With the same kind of proof, one can established that if X is an operator space with the BAP as Banach spaces then $X \otimes_{\min} \mathcal{K}$ has the AP.

5 More general constructions and proofs of Theorems 1.2 - 1.7

In this section, we generalize somehow the definition in the previous construction to allow more completely bounded maps. Let $\mathcal{M}$ be an injective (or hyperfinite) von Neumann algebra of $B(\ell_2)$.

Definition 5.1 *We denote by Y_0 the operator structure on ℓ_2 given by the embedding:*

$$\Phi : \begin{cases} Y_0 & \longrightarrow \quad \bigoplus_{\substack{N\in\mathcal{M};\|N\|\leq 1 \\ i\geq 0}} E_i \\ x & \longmapsto \quad \Big(u_i(N(x)\Big) \end{cases}$$

Thus for $x \in \ell_2 \otimes \mathcal{K}_0$, we have

$$\|x\|_{Y_0 \otimes \mathcal{K}_0} = \sup_{\substack{N\in\mathcal{M};\|N\|\leq 1 \\ i\geq 0}} ||(u_i N \otimes I_{\mathcal{K}})x||_{E_i \otimes \mathcal{K}}.$$

The main purpose for introducing this definition is

Theorem 5.2 *Let $N \in \mathcal{M}$ (recall $\mathcal{M}$ is injective) and let $S : Y_0 \to Y_0$ be a Hilbert-Schmidt map, then $T = N + S$ induces a completely bounded map from Y_0 to Y_0 with*

$$\|T\|_{cb} \le \|N\|_\infty + \|S\|_2.$$

Conversely, let $T : Y_0 \to Y_0$ be a completely bounded map, then there are a $N \in \mathcal{M}$ and a Hilbert-Schmidt map S such that $T = N + S$ and

$$\|N\| \le \|T\|_{cb} \quad \textit{and} \quad \|S\|_2 \le 16\|T\|_{cb}.$$

Proof : We give only a sketch since it is a mere adaptation of the previous arguments.

First, by definition, one easily checks that V induces a completely bounded map and that any Hilbert-Schmidt map is completely bounded.

For the second part, first replace in Lemma 4.4 the condition $uy = 0$ by $\forall N \in \mathcal{M}$, $uNy = 0$. Then, in Lemma 4.5, the condition $qp = 0$ becomes $\forall N \in \mathcal{M}$, $qNp = 0$.

Next, by classical results [4, 6, 3], it follows that $\mathcal{M}$ is injective if and only if its commutant $\mathcal{M}'$ is also injective (or hyperfinite) (see [37] for instance). So $\mathcal{M}'$ can be paved out by an increasing family of finite dimensional sub-von Neumann algebras for the strong operator topology, say $\mathcal{M}' = \overline{\cup A_n}$. Each A_n is of the form $\oplus_{k=1}^{K_n} M_{d_{n,k}}$. Fix n, let p_k be the projection into the k^{th} block of A_n and q_k be any projection in $M_{d_{n,k}}$.Then, $p = \sum_{k=1}^{K_n} q_k p_k$ is a projection in $\mathcal{M}'$. Using those projections p, we can identify $B(\ell_2)$ with $M_{\sum_{k=1}^{K_n} d_{k,n}}(B(\ell_2))$, and we will write $T = (T_{i,j})_{i,j \le K_n}$ for its matrix with respect to p_k's. Obviously, we have $(1-p)Np = 0$ for all $N \in N$, so we can apply the modification of Lemma 4.4 to get that $\|(1-p)Tp\|_2 \le 4\|T\|_{cb}$.

Then, we repeat the averaging argument of Lemma 4.6 for the q_k's ; we integrate over the product $\prod_{k=1}^{K_n} G_{d_{n,k},[d_{n,k}/2+1]}$. The computation gives that

$$\|(\alpha_{i,j} T_{i,j}) - (\delta_{i,j}(V_{i,j} \otimes Id_{M_{d_{n,i}}}))\|_2 \le 4\|T\|_{cb},$$

where $(\alpha_{i,j})$ is a real matrix with entries in $[1/4; 1]$ corresponding to the constants in Lemma 4.6 and $N_n = (\delta_{i,j}(V_{i,j} \otimes Id_{M_{d_{n,i}}}))$ is some block diagonal map which commutes with A_n. Using some Schur multiplier in S_2 to correct the $(\alpha_{i,j})$ leads to

$$\|T - N_n\|_2 \le 16\|T\|_{cb},$$

where N_n commutes with A_n and is bounded by $\|T\|_\infty$ (this exactly corresponds to the normalized trace part in Lemma 4.6).

