COMPLETELY BOUNDED AND IDEAL NORMS OF MULTIPLICATION OPERATORS AND SCHUR MULTIPLIERS

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Abstract. We estimate the completely bounded norms, the completely p-nuclear norms, and the completely p-summing norms of certain multiplication operators and Schur multipliers.

1. Introduction

The goal of this paper is to compute the c.b. norms, completely p-summing norms, and completely p-nuclear norms of multiplication operators and Schur multipliers. Throughout, we use the standard operator space and Banach space terminology and results. The reader is referred to [5, 14, 17] for operator spaces, [12] for Banach spaces, and [4] for operator ideals. $E \otimes F$ refers to the minimal (or spatial) tensor product of operator spaces E and F. M_n stands for the space of $n \times n$ matrices, with its usual operator space structure. The space of operators on ℓ^2 which belong to the Schatten p-class is denoted by $\mathcal{S}^p$, while $\mathcal{S}^p_n$ stands for its $n \times n$ version. $\mathcal{S}^\infty$ denotes the space of compact operators on ℓ^2. We denote the norm of $\mathcal{S}^p$ by $\| \cdot \|_p$, and write $\|T\|_p = \infty$ if $T \notin \mathcal{S}^p$.

As common in the operator space literature, we denote the space of $n \times n$ matrices (that is, $\mathcal{S}^\infty_n$) by M_n. For the sake of brevity, we often use $M_n(E)$ instead of $M_n \otimes E$ (E is an operator space).

The pairing between a matrix space and its conjugate arises from the *parallel duality*: for matrices (finite or infinite) $T = (t_{ij})$ and $S = (s_{ij})$, $\langle T, S \rangle = \mathrm{Tr}\, T\, {}^tS = \sum_{i,j} t_{ij} s_{ij}$ (${}^tS$ is the transposed of S).

We first investigate the *multiplication operators* $\mathbf{M}^{p,q}_{A,B} : \mathcal{S}^p \to \mathcal{S}^q : T \mapsto ATB$, with $A, B \in B(\ell^2)\backslash\{0\}$. If there is no possibility of confusion regarding p and q, we use the notation $\mathbf{M}_{A,B}$. In Section 2, we estimate the c.b. norms of these operators. We prove that $\|\mathbf{M}^{p,q}_{A,B}\| = \|A\|_{2r}\|B\|_{2r}$, with $1/r = |1/p - 1/q|$ (Theorem 2.1). In Section 3, we deal with the completely s-summing and s-nuclear norms of multiplication operators (see below for the definition). Theorem 3.1 shows that $\pi^o_s(\mathbf{M}^{p,q}_{A,B}) = \nu^o_s(\mathbf{M}^{p,q}_{A,B}) = \|A\|_r\|B\|_r$ for certain 4-tuples (p, q, s, r) (see below for the definition and basic properties of $\pi^o_s(\cdot)$ and $\nu^o_s(\cdot)$).

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Next we turn our attention to Schur multipliers. Recall that, for a matrix $\phi = (\phi_{ij})$, the *Schur multiplier* with the *symbol* ϕ (denoted by $\mathbf{S}_\phi$) is the map taking an operator $T = (T_{ij})$ to $(\phi_{ij} T_{ij})$ (the latter matrix is sometimes denoted by $\phi \diamond T$). Note that, if $A = \operatorname{diag}(a_i)$ and $B = \operatorname{diag}(b_j)$, then $\mathbf{M}_{A,B}$ is the Schur multiplier with the symbol $(a_i b_j)_{i,j=1}^\infty$. In Section 4, we complement the results of [15] by computing the c.b. norms of certain Schur multipliers into $\ell^1(\mathbb{N} \times \mathbb{N})$. In Section 5, we deal with completely p-summing norms of Schur multipliers. Theorem 5.1 gives a factorization result for completely p-summing Schur multipliers from $B(\ell^2)$ to $\mathcal{S}^p$ $(1 \leqslant p \leqslant 2)$. In the particular case of $p = 2$, this is a complete description of completely p-summing Schur multipliers. Furthermore, for a Schur multiplier from $B(\ell^2)$ to $\mathcal{S}^2$, we have $\pi_2^o(\mathbf{S}_\phi) = \|\mathbf{S}_\phi\|_{cb}$. Finally, we show that $\pi_p^o(\mathbf{S}_\phi) = \|\phi\|_p$ when $\mathbf{S}_\phi$ is considered as a map from $\mathcal{S}^{p'}$ into $\mathcal{S}^p$, where $1/p + 1/p' = 1$ (Proposition 5.5).

In the commutative case, the operator norms of the diagonal maps from ℓ^p to ℓ^q are easy to compute. The summing norms (and other ideal norms) of such diagonal operators were computed by Garling and Carl [2, 6]. For the convolution operators $T_\mu : f \mapsto f * \mu$, acting on $L^1(G)$ or $C(G)$ (G is a metrizable compact Abelian group), the criteria for being nuclear or p-summing were given in [19] (in fact, the paper also covers convolutions with vector-valued measures). Further information on p-summing norms of convolutors from L_q to L_r can be found in [1]. In the non-commutative setting, certain estimates on ideal norms of formal identities between Schatten spaces were obtained in [3]. A few more ideal norms were estimated in [13].

Recall that the *completely p-summing norm* $\pi_p^o(u)$ $(1 \leqslant p \leqslant \infty)$ of an operator $u : E \to F$ (E and F are operator spaces) is defined as $\pi_p^o(u) = \|I_{\mathcal{S}^p} \otimes u : \mathcal{S}^p \otimes E \to \mathcal{S}^p[F]\|$. This notion was introduced by G. Pisier in [16]. We refer the reader to that article for the necessary background, such as the definition and properties of the space $\mathcal{S}^p[F]$, the non-commutative Pietsch factorization, and a variety of other results. Here, we recall a few basic facts (having their counterparts in the classical theory).

- $\pi_p^o(\cdot)$ is an ideal norm: $\pi_p^o(v_1 u v_2) \leqslant \|v_1\|_{cb} \pi_p^o(u) \|v_2\|_{cb}$. Moreover, $\pi_p^o(u) = \sup_G \pi_p^o(u|_G)$, with the supremum taken over all finite dimensional subspaces of G of the domain of u.
- If $q \geqslant p$, then $\pi_p^o(u) \geqslant \pi_q^o(u)$. In particular, $\pi_p^o(u) \geqslant \pi_\infty^o(u) = \|u\|_{cb}$.
- For $A, B \in \mathcal{S}^p$, $\pi_p(\mathbf{M}_{A,B} : B(\ell^2) \to \mathcal{S}^p) \leqslant \|A\|_{2p} \|B\|_{2p}$ (below, we see that actually we have an equality).

We also need a few easy facts concerning multiplication operators:

- The composition of multiplication operators is again a multiplication operator: $\mathbf{M}_{A_1,B_1} \mathbf{M}_{A_2,B_2} = \mathbf{M}_{A_1 A_2, B_2 B_1}$.
- The adjoint of multiplication operators is again a multiplication operator: $(\mathbf{M}_{A,B})^* = \mathbf{M}_{{}^tB, {}^tA}$.