To conclude, one just needs to proceed as in the end of the proof of 4.3 by taking some the limit N of N_n along some ultrafilter for the strong operator topology. Since each N_n belongs to A'_n, we get that $V \in \mathcal{M}'' = \mathcal{M}$. ■

Suppose $(X_k)_{k\in\mathcal{S}}$ is a sequence of mutually orthogonal subspaces of ℓ_2 s.t. $\ell_2 = (\oplus_k X_k)_2$. Denote by P_k the orthogonal projection onto X_k. For a bounded sequence of complex numbers $\omega = (\omega_i)_{i\in\mathcal{S}}$, we define a bounded map P_ω by setting $P_\omega\xi = \omega_i\xi$ for $\xi \in X_i$.

The von Neumann algebra generated by the P_k's is exactly the space formed of all the P_ω and is injective, so we can apply the previous theorem 5.2. In this situation, we can easily extend to the result to block subspaces of Y_0, that is, to subspaces of the form $Y = \oplus X'_i$, where $X'_i \hookrightarrow \oplus_{k\in S_i} X_k$ ($S_i \cap S_j = \emptyset$ if $i \neq j$). For such a Y, denote by P'_ω a map satisfying $P'_\omega\xi = \omega_i\xi$ for $\xi \in X'_i$.

Theorem 5.3 *Let Y be a block subspace of Y_0. Let $\omega = (\omega_i)$ be a bounded sequence of complex numbers and $S : Y \to Y$ a Hilbert-Schmidt map, then $T = P'_\omega + S$ induces a completely bounded map from Y to Y_0 with*

$$\|T\|_{cb} \leq \sup_{i\in\mathcal{S}} |\omega_i| + \|S\|_2.$$

Conversely, let $T : Y \to Y$ be a completely bounded maps, then there are a complex sequence $\omega = (\omega_i)$ and a Hilbert-Schmidt map S such that $T = P'_\omega + S$ and

$$\sup_{i\geq 0} |\omega_i| \leq \|T\|_{cb} \quad \textit{and} \quad \|S\|_2 \leq 16\|T\|_{cb}.$$

Moreover, the maps $T \mapsto \omega_i$ are linear.

We are now in a position to prove Theorems 1.2 to 1.6.

Proof of Theorem 1.2 : The space is Y_0 built of infinitely many 1-dimensional blocks, say $X_k = \text{span}(e_k)$ with $\|e_k\| = 1$. Then $(e_k)_{k\in\mathbb{N}}$ is a completely bounded basis and Theorem 1.2 is a simple reformulation of 5.3. ■

Proof of Theorem 1.3 : Denote by X the space Y_0 for which $\dim X_k = k$ for $k = 1, 2, \ldots$. It is clear that $(P_k)_{k\geq 1}$ forms a completely unconditional F.D.D for X.

Suppose, for the sake of contradiction, that X has a complete local basis structure. Then there is a constant $C > 0$ such that for any finite dimensional subspace $E \subset X$, there is a finite dimensional superspace $F \supset E$ that admits a C-complete basis.

Apply the preceding for $E = X_k$ to get completely bounded basis projections Q_1,..., Q_N from $F \supset X_k$ to itself. Let $A_i = P_k Q_i P_k$, for $i \geq 1$. Since Q_i is a projection, we have

$$A_i^2 - A_i = P_k Q_i (P_k - Id) Q_i P_k.$$

By Lemma 4.5, we get that $\|(P_k - Id)Q_i P_k\|_2 \leq 4C$, so $\|A_i^2 - A_i\|_2 \leq 4C^2$.

Moreover A_i is completely bounded with norm less than C, by Theorem 5.3, we must have $A_i = \lambda_i P_k + S_i$ with $\|S_i\|_2 \leq 16C$ and $|\lambda_i| \leq C$. But

$$A_i^2 - A_i = (\lambda_i^2 - \lambda_i)P_k + \lambda_i(S_i P_k + P_k S_i) + S_i^2 - S_i,$$

thus $\|(\lambda_i^2 - \lambda_i)P_k\|_2 \leq KC^2$ for some universal constant K. Therefore, we obtain

$$|\lambda_i^2 - \lambda_i| \leq \frac{KC^2}{\sqrt{k}}.$$

In the same way, considering $A_{i+1} - A_i$ and remarking that $Q_{i+1} - Q_i$ is a rank one map and hence Hilbert-Schmidt, we conclude

$$|\lambda_{i+1} - \lambda_i| \leq \frac{K'C}{\sqrt{k}}.$$

This works also if $i = 0$ with $\lambda_0 = 0$!