One more ideal of operators will be investigated. Following Section 3.1.3 of [7], we say that an operator $u : X \to Y$ is *p-completely nuclear* $(1 \leqslant p \leqslant \infty)$ if it has a factorization $u = w\mathbf{M}_{A,B}v$, with $v : X \to \mathcal{S}^\infty$, $w : \mathcal{S}^p \to Y$, and $\mathbf{M}_{A,B} : \mathcal{S}^\infty \to \mathcal{S}^p$, with $A, B \in \mathcal{S}^{2p}$. The p-completely nuclear norm $\nu_p^o(u)$ is defined as the infimum of $\|w\|_{cb}\|A\|_{2p}\|B\|_{2p}\|v\|_{cb}$, running over all the factorizations as above. Denote the class of completely p-nuclear operators by $\mathcal{N}_p^o$. The factorization results for completely p-summing operators show that, for any u, $\pi_p^o(u) \leqslant \nu_p^o(u)$. Furthermore, it was shown by M. Junge [7] that $\Pi_p^o(X, Y)$ and $\mathcal{N}_{p'}^o(Y, X)$ $(1/p + 1/p')$ are in trace duality, provided the spaces X and Y are finite dimensional.

To finish this Introduction, recall that, by [16], $\mathcal{S}_n^p = C_n^p \otimes_h R_n^p$, where C_n^p (respectively, R_n^p) is the column (row) subspace of $\mathcal{S}_n^p$. Moreover, for any operator space X, $\mathcal{S}_n^p[X] = C_n^p \otimes_h X \otimes_h R_n^p$. In the infinite dimensional case, we have $\mathcal{S}^p = C^p \otimes_h R_p$, and $\mathcal{S}^p[X] = C^p \otimes_h X \otimes_h R^p$, where C^p and R^p are the column and row subspaces of $\mathcal{S}^p$.

It is known that $C_n^\infty = C_n$, and $R_n^\infty = R_n$ (the usual column and row spaces). Moreover, $C_n^{p*} = R_n^p = C_n^{p'}$ $(1/p + 1/p' = 1)$, and $(C_n^{p_0}, C_n^{p_1})_\theta = C_n^p$ for $1/p = (1-\theta)/p_0 + \theta/p_1$. These results remain true for C^p and R^p.

2. Completely bounded norms

Here we compute the c.b. norms of certain multiplication operators $\mathbf{M}_{A,B}$, with $A, B \in B(\ell^2)\backslash\{0\}$. Our main result is

Theorem 2.1. *Suppose* $1 \leqslant p, q \leqslant \infty$, *and* $1/r = |1/p - 1/q|$. *Then* $\|\mathbf{M}_{A,B}\|_{CB(\mathcal{S}^p,\mathcal{S}^q)} = \|A\|_{2r}\|B\|_{2r}$. *If either* A *or* B *does not belong to* $\mathcal{S}^r$, *then* $\mathbf{M}_{A,B} \notin CB(\mathcal{S}^p, \mathcal{S}^q)$.

The following lemma will be used extensively throughout this paper, in order to reduce our problems to finite dimensional ones. Denote by p_n the projection onto the linear span of the first n elements of the canonical basis of ℓ^2.

Lemma 2.2. *Suppose* $A, B \in B(\ell^2)$, *and* $\mathcal{E}$ *and* $\mathcal{F}$ *are either* $\mathcal{S}^r$ $(1 \leqslant r \leqslant \infty)$ *or* $B(\ell^2)$. *Then* $\|M_{A,B}\|_{CB(\mathcal{E},\mathcal{F})} = \sup_n \|M_{p_nA,Bp_n}\|_{CB(\mathcal{E},\mathcal{F})}$. *Moreover, for any* $1 \leqslant p < \infty$, $\pi_p^o(M_{A,B} : \mathcal{E} \to \mathcal{F}) = \sup_n \pi_p^o(M_{p_nA,Bp_n} : \mathcal{E} \to \mathcal{F})$.

A further technical lemma is needed. Below, $\mathbf{P}_n$ is the truncation operator $T \mapsto p_nTp_n$ (in other words, $\mathbf{P}_n = \mathbf{M}_{p_n,p_n}$).

Lemma 2.3. *Suppose* $1 \leqslant p < \infty$, $N \in \mathbb{N}$, *and* $y \in \mathcal{S}_N^p[\mathcal{F}]$, *where* $\mathcal{F}$ *is either* $\mathcal{S}^q$ $(1 \leqslant q \leqslant \infty)$ *or* $B(\ell^2)$. *Then* $\|y\| = \lim_n \|(I_{\mathcal{S}_N^p} \otimes \mathbf{P}_n)(y)\|$.

Proof. The case of $\mathcal{F} = \mathcal{S}^q$ (with $1 \leqslant q \leqslant \infty$) is easy. Indeed, the map $z \mapsto (I_{\mathcal{S}_N^p} \otimes \mathbf{P}_n)(z)$ is contractive. Furthermore, we can approximate y by elements of the form $\sum_{i=1}^M a_i \otimes u_i$, with $a_i \in \mathcal{S}_N^p$ and $u_i \in \mathcal{F}$. As $\lim_n \mathbf{P}_n u_i = u_i$ for each i, we are done.

The case of $\mathcal{F} = B(\ell^2)$ is slightly more complicated. Suppose, for the sake of contradiction, that $y \in \mathcal{S}_N^p[B(\ell^2)]$ is such that $\|y\| > 1$, yet there exists $\delta > 0$ s.t. $\|y_n\| < (1-\delta)^3$ for any n (here, $y_n = (I_{\mathcal{S}_N^p} \otimes \mathbf{P}_n)(y)$). Viewing M_n as the upper left corner of $B(\ell^2)$, we can consider y_n as an element of $\mathcal{S}_N^p[M_n]$. By Theorem 1.5 of [16], for each n there exist non-negative $C_n, D_n \in \mathcal{S}_N^{2p}$ and $v_n \in M_N(M_n)$, such that $\|C_n\| = \|D_n\| = \|v_n\| < 1-\delta$, and $y_n = (C_n \otimes I_{\ell_n^2})v_n(D_n \otimes I_{\ell_n^2})$. By compactness, there exists $n_1 < n_2 < \ldots$ s.t. the sequences (C_{n_k}) and (D_{n_k}) converge to $\tilde{C}$, respectively $\tilde{D}$. Let $C = \sigma + \tilde{C}$ and $D = \sigma + \tilde{D}$, where $\sigma > 0$ is so small that C and D lie in the unit ball of $\mathcal{S}_N^{2p}$. Let

$$u_k = (C^{-1}C_{n_k} \otimes I_{\ell_{n_k}^2})v_{n_k}(D_{n_k}D^{-1} \otimes I_{\ell_{n_k}^2}).$$