To summarize, if k is big enough, we must have

$$|\lambda_i - 1| \leq 1/10 \ \text{ or } \ |\lambda_i| \leq 1/10 \qquad (1)$$
$$\text{and} \qquad |\lambda_{i+1} - \lambda_i| \leq 1/10. \qquad (2)$$

But we have $\lambda_0 = 0$ and $\lambda_N = 1$ as $Q_N = Id_F$. This yields a contradiction since by (2), then there must be a i with $|\lambda_i - 1/2| \leq 1/10$, which is impossible by (1). So X does not have a complete local basis structure. ∎

Proof of Theorem 1.4 : Fix $n \in \mathbb{N} \cup \{\infty\}$, and consider the space $Y_0 = \oplus_{1 \leq k \leq n} X_k$ with $\dim X_k = \infty$. As before, P_k stands for the orthogonal projection onto X_k. Suppose $U \in CB(X)$. By Lemma 4.5 and Theorem 5.3, the operators $(I - P_k)UP_k$ and $P_k U(I - U_k)$ are compact for any k, and $P_k U P_k = f_k(U)P_k + S_k$, where $f_k(U) \in \mathbb{C}$ (with $|f_k(u)| \leq ||U||_{cb}$) and S_k is compact. Clearly, f_k is a linear functional, and $|f_k(U)| \leq ||U|| \leq ||U||_{cb}$. Moreover, $f_k(P_k) = 1$, hence f_k is non-trivial. To show that it is multiplicative, consider

$U_1, U_2 \in CB(X)$. Then $P_k U_1 U_2 P_k = P_k U_1 P_k U_2 P_k + P_k U_1 (I - P_k) U_2 P_k$. The second term in this sum is compact, and

$$\begin{aligned} P_k U_1 P_k U_2 P_k &= \big(f_k(U_1) P_k + P_k(U_1 - f_k(U_1) P_k)\big)\big(f_k(U_2) P_k \\ &+ P_k(U_2 - f_k(U_2) P_k)\big) = f_k(U_1) f_k(U_2) P_k \\ &+ P_k\big(f_k(U_1)(U_2 - f_k(U_2)P_k) + f_k(U_2)(U_1 - f_k(U_1)P_k) \\ &+ (U_1 - f_k(U_1)P_k)(U_2 - f_k(U_2)P_k)\big) P_k. \end{aligned}$$

The above equality shows that $P_k U_1 U_2 P_k - f_k(U_1) f_k(U_2) P_k$ is compact, hence $f_k(U_1 U_2) = f_k(U_1) f_k(U_2)$.

To prove the first point, it suffices to show that, if n is finite, then every multiplicative functional on $CB(X)$ coincides with f_k for some k. We use some ideas from [1]. If f is a non-trivial multiplicative functional on $CB(X)$, then $f(P_k) = 1$ for exactly one value of k, and $f(P_k) = 0$ otherwise. Suppose, without loss of generality, that $f(P_0) = 1$. Then $f(I_X - P_0) = 0$, hence $f(U) = f(P_0 U P_0)$ for any $U \in CB(X)$. However, $P_0 U P_0 = f_0(U) P_0 + S_0$, where S_0 is compact. Moreover, the c.b. norm on the set of compact operators on X_0 is equivalent to $\|\cdot\|_2$, hence S is a norm limit of finite rank operators.

To complete the proof of the first part, we have to show that $f(S) = 0$ whenever S is a rank 1 operator. Suppose first that $S = \xi \otimes \eta$ (that is, $Sx = \langle x, \xi \rangle \eta$) with $\langle \eta, \xi \rangle = 0$. Then $S^2 = 0$, hence $f(S) = 0$.

Now consider $S = \xi \otimes \eta$ for generic ξ and η. Find ξ' and η' s.t. $\langle \xi', \eta \rangle = \langle \xi, \eta' \rangle = 0$ and $\langle \xi', \eta' \rangle = 1$. Then $S = S_2 S_1$, where $S_1 = \xi \otimes \eta'$ and $S_2 = \xi' \otimes \eta$. By the reasoning above, $f(S_1) = f(S_2) = 0$, hence, by multiplicativity, $f(S) = 0$.

We turn to the second part. Denote by X the space Y_0 for which the blocks $X_0, X_1, X_2, \dots$ are infinite dimensional. To proceed, we need to recall a folklore result.

Lemma 5.4 *There exists an uncountable family of infinite subsets $A \subset \mathbb{N}$ such that both A and $\mathbb{N} \backslash A$ are infinite, and the intersection of any two elements of the family is finite.*

Just think of real numbers as limits of Cauchy sequences of rational numbers.

Suppose $\mathcal{F} = (A_i)_{i \in \mathcal{I}}$ is an uncountable collection of subsets of $\mathbb{N}$ such that $A_i \cap A_j$ is finite whenever $i \neq j$. Find a free ultrafilter $\mathfrak{U}$ on $\mathbb{N}$, and

$B \in \mathfrak{U}$ s.t. both B and $\mathbb{N}\backslash B$ are infinite. For any $i \in \mathcal{I}$ find a bijection π_i on $\mathbb{N}$ s.t. $\pi_i(B) = A_i$, and define the ultrafilter $\mathfrak{U}_i = \{\pi_i(C) \,|\, C \in \mathfrak{U}\}$. Consider the functional $\phi_i(T) = \lim_{\mathfrak{U}_i} f_k(T)$. Then $|\phi_i(T)| \leq ||T||$ for any $T \in CB(X)$, hence ϕ_i is indeed a bounded linear functional on $CB(X)$. Moreover, it is non-trivial: $f_k(I_X) = 1$, hence $\phi_i(I_X) = 1$. Thus, $||\phi_i|| = 1$.

Finally, we show that $||\phi_i - \phi_j|| \geq 1$. Indeed, fix i, and consider the orthogonal projection Q onto $\mathrm{span}[X_k \,|\, k \in A_i]$. Then Q is completely contractive, and

$$f_k(Q) = \begin{cases} 1 & k \in A_i \\ 0 & k \notin A_i \end{cases} .$$

Thus, $\phi_i(Q) = 1$, since $A_i \in \mathfrak{U}_i$. On the other hand, $A_i \notin \mathfrak{U}_j$ for $j \neq i$ (otherwise, the finite set $A_i \cap A_j$ must belong to $\mathfrak{U}_j$, which is impossible). Thus, $\mathbb{N}\backslash A_i$ is an element of $\mathfrak{U}_j$, hence $\phi_j(Q) = 0$. ■

In the preceding proof, we could have used more informations given by the proof of 4.3 about the functionals f_k but we prefer this more algebraic approach.

Proof of Theorem 1.5 : Find a type III hyperfinite factor $\mathcal{M} \hookrightarrow B(\ell_2)$. By Theorem 5.2, there exists an operator space X, isometric to ℓ_2, and such that $CB(X) = N + S_2$. Suppose $P \in \mathcal{M}$ is a proper orthogonal projection. Find a partial isometry $U \in \mathcal{M}$ such that $\ker U = \{0\}$ and $\mathrm{ran}\, U = Z = P(\ell_2)$. Clearly, U is a complete isometry, hence Z is completely isometric to X, and completely contractively complemented in it.

Now find $e \in Z^\perp$ with $\|e\| = 1$, and consider an orthogonal projection $Q = P + e \otimes e$. Denote the range of Q by Y. Clearly, Q is completely bounded, hence Y is completely complemented in X. Moreover, Z is completely complemented in Y.

Suppose, for the sake of contradiction, that $T : X \to Y$ is a complete isomorphism. Then $T = V + S$, with $V \in N$ and $S \in S_2$. We first show that V is an isomorphism on a subspace of ℓ_2 of finite codimension. Indeed, otherwise there exists, for every $\varepsilon > 0$, an infinite dimensional subspace E s.t. $\|V|_E\| < \varepsilon$. Then $V + S$ is not an isomorphism, which is impossible.

Now we show that V has a trivial kernel. Indeed, let p be the orthogonal projection onto $(\ker V)^\perp$. Then $p \in \mathcal{M}$. Since every proper projection in $\mathcal{M}$ is "infinitely divisible", p must be 0 or infinite dimensional. The latter is impossible by the previous paragraph.

Let R be the orthogonal projection onto $F = V(\ell_2)$. Define $W : F \to Y$ by setting $W\eta = TV^{-1}\eta = \eta + S'\eta$ for $\eta \in F$. Here, $S' = SV^{-1} \in S_2$.

$RW = Id_F + RS'$ is a Fredholm operator of index 0 (see Proposition 2.c.10 of [22]), hence R is an isomorphism on a finite codimensional subspace of Y. Note that, for $\eta \in F \ominus R(Y)$, $\|\eta - W\eta\| \geq \|\eta\|$. Hence, $\dim F/R(Y) < \infty$. However, $RP \in N$, hence the orthogonal projection onto $R(Z)$ belongs to N. Reasoning as before, we conclude that $\dim F/R(Z) \notin (0, \infty)$. Hence, $F = R(Z)$. Thus, $\dim \ker(RW) > 0$ and $\operatorname{ran}(RW) = F$. Therefore, the index of RW is non-zero, which yields a contradiction. ■

Proof of Theorem 1.6 : This is just a local version of the Definition 4.1 : fix n, instead of choosing a dense family (u_i) in the ball of Hilbert-Schmidt maps, we take a dense family among the Hilbert-Schmidt maps of rank $2n$.