Note that $\|u_k\| \leqslant 1-\delta$ for each k, and $y_{n_k} = (C \otimes I_{\ell_{n_k}^2})u_k(D \otimes I_{\ell_{n_k}^2})$, with $\|u_k\|_{M_N(M_{n_k})} < 1$. We can view (u_k) as a sequence in $M_N(B(\ell^2))$. Passing to a further subsequence if necessary, we can assume that (u_k) converges weak* to u in the unit ball of $M_N(B(\ell^2))$. Then weak$^* - \lim y_{n_k} = (C \otimes I_{\ell^2})u(D \otimes I_{\ell^2})$ lies in the unit ball of $\mathcal{S}_N^p[B(\ell^2)]$. On the other hand, weak$^* - \lim y_{n_k} = y$, which is assumed to have norm greater than 1. This yields a contradiction. ■

Proof of Lemma 2.2. We work with the completely p-summing norm, as the case of the c.b. norm is similar. We show that, if $\pi_p^o(M_{A,B}) > 1$, then $\pi_p^o(M_{p_nA,Bp_n}) \geqslant 1$ for n sufficiently large. Find $N \in \mathbb{N}$ and $x \in S_N^p \otimes \mathcal{E}$ s.t. $\|x\| < 1$, and

$$\|(I_{M_N} \otimes \mathbf{M}_{A,B})x\|_{\mathcal{S}_N^p[\mathcal{F}]} = \|(I_{\ell_N^2} \otimes A)x(I_{\ell_N^2} \otimes B)\|_{\mathcal{S}_N^p[\mathcal{F}]} > 1.$$

By Lemma 2.3,

$$\begin{aligned}\|(I_{\ell_N^2} \otimes A)x(I_{\ell_N^2} \otimes B)\| &= \lim_n \|(I_{\mathcal{S}_N^p} \otimes \mathbf{P}_n)((I_{\ell_N^2} \otimes A)x(I_{\ell_N^2} \otimes B))\| \\ &= \lim_n \|(I_{\ell_N^2} \otimes p_nA)x(I_{\ell_N^2} \otimes Bp_n)\|,\end{aligned}$$

yielding the desired result. ■

The next lemma yields upper estimates the c.b. norms of the multiplication operators.

Lemma 2.4. *For $1 \leqslant p, q \leqslant \infty$ and $1/r = |1/p - 1/q|$, $\|\mathbf{M}_{A,B}\|_{CB(\mathcal{S}^p,\mathcal{S}^q)} \leqslant \|A\|_{2r}\|B\|_{2r}$. The same estimate holds when $\mathcal{S}^\infty$ (when p or q equals ∞) is replaced by $B(\ell^2)$.*

Proof. By Lemma 2.2, it suffices to show that $\|\mathbf{M}_{A,B}\|_{CB(\mathcal{S}_n^p,\mathcal{S}_n^q)} \leqslant \|A\|_{2r}\|B\|_{2r}$, for $n \times n$ matrices A and B. By [23], the natural identifications $\Phi : \mathcal{S}_n^{2r} \to CB(C_n^p, C_n^q)$ and $\Psi : \mathcal{S}_n^{2r} \to CB(R_n^p, R_n^q)$ are isometries. In this notation, $\mathbf{M}_{A,B} = \Phi(A) \otimes \Psi(B)$. Indeed, by polar decomposition we can assume that A and B are diagonal: write $A = \mathrm{diag}\,((a_i)_{i=1}^n)$, $B = \mathrm{diag}\,((b_i)_{i=1}^n)$. The matrix units in $\mathcal{S}_n^p$ are denoted by E_{ij} $(1 \leqslant i, j \leqslant n)$. Identify C_n^p and R_n^p with $\mathrm{span}[E_{i1} : 1 \leqslant i \leqslant n]$ and $\mathrm{span}[E_{1j} : 1 \leqslant j \leqslant n]$, respectively. E_{ij} can be identified with $E_{i1} \otimes E_{1j}$. For $\xi = \sum_i \xi_i E_{i1}$,

$A\xi = \sum_i a_i \xi_i E_{i1}$. The action of B can be described in a similar way. Thus $\mathbf{M}_{A,B} E_{ij} = \Phi(A) E_{i1} \otimes \Psi(B) E_{j1}$. By the properties of tensor products,

$$\begin{aligned} \|\mathbf{M}_{A,B}\|_{cb} &= \|\Phi(A) \otimes \Psi(B)\|_{CB(C_n^p \otimes_h R_n^p, C_n^q \otimes_h R_n^q)} \\ &\leqslant \|\Phi(A)\|_{CB(C_n^p, C_n^q)} \|\Psi(B)\|_{CB(R_n^p, R_n^q)} \leqslant \|A\|_{2r} \|B\|_{2r}, \end{aligned}$$

as desired. ■

Lemma 2.5. *For every $A, B \in B(\ell^2) \backslash \{0\}$, $\|\mathbf{M}_{A,B}\|_{CB(\mathcal{S}^1, \mathcal{S}^\infty)} = \|\mathbf{M}_{A,B}\|_{CB(\mathcal{S}^1, B(\ell^2))} = \|A\|_2 \|B\|_2$.*

Proof. We only need to consider the finite dimensional version: $\|\mathbf{M}_{A,B}\|_{CB(\mathcal{S}_n^1, \mathcal{S}_n^\infty)} = \|A\|_2 \|B\|_2$. By polar decomposition, we can assume that $A = \operatorname{diag}\left((a_i)_{i=1}^n\right)$ and $B = \operatorname{diag}\left((b_i)_{i=1}^n\right)$. Then $\mathbf{M}_{A,B} E_{ij} = a_i b_j E_{ij}$, where E_{ij} are the matrix units. The canonical isometry between $CB(E, F)$ and $E^* \otimes F$ (for finite dimensional E and F) allows us to identify the operator $u = \mathbf{M}_{A,B} \in CB(\mathcal{S}_n^1, \mathcal{S}_n^\infty)$ with $\tilde{u} = \sum_{i,j=1}^n a_i b_j E_{ij} \otimes E_{ij} \in M_n \otimes M_n$. The space $M_n \otimes M_n$ can be identified with M_{n^2}, with $E_{ij} \otimes E_{k\ell}$ corresponding to $E_{ik, j\ell}$. Therefore,

$$\begin{aligned} \|u\|_{cb} = \|\tilde{u}\| = \|(\sum_{i=1}^n a_i E_{ii,1})(\sum_{j=1}^n b_j E_{1,jj})\| = \|\sum_{i=1}^n a_i E_{ii,1}\| \cdot \|\sum_{j=1}^n b_j E_{1,jj}\| \\ = \big(\sum_{i=1}^n |a_i|^2\big)^{1/2} \big(\sum_{j=1}^n |b_j|^2\big)^{1/2} = \|A\|_2 \|B\|_2. \end{aligned}$$

■

Next we work with the "weighted transposition" operator. We denote by $\mathbf{\Theta}$ the transposition operator $A \mapsto {}^t\!A$ on the space of $n \times n$ matrices. In other words, $\mathbf{\Theta} E_{ij} = E_{ji}$, where $(E_{ij})_{i,j=1}^n$ are the matrix units. The same operator, acting from $\mathcal{S}_n^p$ to $\mathcal{S}_n^q$, is denoted by $\mathbf{\Theta}^{p,q}$.