With this new definition, (1) follows directly from Remark 3; indeed, the c.b. norm of the formal identity $\text{MIN}(\ell_2) \to X$ is then the supremum of the c.b norm of $u_i : \text{MIN}(\ell_2) \to E_i$, but since these maps have rank less than $2n$, we have by the Cauchy-Schwartz inequality $\|u_i\|_1 \leq \sqrt{2n}\|u_i\|_2$. Lemmas 4.4 remain true with the difference that the dimensions appearing must be $2n$ (or less). (2) is then a particular case of it : let $(e_i)_{i\leq n}$ and $(f_i)_{i\leq n}$ be an orthonormal basis for E and F. Take for y the map such that $y(e_i) = (e_i - f_i)$ and $y(f_i) = 0$, and set u to satisfy $u(e_i) = u(f_i) = e_i$. We have $\|y\|_\infty \|u\|_2 = 2\sqrt{n}$, and $uy = 0$ but $\operatorname{Tr}(uTy) = n$. The last assertion is obtained from the previous one and an easy perturbation argument.

For the last statement, it suffices to remark that if some operator spaces $(X_n)_{n\geq 2}$ have the OAP then $\ell_2((X_n)_{n\geq 2})$ also have this property. If P_k denotes the block projection onto $\ell_2((X_n)_{k\geq n\geq 2})$ then the sequence (P_k) converges pointwise to the identity on $\ell_2((X_n)_{n\geq 2})$ and consists only of completely contractive maps. Thus if $x \in \mathcal{K} \otimes_{\min} \ell_2((X_n)_{n\geq 2})$, there is k such that $\|Id_{\mathcal{K}} \otimes P_k(x) - x\| \leq 1/2$. Since $Id_{\mathcal{K}} \otimes P_k(x)$ has a finite support using the OAP for each X_n for $n \leq k$ gives a finite rank map ϕ such that $\|Id_{\mathcal{K}} \otimes P_k(x) - Id_{\mathcal{K}} \otimes \phi(x)\| \leq 1/2$, this completes the proof by homogeneity. ■

Corollary 5.5 *Let X' be a subspace of a quotient of X appearing in Theorem 1.6 then the completely nuclear and completely integral norms of operators on an infinite dimensional X' do not coincide.*

Proof : See Chapter 12 of [7] for definitions. Indeed, $I_{X'}$ does not belong to the point-norm closure of $\lambda B_{F(X')}$ for $\lambda < n^{1/4}/4$, where $B_{F(X')}$ is the set of all finite rank operators on X' with c.b. norm not exceeding 1 (see also [19]). However, X' is Hilbertian, hence the space of completely nuclear operators on

X' can be identified with the predual of $CB(X')$. By Hahn-Banach theorem and the duality between classes of c.b. maps (once again, see Chapter 12 of [7]), there exists $u : X' \to X'$ for which

$$i(u) = \sup\{|\mathrm{Tr}\,(vu)| \,|\, v \in CB(X'),\ \text{rank } v < \infty,\ ||v||_{cb} \leq 1\} \leq 1,$$

but

$$\nu(u) \geq \mathrm{Tr}\,(I_{X'}u) = \mathrm{Tr}\,(u) \geq n^{1/2}/8.$$

■

Proof of Theorem 1.7 : Once again, this is an adaptation of Definition 4.1.

We refer to [10, 24] for discussion about symmetric spaces and property $\mathcal{P}$. The proof uses some ideas of [24].

Proof of (2) $\Rightarrow$ (1): if $V = \ell_\infty$, any homogeneous Hilbertian operator space X suits. Otherwise, assume w.l.o.g. that $\|id : \ell_2 \to V\|_{cb} = 1$. Denote by V_n the subspace of V spanned by the first n vectors of its basis. Define a dense sequence of linear operators $u_i : \ell_2 \to \ell_2^{n_i}$ with $\|u_i\|_{V^*_{n_i}} = 1$. For $x \in X_0 \otimes \mathcal{K}_0$, as in 4.1 define

$$\|x\| = \sup_i \|(u_i \otimes I_{\mathcal{K}})x\|_{E_i \otimes \mathcal{K}}.$$

As before, we see that X_0 is isometric to ℓ_2.