Lemma 2.6. *For $1 \leqslant s \leqslant 2$, $\|\mathbf{\Theta}\|_{CB(\mathcal{S}_n^s, \mathcal{S}_n^{s'})} = 1$.*

Together with Lemma 2.4, this lemma immediately implies

Corollary 2.7. *Suppose $1 \leqslant q \leqslant \infty$, $1 \leqslant r \leqslant 2$, and $1/t = |1 - 1/r - 1/q|$. Suppose, furthermore, that the sequences $\alpha = (\alpha_i)_{i=1}^n$ and $\beta = (\beta_j)_{j=1}^n$ satisfy $\|\alpha\|_{2t} = \|\beta\|_{2t} = 1$. Then the operator $T : \mathcal{S}^r \to \mathcal{S}^q : E_{ji} \mapsto \alpha_i \beta_j E_{ij}$ is completely contractive.*

Proof of Lemma 2.6. Clearly, $\|\mathbf{\Theta}\|_{cb} \geqslant \|\mathbf{\Theta} E_{11}\| = 1$. To prove the converse inequality, note that, for $s = 2$, $\|\mathbf{\Theta}\|_{CB(\mathcal{S}_n^2)} = \|\mathbf{\Theta}\|_{B(\mathcal{S}_n^2)} = 1$, as the operator Hilbert space $\mathcal{S}_n^2$ is 1-homogeneous. For $s = 1$, we identify $\mathbf{\Theta} \in CB(\mathcal{S}_n^1, M_n)$ with an element $\sum_{ij} E_{ji} \otimes E_{ij} \in M_n(M_n)$, which has norm 1. We complete the proof by using complex interpolation. ■

Lemma 2.8. *For every $A, B \in B(\ell^2) \backslash \{0\}$, $\|\mathbf{M}_{A,B}\|_{CB(\mathcal{S}^\infty, \mathcal{S}^1)} = \|A\|_2 \|B\|_2$.*

Proof. As before, we can only consider the $n \times n$ case. The upper estimate for $\|\mathbf{M}_{A,B}\|_{CB(\mathcal{S}^\infty,\mathcal{S}^1)}$ follows from Lemma 2.4. To prove the lower estimate, assume $A = \mathrm{diag}\,((a_i)_{i=1}^n)$ and $B = \mathrm{diag}\,((b_j)_{j=1}^n)$ with a_i, b_j non-negative, and $\|A\|_2 = \|B\|_2 = 1$.

Identify the transposition map $\mathbf{\Theta} : \mathcal{S}_n^1 \to M_n : E_{ij} \mapsto E_{ji}$ with $\tilde{u} = \sum_{i,j=1}^n E_{ij} \otimes E_{ji} \in M_n \otimes M_n$, which has norm 1. Thus, it suffices to show that $\|\tilde{v}\| \geqslant 1$, where

$$\tilde{v} = (I_{M_n} \otimes \mathbf{M}_{A,B})\tilde{u} = \sum_{i,j=1}^n E_{ij} \otimes a_j b_i E_{ji} \in M_n \otimes \mathcal{S}_n^1.$$

But $\tilde{v}$ corresponds to $v : M_n \to M_n : E_{ji} \mapsto a_j b_i E_{ij}$. To show that $\|v\|_{cb} \geqslant 1$, note that $(I_{M_n} \otimes v)\tilde{u} = \sum_{i,j=1}^n E_{ij} \otimes a_j b_i E_{ij}$ can be identified with $(\sum_i b_i E_{ii,1})(\sum_j a_j E_{1,jj}) \in M_{n^2}$, the latter having norm one. ■

Proof of Theorem 2.1. The case of $p = q$ is trivial. Suppose $p < q$, and show that $\|\mathbf{M}_{A,B}\|_{CB(\mathcal{S}^p,\mathcal{S}^q)}\| = \|A\|_{2r}\|B\|_{2r}$, where $1/r = 1/p - 1/q$. The upper estimate for $\|\mathbf{M}_{A,B}\|_{cb}$ was established in Lemma 2.4. It suffices to obtain the lower estimate when A and B are positive $n \times n$ matrices. Let $1/r_1 = 1 - 1/p$ and $1/r_2 = 1/q - 1/\infty$. Find $A_1, B_1 \in \mathcal{S}_n^{2r_1}$ and $A_2, B_2 \in \mathcal{S}_n^{2r_2}$ so that $\|A_1\|_{2r_1} = \|B_1\|_{2r_1} = \|A_2\|_{2r_2} = \|B_2\|_{2r_2} = 1$, $\|A_0\|_2 = \|A\|_{2r}$, and $\|B_0\|_2 = \|B\|_{2r}$, where $A_0 = A_2 A A_1$ and $B_0 = B_1 B B_2$. Then $\mathbf{M}_{A_0,B_0} = \mathbf{M}_{A_2,B_2}\mathbf{M}_{A,B}\mathbf{M}_{A_1,B_1}$, hence

$$\|\mathbf{M}_{A_0,B_0}\|_{CB(\mathcal{S}_n^1,\mathcal{S}_n^\infty)} \leqslant \|\mathbf{M}_{A_1,B_1}\|_{CB(\mathcal{S}_n^1,\mathcal{S}_n^p)}\|\mathbf{M}_{A,B}\|_{CB(\mathcal{S}_n^p,\mathcal{S}_n^q)}\|\mathbf{M}_{A_2,B_2}\|_{CB(\mathcal{S}_n^q,\mathcal{S}_n^\infty)}$$

By Lemma 2.5, $\|\mathbf{M}_{A_0,B_0}\|_{CB(\mathcal{S}_n^1,\mathcal{S}_n^\infty)} = \|A_0\|_2\|B_0\|_2 = \|A\|_{2r}\|B\|_{2r}$. By Lemma 2.4, $\mathbf{M}_{A_1,B_1}$ and $\mathbf{M}_{A_2,B_2}$ are complete contractions. Thus, $\|\mathbf{M}_{A,B}\|_{cb} \geqslant \|A\|_{2r}\|B\|_{2r}$.

The case of $p > q$ is handled similarly, except that one uses Lemma 2.8 instead of Lemma 2.5. ■

3. OPERATOR p-SUMMING NORMS

Throughout this section, we use the notation $w' = w/(w-1)$, for $w \in [1,\infty]$ (in other words, w' satisfies $1/w + 1/w' = 1$). The main result of this section is

Theorem 3.1. *Consider $\mathbf{M}_{A,B}$ as an operator from $\mathcal{S}^{p_1}$ to $\mathcal{S}^{p_2}$. In each of the following situations, $\pi_p^o(\mathbf{M}_{A,B}) = \nu_p^o(\mathbf{M}_{A,B}) = \|A\|_{2r}\|B\|_{2r}$.*

1. $p_1' \leqslant p \leqslant p_2$, $1/r = 1/p_1 + 2/p - 1/p_2$.
2. $p_2 \leqslant p \leqslant p_1$, $1/r = 1/p_1 + 1/p_2$.
3. $p_2' \leqslant \min\{2,p\} \leqslant \max\{2,p\} \leqslant p_1'$, $1/r = 1/p_1' + 1/p_2'$.
4. $p_1 \leqslant p \leqslant p_2' \leqslant 2$, $1/r = 1/p_2 + 2/p - 1/p_1$.
5. $p_1 = \infty$, $1/r = 1/p + |1/p - 1/p_2|$. *In particular, if $p_1 = \infty \geqslant p \geqslant p_2$, then* $\nu_p^o(\mathbf{M}_{A,B}) = \pi_p^o(\mathbf{M}_{A,B}) = \|\mathbf{M}_{A,B}\|_{cb} = \|A\|_{2p}\|B\|_{2p}$.

Note that part (5) a strengthening of the Maurey Factorization Theorem for the class of multiplication operators (see Chapter 10 of [4] for the classical case, and [9, 10] for the recent non-commutative results).

Observing that $\mathbf{M}_{I,I}$ is the identity operator on $\mathcal{S}_n^q$, we conclude:

Corollary 3.2. *For* $n \in \mathbb{N}$, $q \in [2,\infty]$, *and* $q/(q-1) \leqslant p \leqslant q$, $\pi_p^o(I_{\mathcal{S}_n^q}) = n^{2/p}$. *More generally, for such* p *and* q, $\pi_p^o(I_{\mathcal{S}^q_{n_1 n_2}}) = (n_1 n_2)^{1/p}$.

Remark 3.3. For $p = 2$, this result follows from Chapter 6 of [16], where it was shown that $\pi_2^o(I_E) = \sqrt{\dim E}$ for any finite dimensional operator space E. For $p \neq 2$, the operator p-summing norms of operators on OH, and some other homogeneous Hilbertian spaces, were recently computed by M. Junge, Q. Xu, and K.-L. Yew [11, 24].

In the classical case, the ideal norms (including the p-summing norms) of $I_{\ell_n^q}$ can be found in e.g. [2, 6]. Furthermore, by Lemma 5.2 of [3],

$$\pi_2(id : \mathcal{S}^p_{n_1 n_2} \to \mathcal{S}^q_{n_1 n_2}) = \sqrt{n_1 n_2}\,\frac{\max\{1, \min\{n_1, n_2\}^{1/q-1/2}\}}{\max\{1, \min\{n_1, n_2\}^{1/p-1/2}\}}.$$

By Lemma 2.2, it suffices to establish the finite dimensional case of Theorem 3.1. Accordingly, we restrict ourselves to the finite dimensional situation for the rest of this section. Furthermore, we assume that $A = (\mathrm{diag}\,(a_i)_{i=1}^n)$ and $B = (\mathrm{diag}\,(b_j)_{j=1}^n)$, with (a_i) and (b_j) positive.

Lemma 3.4. *In each of the following cases,* $\nu_p^o(\mathbf{M}_{A,B}^{p_1,p_2}) \leqslant \|A\|_{2r}\|B\|_{2r}$*:*

(1)
$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p} + \left|\frac{1}{p} - \frac{1}{p_2}\right| = \begin{cases} 1/p_1 + 1/p_2 & p \geqslant p_2 \\ 2/p + 1/p_1 - 1/p_2 & p \leqslant p_2 \end{cases}.$$

(2) $p_1 \leqslant 2 \leqslant p_2$,
$$\frac{1}{r} = \frac{1}{p_1'} + \frac{1}{p} + \left|\frac{1}{p} - \frac{1}{p_2'}\right| = \begin{cases} 1/p_1' + 1/p_2' & p \geqslant p_2' \\ 2/p + 1/p_1' - 1/p_2' & p \leqslant p_2 \end{cases}.$$

Lemma 3.5. *In each of the following cases,* $\pi_p^o(\mathbf{M}_{A,B}^{p_1,p_2}) \geqslant \|A\|_{2r}\|B\|_{2r}$*:*

(1)
$$\frac{1}{r} = \frac{1}{p_2'} + \frac{1}{p} - \left|\frac{1}{p} - \frac{1}{p_1'}\right| = \begin{cases} 1/p_1 - 1/p_2 + 2/p & p \geqslant p_1' \\ 1/p_1' + 1/p_2' & p \leqslant p_1' \end{cases}.$$

(2)
$$\frac{1}{r} = \frac{1}{p_2} + \frac{1}{p} - \left|\frac{1}{p} - \frac{1}{p_1}\right| = \begin{cases} 1/p_1 + 1/p_2 & p_1 \geqslant p \\ 2/p + 1/p_2 - 1/p_1 & p_1 \leqslant p \end{cases}.$$

(3) $p_1 = \infty$, $1/r = 1/p + |1/p - 1/p_2|$.

Proof of Lemma 3.4. (1) It suffices to consider the case of $\|A\|_{2r} = \|B\|_{2r} = 1$. Write $A = A_3A_2A_1$ and $B = B_1B_2B_3$, where $\|A_1\|_{2p_1} = \|B_1\|_{2p_1} = 1$, $\|A_2\|_{2p} = \|B_2\|_{2p} = 1$, and $\|A_3\|_{2q} = \|B_3\|_{2q} = 1$, where $1/q = |1/p - 1/p_2|$. We regard $\mathbf{M}_{A_1,B_1}$, $\mathbf{M}_{A_2,B_2}$, and $\mathbf{M}_{A_3,B_3}$ as acting from $\mathcal{S}^{p_1}$ to $\mathcal{S}^\infty$, from $\mathcal{S}^\infty$ to $\mathcal{S}^p$, and from $\mathcal{S}^p$ to $\mathcal{S}^{p_2}$, respectively. Then $\nu_p^o(\mathbf{M}_{A,B}) \leqslant \|\mathbf{M}_{A_1,B_1}\|_{cb}\nu_p^o(\mathbf{M}_{A_2,B_2})\|\mathbf{M}_{A_3,B_3}\|_{cb} \leqslant 1$.

(2) The factorization $\mathbf{M}^{p_1,p_2}_{A,B} = \mathbf{\Theta}^{p_2',p_2} \circ \mathbf{M}^{p_1',p_2'}_{{}^tB,{}^tA} \circ \mathbf{\Theta}^{p_1,p_1'}$, Lemma 2.6, and part (1) yield the result. ■

Proof of Lemma 3.5. By Lemma 2.2, it suffices to estimate $\pi_p^o(\mathbf{M}_{A,B})$ from below when A and B are $n \times n$ matrices. By scaling and approximation, we can assume that $A, B \in M_n$, $\|A\|_{2r} = \|B\|_{2r} = 1$, $A = \mathrm{diag}\,((a_i)_{i=1}^n)$, and $B = \mathrm{diag}\,((b_j)_{j=1}^n)$, with a_i, b_j positive.