Suppose $u : \mathrm{MIN}(\ell_2) \to X_0$ is a linear map. We shall show that $\|u\|_{cb} \leq \|u\|_V$. Indeed, by the theory of operator ideals

$$\|u\|_{cb} \leq \sup_i \|u_i u\|_1 \leq \|u\|_V.$$

Consider $T : X' \to X$. Reasoning as in the proof of Lemma 4.4, we conclude that $\|T\|_{cb} \gtrsim \|uTv\|_1$, where the supremum runs over all rank n operators u, v with $\|u\|_{V_n^*} = \|v\| = 1$ and $uv = 0$. Thus (as in Lemma 4.5), $\|T\|_{cb} \gtrsim \|pTq\|_V$ whenever p and q are mutually orthogonal finite rank projections. The proof is then completed as in Theorem 4.3.

Proof of (1) $\Rightarrow$ (2): suppose X is such that $T = \lambda I_X + S$ (with $S \in S_V$) for any $T \in CB(X)$. Then $\mathcal{P}$ must be satisfied, because the completely bounded norm of an operator is the supremum of its restriction to finite dimensional subspaces. Reasoning as in the proof of Theorem 4.8 and Corollary 4.10, we conclude that, for any zero trace $y \in Y^* \hat{\otimes} Y$, $\|y\|_{Y^* \hat{\otimes} Y} \sim \|\Phi(y)\|_{V^*}$. But since $\Phi(y)$ is compact, we have $\|\Phi(y)\|_{CB(Y)} \sim \|\Phi(y)\|_V$. As Φ is completely contractive, we have that for some universal constant

$\|\Phi(y)\|_V \leq C\|\Phi(y)\|_{V^*}$ for any zero trace finite rank map y. Moreover, by considering a sequence $(a_1, -a_1, a_2, -a_2, \ldots)$ instead of $(a_1, a_2, \ldots)$, we see that $\|\Phi(y)\|_V \leq 2C\|\Phi(y)\|_{V^*}$ for any finite rank y. Hence, the natural embedding of V^* into V is continuous. Since $\ell_2 = (V, V^*)_{1/2}$, we are done.
■

6 The non-Hilbertian case: proof of Theorem 1.8

This section is devoted to the proof of Theorem 1.8. We start with a simple application of the OAP.

Lemma 6.1 *Let X be an operator space with the OAP. Let $(F_i)_{i\geq 0}$ be a sequence of finite dimensional subspaces of X and $(E_i)_{i\geq 0}$ be a sequence of finite dimensional 1-exact operator spaces and $f : \mathbb{N} \to \mathbb{R}^+\backslash[0,2]$ be a function with* $\lim_{n\to\infty} f(n) = \infty$. *Then, there is a finite rank map $\phi : X \to X$ such that*

$$\phi_{|F_0} = Id_{F_0}$$
$$\forall\, v_i : X \to E_i,\ \|v_i\phi_{|F_i}\|_{cb} \leq \|v_i\|_{cb} f(i).$$

Proof : Since the E_i are finite dimensional and 1-exact, for all $\epsilon > 0$ (say $\epsilon < 1/4$), there exist $n_i \in \mathbb{N}$ such that for any operator space Y :

$$\|v : Y \to E_i\|_{cb} \leq (1+\epsilon)\|v \otimes Id_{M_{n_i}} : M_{n_i}(Y) \to M_{n_i}(E_i)\|. \qquad (*)$$

Let $(x_{i,k})$, be a finite ϵ_i-net of the unit sphere of $M_{n_i}(F_i)$ and define the norm one element $x_i = \oplus x_{i,k} \in \mathcal{K}_0 \otimes X$. We put

$$x = x_0 \oplus_{i\geq 1} \frac{x_i}{f(i)}.$$

It belongs to $\mathcal{K} \otimes_{\min} X$ as $f \to \infty$. By the OAP of X, there is a finite rank map $\phi : X \to X$ such that

$$\|(\phi \otimes Id_{\mathcal{K}})x - x\| \leq \epsilon.$$

Moreover, by a perturbation argument, we can assume that $\phi_{|F_0} = Id_{F_0}$. In particular, using the x_i part of x gives that for any $v_i : X \to E_i$, we have

$$\|(v_i\phi \otimes Id_{\mathcal{K}})x_i/f(i)\| \leq \epsilon\|v_i\|_{cb} + \|(v_i \otimes Id_{\mathcal{K}})x_i/f(i)\| \leq (1/f(i) + \epsilon)\|v_i\|_{cb}.$$

Since x_i is built of an ϵ_i-net of $M_{n_i}(F_i)$ (say $\epsilon_i \leq \epsilon/(3n_i)$), by $(*)$, we have

$$\|v_i\phi_{|F_i}\|_{cb} \leq (1+\epsilon)\|(v_i\phi \otimes Id_{M_{n_i}})x_i\|.$$

Finally, we have

$$\|v_i\phi_{|F_i}\|_{cb} \leq (1+\epsilon)(1+\epsilon f(i))\|v_i\|_{cb} \leq f(i)\|v_i\|_{cb}.$$

■

Proof of Theorem 1.8 : The main idea is to use Dvoretsky's Theorem to get copies of ℓ_2^n inside X and then to introduce a construction similar to 4.1 ; the main point is to extend the maps u_i to the whole of X in a suitably good way so that we have analogues of Lemmas 4.4 and 4.6.

Fix an integer $n > 2^{22}$. First, consider an isometric embedding $X \hookrightarrow C(K)$, for some locally compact space K. Since $C(K)$ has the MAP, using Dvoretsky's Theorem, we can find 2-isomorphic copies Z_k of $\ell_2^{n^k}$ in X , and finite rank maps $T_k : C(K) \to C(K)$ such that

$$T_{k|Z_j} = \begin{cases} Id_{Z_k} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \quad \text{and} \quad \|T_k\| \leq 4.$$

Let $Id_k : Z_k \to \ell_2^{n^k}$ be an isomorphism such that $\|Id_k\| \leq 1$ and $\|Id_k^{-1}\| \leq 2$. Since these maps (Id_k) are Hilbert-Schmidt, we can find extensions $\overline{Id_k} : C(K) \to \ell_2^{n^k}$, with $\pi_2(\overline{Id_k}) \leq n^{k/2}$. Set $\widetilde{Id_k} = \overline{Id_k}T_k$, they satisfy

$$\widetilde{Id_k}_{|Z_j} = \begin{cases} Id_k & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \quad \text{and} \quad \pi_2(\widetilde{Id_k}) \leq 4n^{k/2}.$$

For $k \geq 2$, let $(p_{k,i})_{i=1}^n$ be partial isometries of $\ell_2^{n^k}$ onto $\ell_2^{n^{k-1}}$, such that $\sum_{i=1}^n p_{k,i}^* p_{k,i} = Id_{\ell_2^{n^k}}$. ¿From now on, we use the notation ϵ for ± 1.

By Theorem 3.1, we can find a family $(E_{k,i,\epsilon})_{k\geq 2, i\leq n, \epsilon\in\{\pm 1\}}$ of well separated operator space structures on $(\ell_2^{n^{k-1}})$. We set

$$u_{k,i,\epsilon} = \frac{\widetilde{Id_{k-1}} + \epsilon p_{k,i}\widetilde{Id_k}}{8n^{k/2}} \ : \ C(K) \to \ell_2^{n^{k-1}},$$

these maps are all 2-summing with norm less than 1.

We define the operator space structure on X via the following embedding

$$\Phi : \begin{cases} X & \longrightarrow & \min C(K) \underset{\substack{k\geq 2\\ 1\leq i\leq n\\ \epsilon=\pm 1}}{\oplus} E_{k,i,\epsilon} \\ x & \longmapsto & \Big(x, (u_{k,i,\epsilon}(x))\Big) \end{cases}.$$

Since all maps $u_{k,i,\epsilon}$ are contractive, this is an isometric embedding and we indeed get an operator space structure on X.

We show that this structure fails the OAP. For the sake of contradiction, assume X has the OAP, then by the previous lemma there is a finite rank map $\phi : X \to X$ such that $\phi_{|Z_1+Z_2} = Id_{Z_1+Z_2}$ and for $k \geq 2$

$$\|u_{k,i,\epsilon}\phi_{|Z_{k-1}+Z_k}\|_{cb(Z_{k-1}+Z_k,E_{k,i,\epsilon})} \leq k\|u_{k,i,\epsilon}\|_{cb} \leq k.$$

We show by induction the minoration

$$(\mathcal{P}_k) \qquad \operatorname{Re}\operatorname{Tr}(\widetilde{Id_k}\phi Id_k^{-1}) \geq \Big(\frac{n}{2}\Big)^k.$$

This is obviously true for $k = 1$ and $k = 2$, since $\phi_{|Z_k} = Id_{Z_k}$, and $\widetilde{Id_k}\phi Id_k^{-1} = Id_{\ell_2^{n^k}}$ for $k = 1, 2$.

Assume $k \geq 3$ and $(\mathcal{P}_{k-1})$.