(1) Let $1/s = 1/p_2 + 1/p' + |1/p_1 - 1/p'|$, and note that $1/r + 1/s = 2$. Find positive operators $A_1 = (\mathrm{diag}\,(a_i^{(1)})_{i=1}^n)$ and $B_1 = (\mathrm{diag}\,(b_j^{(1)})_{j=1}^n)$ s.t.

$$\|A_1 A\|_1 = \|A_1\|_{2s} = \|A\|_{2r} = 1 = \|BB_1\|_1 = \|B_1\|_{2s} = \|B\|_{2r}.$$

Then $\mathbf{M}_{A_1B_1}\mathbf{M}_{AB}E_{ij} = a_i a_i^{(1)} b_j b_j^{(1)} E_{ij}$ for any matrix unit E_{ij}, hence the trace of $\mathbf{M}_{A_1B_1}\mathbf{M}_{AB}$ (acting on $\mathcal{S}_n^{p_1}$) equals $\sum_{i,j} a_i a_i^{(1)} b_j b_j^{(1)} = \|A_1 A\|_1 \|BB_1\|_1 = 1$. By the trace duality,

$$1 = \mathrm{Tr}\,(\mathbf{M}_{A_1B_1}\mathbf{M}_{AB}) \leqslant \pi_p^o(\mathbf{M}_{AB})\nu_{p'}^o(\mathbf{M}_{A_1B_1}).$$

By Lemma 3.4, $\nu_{p'}^o(\mathbf{M}_{A_1B_1}) \leqslant \|A_1\|_{2s}\|B_1\|_{2s} = 1$, which yields the desired estimate for $\pi_p^o(\mathbf{M}_{AB})$.

(2) Suppose, for the sake of contradiction, that $\pi_p^o(\mathbf{M}_{A,B}) < 1$. Let $1/s = |1/p_2 - 1/p|$, $1/t = |1/p - 1/p_1|$, and $q = \min\{p, p_2\}$. Find sequences $a^{(1)} = (a_i^{(1)})_{i=1}^n$, $b^{(1)} = (b_j^{(1)})_{j=1}^n$, $a^{(2)} = (a_i^{(2)})_{i=1}^n$, and $b^{(2)} = (b_j^{(2)})_{j=1}^n$ in such a way that

$$\|(a_i^{(1)})\|_{2t} = \|(a_i^{(2)})\|_{2s} = \|(a_i a_i^{(1)} a_i^{(2)})\|_{p_2} = \|(b_j^{(1)})\|_{2t} = \|(b_j^{(2)})\|_{2s} = \|(b_j^{(1)} b_j b_j^{(2)})\|_q = 1$$

(this is possible, since $\|(a_i)\|_{2r} = \|(b_j)\|_{2r} = 1$, and $1/r + 1/s + 1/t = 2/q$). Consider $u = \sum_{ij} a_i^{(1)} b_j^{(1)} E_{ji} \otimes E_{ij} \in \mathcal{S}_n^p \otimes \mathcal{S}_n^{p_1}$. By Corollary 2.7, $\|u\| \leqslant 1$. Let

$$v = (I_{\mathcal{S}_n^p} \otimes \mathbf{M}_{A,B})(u) = \sum_{ij} a_i a_i^{(1)} b_j b_j^{(1)} E_{ji} \otimes E_{ij}.$$

If $\pi_p^o(\mathbf{M}_{A,B}) < 1$, then $\|v\|_{\mathcal{S}_n^p[\mathcal{S}_n^{p_2}]} < 1$. We achieve the desired contradiction by showing that $\|v\|_{\mathcal{S}_n^p[\mathcal{S}_n^{p_2}]} \geqslant 1$.

Suppose first $p \geqslant p_2$. By [22],

$$\|v\|_{\mathcal{S}_n^p[\mathcal{S}_n^{p_2}]} = \sup_{\|C\|_{2s}, \|D\|_{2s} \leqslant 1} \|(\mathbf{M}_{C,D} \otimes I_{\mathcal{S}_n^{p_2}})v\|_{\mathcal{S}_n^{p_2}[\mathcal{S}_n^{p_2}]}. \tag{3.1}$$

Taking $C = (\mathrm{diag}\,(b_j^{(2)}))$ and $D = (\mathrm{diag}\,(a_i^{(2)}))$, we have:

$$\begin{aligned}\|v\|_{\mathcal{S}_n^p[\mathcal{S}_n^{p_2}]} &\geqslant \Big\| \sum_{i,j=1}^n E_{ji} \otimes a_i a_i^{(1)} a_i^{(2)} b_j b_j^{(1)} b_j^{(2)} E_{ij} \Big\|_{\mathcal{S}_n^{p_2}[\mathcal{S}_n^{p_2}]} \\ &= \Big(\sum_{i,j=1}^n |a_i a_i^{(1)} a_i^{(2)} b_j b_j^{(1)} b_j^{(2)}|^{p_2} \Big)^{1/p_2} = \|(a_i a_i^{(1)} a_i^{(2)})\|_{p_2} \|(b_j b_j^{(1)} b_j^{(2)})\|_{p_2} = 1,\end{aligned}$$

which is the estimate we need.

For $p \leqslant p_2$, the proof proceeds along similar lines, with one exception: instead of (3.1), we use the inequality

$$\|v\|_{\mathcal{S}_n^p[\mathcal{S}_n^{p_2}]} \geqslant \|(I_{\ell_n^2} \otimes C)v(I_{\ell_n^2} \otimes D)\|_{\mathcal{S}_n^p[\mathcal{S}_n^p]}$$

whenever C and D are in the unit ball of $\mathcal{S}_n^{2s}$ (this follows from the fact that $\mathbf{M}_{C,D} : \mathcal{S}_n^{p_2} \to \mathcal{S}_n^p$ is completely contractive, and Corollary 1.2 of [16]). In particular,

$$\begin{aligned} &\|v\|_{\mathcal{S}_n^p[\mathcal{S}_n^{p_2}]} \geqslant \|(I_{\ell_n^2} \otimes A_2)v(I_{\ell_n^2} \otimes B_2)\|_{\mathcal{S}_n^p[\mathcal{S}_n^p]} \\ &= \|\sum a_i a_i^{(1)} a_i^{(2)} b_j b_j^{(1)} b_j^{(2)} E_{ji} \otimes E_{ij}\|_{\mathcal{S}_n^p[\mathcal{S}_n^p]} = \|(a_i a_i^{(1)} a_i^{(2)})\|_p \|(b_j b_j^{(1)} b_j^{(2)})\|_p = 1. \end{aligned}$$