We repeat the argument of Lemma 4.4. Consider the maps

$$t_{k,i,\epsilon} = Id_{k-1}^{-1} - \epsilon Id_k^{-1}p_{k,i}^* : \ell_2^{n^{k-1}} \to Z_{k-1} + Z_k \subset X \subset C(K).$$

They are adjusted so that

$$\begin{aligned} u_{k,i,\epsilon}t_{k,i,\epsilon} &= \Big(\frac{\widetilde{Id_{k-1}} + \epsilon p_{k,i}\widetilde{Id_k}}{8n^{k/2}}\Big).\Big(Id_{k-1}^{-1} - \epsilon Id_k^{-1}p_{k,i}^*\Big) \\ &= c_n(Id_{\ell_2^{n^{k-1}}} + 0 + 0 - \epsilon^2 p_{k,i}p_{k,i}^*) \\ &= 0. \end{aligned}$$

Therefore, we have

$$\|t_{k,i,\epsilon}\|_{cb(E_{k,i,\epsilon}^*,X)} = \max\{\|t_{k,i,\epsilon}\|, \sup_{(l,j,\epsilon')\neq(k,i,\epsilon)} \|u_{l,j,\epsilon'}t_{k,i,\epsilon}\|_2\}.$$

Since the maps $u_{l,j,\epsilon'}$ are 2-summing, we get that $\|t_{k,i,\epsilon}\|_{cb} \leq \|t_{k,i,\epsilon}\| \leq 4$.

On the other hand, we have

$$\|u_{k,i,\epsilon}\phi t_{k,i,\epsilon}\|_1/8 \le \|u_{k,i,\epsilon}\phi t_{k,i,\epsilon}\|_{cb} \le \|u_{k,i,\epsilon}\phi_{|Z_{k-1}+Z_k}\|_{cb}\|t_{k,i,\epsilon}\|_{cb} \le 8k.$$

And, the classical polarization argument leads to the identity

$$u_{k,i,1}\phi t_{k,i,1} + u_{k,i,-1}\phi t_{k,i,-1} = \frac{1}{4n^{k/2}}(\widetilde{Id_{k-1}}\phi Id_{k-1}^{-1} - p_{k,i}\widetilde{Id_k}\phi Id_k^{-1}p_{k,i}^*).$$

Taking the real part of the trace and the previous majoration give

$$\mathrm{Re}\,\mathrm{Tr}\,\frac{1}{4n^{k/2}}(\widetilde{Id_{k-1}}\phi Id_{k-1}^{-1} - p_{k,i}\widetilde{Id_k}\phi Id_k^{-1}p_{k,i}^*) \le 128k.$$

Summing on i from 1 to n and using

$$\sum p_{k,i}^* p_{k,i} = Id_{\ell_2^{n^k}},$$

we have

$$\begin{aligned}\mathrm{Re}\,\mathrm{Tr}\,\widetilde{Id_k}\phi Id_k^{-1} &\ge n\mathrm{Re}\,\mathrm{Tr}\,\widetilde{Id_{k-1}}\phi Id_{k-1}^{-1} - 512kn^{k/2+1}\\ &\ge \left(\frac{n}{2}\right)^k(2-512kn^{-k/2+1}) \ge \left(\frac{n}{2}\right)^k,\end{aligned}$$

provided that $n > 2^{22}$ and $k \ge 3$.

So the majoration $(\mathcal{P}_k)$ holds for all $k \ge 1$.

To conclude, we show that this is impossible. Indeed $\widetilde{Id_k}\phi Id_k^{-1}$ is 2-summing with norm less that $16n^{k/2}\|\phi\|$ (because $\pi_2(\widetilde{Id_k}) \le 4n^{k/2}$), then, by the Cauchy-Schwartz inequality, we get

$$\begin{aligned}\left(\frac{n}{2}\right)^k \le \mathrm{Re}\,\mathrm{Tr}\,\widetilde{Id_k}\phi Id_k^{-1} &\le \|\widetilde{Id_k}\phi Id_k^{-1}\|_2\sqrt{\mathrm{rank}\,\widetilde{Id_k}\phi Id_k^{-1}}\\ &\le 16n^{k/2}\|\phi\|\sqrt{\mathrm{rank}\,\phi}.\end{aligned}$$

So $\|\phi\|\sqrt{\mathrm{rank}\,\phi} \ge 1/16\left(\frac{\sqrt{n}}{2}\right)^k$, for all $k \ge 1$, that gives the contradiction.

For the statement about dual spaces, if $X = Y^*$, it suffices to observe that the maps $u_{k,i,\epsilon}$ can be chosen to be weak*-continuous (see [16]). In this case, if the net (x_i) in the unit ball of $X \otimes M_s$ converges to x in the $\sigma(Y,X)$ topology, then $||x|| \le 1$. Thus, by [20], X is a dual operator space. ■

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