(3) In the case of $p_2 \leqslant p$, the inequality $\pi_p^o(T) \geqslant \|T\|_{cb}$ (valid for any operator T), and the equality $\|\mathbf{M}_{A,B}\|_{CB(\mathcal{S}^\infty,\mathcal{S}^{p_2})} = \|A\|_{2p_2}\|B\|_{2p_2}$ (Theorem 2.1), yield the proof. For $p_2 \geqslant p$, Remark 5.10 of [16] yields a factorization $\mathbf{M}_{A,B} = w\mathbf{M}_{A_1,B_1}$ via $\mathcal{S}_n^p$, with $\|A_1\|_{2p} = \|B\|_{2p} = 1$, and $\|w\|_{cb} = \pi_p^o(\mathbf{M}_{A,B})$. As the operators A and B are invertible, so are A_1 and B_1, and therefore, $w = \mathbf{M}_{A,B}\mathbf{M}_{A_1,B_1}^{-1} = \mathbf{M}_{AA_1^{-1},B_1^{-1}B}$. Then $\|w\|_{cb} = \pi_p^o(\mathbf{M}_{A,B}) = \|AA_1^{-1}\|_{2s}\|B_1^{-1}B\|_{2s}$, with $1/s = 1/p - 1/p_2$. As $1/r = 2/p - 1/p_2 = 1/s + 1/p$,

$$\|A\|_{2r}\|B\|_{2r} \leqslant \|AA_1^{-1}\|_{2s}\|A_1\|_{2p}\|B_1\|_{2p}\|B_1^{-1}B\|_{2s} \leqslant \pi_p^o(\mathbf{M}_{A,B}^{\infty,p_2}).$$

■

Proof of Theorem 3.1. For $i, j \in \{0, 1\}$, case $(2i + j + 1)$ of our theorem follows from Lemma 3.4(i) and Lemma 3.5(j). Case (5) follows from Lemma 3.4(1), Lemma 3.5(3), and Theorem 2.1. ■

4. Toeplitz Schur multipliers into ℓ^1

Here, we consider the c.b. norms of Toeplitz Schur multipliers from a matrix space into $\ell^1(\mathbb{N} \times \mathbb{N})$. For a sequence $m = (m_s)_{s\in\mathbb{Z}}$, we define the *Toeplitz Schur multiplier* $T_m : (a_{ij})_{i,j\in\mathbb{N}} \mapsto m_{j-i}a_{ij}$. The result below (essentially contained in [15]) describes the c.b. norm of T_m as acting from $\mathcal{S}^1$ to $\ell^1(\mathbb{N} \times \mathbb{N})$.

Proposition 4.1. *In the above notation,* $\|T_m\|_{cb} = \|T_m\| = \|m\|_{\ell^1}$.

Proof. It is shown in [15] that $\|T_m\| = \|m\|_{\ell^1}$. It remains to show that $\|T_m\|_{cb} \leqslant \|m\|_{\ell^1}$. By an extreme point argument, it suffices to consider the case of $m_s = 1$ for $s = p$, $m_s = 0$ otherwise. For such an m, we have to show that $\|I_{\mathcal{S}^1} \otimes T_m : \mathcal{S}^1[\mathcal{S}^1] \to \mathcal{S}^1[\ell^1(\mathbb{N} \times \mathbb{N})\| \leqslant 1$. Note that $\mathcal{S}^1[\mathcal{S}^1]$ is isometric to $\mathcal{S}^1(\mathbb{N} \times \mathbb{N})$. The extreme points of the closed unit ball of the latter space are of the form $x \otimes y$, where x and y are unit vectors in $\ell^2(\mathbb{N} \times \mathbb{N})$. Write $x = (x_j)$ and $y = (y_k)$, where $x_j, y_k \in \ell^2$, and $\sum_j \|x_j\|^2 = \sum_k \|y_k\|^2 = 1$. Then

$$\big(I_{\mathcal{S}^1} \otimes T_m(x \otimes y)\big)_{jk} = \begin{cases} x_j \otimes y_k & j - k = p \\ 0 & \text{otherwise} \end{cases}.$$

If $p \geqslant 0$, we have:

$$\begin{aligned}\|T_m(x \otimes y)\| &= \sum_{j-k=p} \|x_j\| \|y_k\| = \sum_{k=1}^{\infty} \|x_{k+p}\| \|y_k\| \\ &\leqslant \big(\sum_{k=1}^{\infty} \|x_{k+p}\|^2\big)^{1/2} \big(\sum_{k=1}^{\infty} \|y_k\|^2\big)^{1/2} \leqslant 1.\end{aligned}$$

The case of $p \leqslant 1$ is handled similarly. ■

Next we consider a Schur multiplier from the upper triangular trace class matrices to ℓ^1. More precisely, we consider the subspace $\mathcal{HS}^1$ of $\mathcal{S}^1$, consisting of all trace class matrices (a_{ij}) s.t. $a_{ij} = 0$ for $i > j$. A sequence $(m_s)_{s \geqslant 0}$ defines a multiplier $T_m : \mathcal{HS}^1 \to \ell^1(\mathbb{N} \times \mathbb{N}) : (a_{ij}) \mapsto (m_{j-a} a_{ij})$. Following [15], we define

$$\rho(m) = \Big(|m_0|^2 + |m_1|^2 + \sup_{r \geqslant 1} \sum_{q=1}^{\infty} \big(\sum_{s=rq+1}^{r(q+1)} |m_s|^2 \big)^2 \Big)^{1/2}.$$

Proposition 4.2. *In the above notation, there exists a constant c such that*

$$c\rho(m) \geqslant \|T_m\|_{cb} \geqslant \|T_m\| \geqslant \frac{\rho(m)}{\sqrt{3}}.$$

Sketch of a proof. It was shown in [15] (Theorem 3.3) that $c\rho(m) \geqslant \|T_m\| \geqslant \rho(m)/\sqrt{3}$. In fact, that proof can be easily modified to yield $c\rho(m) \geqslant \|T_m\|_{cb}$. ■

Finally, we consider the Schur multipliers from $B(\ell^2)$ (or $\mathcal{S}^\infty$) to $\ell^1(\mathbb{N} \times \mathbb{N})$. The space of bounded (completely bounded) Schur multipliers between matrix spaces E and F shall be denoted by $B^S(E, F)$ (resp. $CB^S(E, F)$).

The space $\ell^1(\ell^2)$ is viewed as the space of matrices $a = (a_{ij})$, equipped with the norm $\|a\|_{\ell^1(\ell^2)} = \sum_i (\sum_j |a_{ij}|^2)^{1/2}$. The space ${}^t\ell^1(\ell^2)$ is equipped with the norm $\|a\|_{{}^t\ell^1(\ell^2)} = \|{}^t a\|_{\ell^1(\ell^2)}$. Finally, we define the space $\mathcal{X} = \ell^1(\ell^2) + {}^t\ell^1(\ell^2)$. That is,

$$\|a\|_{\mathcal{X}} = \inf_{a=b+c} \|b\|_{\ell^1(\ell^2)} + \|c\|_{{}^t\ell^1(\ell^2)}.$$

In this notation, we have:

Proposition 4.3. *The following spaces are isomorphic (via the natural identity): (1) $\mathcal{X}$, (2) $B^S(B(\ell^2), \ell^1(\mathbb{N} \times \mathbb{N}))$, (3) $CB^S(B(\ell^2), \ell^1(\mathbb{N} \times \mathbb{N}))$, (4) $B^S(\mathcal{S}^\infty, \ell^1(\mathbb{N} \times \mathbb{N}))$, (5) $CB^S(\mathcal{S}^\infty, \ell^1(\mathbb{N} \times \mathbb{N}))$.*

Proof. It is easy to see that the spaces (2) and (4) are isometric, as are (3) and (5). It is shown in [15] that (1) and (3) are isomorphic. Moreover, $id : CB^S(B(\ell^2), \ell^1(\mathbb{N} \times \mathbb{N})) \to B^S(B(\ell^2), \ell^1(\mathbb{N} \times \mathbb{N}))$ is a contraction. By duality, it suffices to show that $id : \mathcal{X} \to CB^S(c_0(\mathbb{N} \times \mathbb{N}), \mathcal{S}^1)$ is a contraction. By symmetry, we need to prove that $id : \ell^1(\ell^2) \to CB^S(c_0(\mathbb{N} \times \mathbb{N}), \mathcal{S}^1)$ is a contraction. By an extreme point argument, it suffices to show that $\|\mathbf{S}_a\|_{CB^S(c_0(\mathbb{N}\times\mathbb{N}),\mathcal{S}^1)} \leqslant 1$ whenever the matrix $a = (a_{ij})$ is such that $a_{ij} = 0$ for $i > 1$, and $\sum_j |a_{1j}|^2 = 1$. This, in turn, is equivalent to proving that, for any family (x_{ij}) of contractive $n \times n$ matrices, $\| \sum_j a_{1j} E_{1j} \otimes x_{1j} \|_{\mathcal{S}^1 \otimes M_n} \leqslant 1$. The matrix units E_{1j} span a copy of C (the column space) in $\mathcal{S}^1$. Thus, we need to show that $\| \sum_j a_{1j} E_{j1} \otimes x_{1j} \|_{\mathcal{S}^\infty \otimes M_n} \leqslant 1$. However,

$$\| \sum_j a_{1j} E_{j1} \otimes x_{1j} \|^2_{\mathcal{S}^\infty \otimes M_n} = \| \sum_j |a_{1j}|^2 x_{1j}^* x_{1j} \| \leqslant \sum_j |a_{1j}|^2 \| x_{1j}^* x_{1j} \| \leqslant 1,$$

as desired. ■

5. Schur multipliers into Schatten spaces

In this section we consider Schur multipliers from $\mathcal{S}^p$ (or $B(\ell^2)$) to $\mathcal{S}^q$.

Theorem 5.1. *Consider $1 \leqslant p \leqslant 2$, a matrix $\phi = (\phi_{ij})$, and the following statements:*

1. $\pi_p^o(\mathbf{S}_\phi) \leqslant 1$, *where $\mathbf{S}_\phi$ is a Schur multiplier from $B(\ell^2)$ (or $\mathcal{S}^\infty$) to $\mathcal{S}^p$.*
2. *There exist sequences $a = (a_i)_{i \in \mathbb{N}}$ and $b = (b_j)_{j \in \mathbb{N}}$ in the unit ball of ℓ^{2p}, such that $|\phi_{ij}| \leqslant |a_i b_j|$ for any $i, j \in \mathbb{N}$.*
3. *There exists a matrix ψ, the sequences $a' = (a'_i)_{i \in \mathbb{N}}$ and $b' = (b'_j)_{j \in \mathbb{N}}$ in the unit ball of ℓ^4, and the sequences $a'' = (a''_i)_{i \in \mathbb{N}}$ and $b'' = (b''_j)_{j \in \mathbb{N}}$ in the unit ball of ℓ^{2r} $(1/r = 1/p - 1/2)$, such that $|\psi_{ij}| \leqslant |a'_i b'_j|$, and $\mathbf{S}_\phi = \mathbf{M}_{\mathrm{diag}\,(a''_i), \mathrm{diag}\,(b''_i)} \mathbf{S}_\psi$. Here, $\pi_2^o(\mathbf{S}_\psi) \leqslant 1$ ($\mathbf{S}_\psi$ is viewed as a map from $B(\ell^2)$ to $\mathcal{S}^2$), and furthermore, $\|\mathbf{M}_{\mathrm{diag}\,(a''_i), \mathrm{diag}\,(b''_i)}\|_{CB(\mathcal{S}^2, \mathcal{S}^p)} \leqslant 1$.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3). For $p = 2$, (2) $\Leftrightarrow$ (1).

By Theorem 5.5 and Remark 5.6, the implication (2) $\Rightarrow$ (1) fails for $p = 1$. We do not know whether it is true for $p \in (1, 2)$.

The next result shows that, for $p = 2$, the c.b. norms and the completely 2-summing norms of Schur multipliers from $B(\ell^2)$ to $\mathcal{S}^2$ are equivalent.

Theorem 5.2. *There is a constant C such that $\pi_2^o(\mathbf{S}_\phi) \leqslant C \|\mathbf{S}_\phi\|_{cb}$ for any Schur multiplier $\mathbf{S}_\phi$ from $B(\ell^2)$ (or $\mathcal{S}^\infty$) to $\mathcal{S}^2$.*

Remark 5.3. The classical Grothendieck Theorem tells us that any bounded operator from $C(K)$ to a Hilbert space is 2-summing. This is no longer true in the non-commutative case. Indeed, by Corollary 5 of of [8], for every N there exists a projection P from $B(\ell^2)$ to OH_N s.t. $\|P\|_{cb} \leqslant c\sqrt{N/(1 + \log N)}$. On the other hand, by Theorem 6.13 of [16], $\pi_2^o(P) \geqslant \pi_2^o(id_{OH_N}) = \sqrt{N}$. See [9, 10] for some positive results in this direction.

Considering a multiplication operator from $L^\infty(\mu)$ to $L^p(\mu)$, instead of a Schur multiplier, we complement the known results by computing the completely p-summing norm precisely.

Theorem 5.4. *Suppose μ is a σ-finite measure, and ϕ is a μ-measurable function. Denote by M_ϕ the operator of multiplication by ϕ, acting from $L^\infty(\mu)$ to $L^p(\mu)$. Then, for $1 \leqslant p \leqslant \infty$, $\pi_p(M_\phi) = \pi_p^o(M_\phi) = \|M_\phi\| = \|M_\phi\|_{cb} = \|\phi\|_p$.*

For the sake of completeness, we include a formula for the completely p-summing norms of Schur multipliers from $\mathcal{S}^{p'}$ to $\mathcal{S}^p$.

Proposition 5.5. *Suppose ϕ is a matrix, determining the Schur multiplier $\mathbf{S}_\phi$ from $\mathcal{S}^{p'}$ to $\mathcal{S}^p$ ($1 \leqslant p \leqslant \infty$, $1/p + 1/p' = 1$). Then $\pi_p^o(\mathbf{S}_\phi) = \|\phi\|_p$. For $p = 1$, the result is also true for Schur multipliers from $B(\ell^2)$ to $\mathcal{S}^1$.*

Note that, by applying this theorem to $\mathbf{M}_{A,B}$, we recover a particular case of Theorem 3.1 (that of $p = p_2 = p_1/(p_1 - 1)$).

Remark 5.6. In [18, 22], $\|\mathbf{S}_\phi\|_{cb}$ was computed for $1 \leqslant p \leqslant 2$. A counterpart of the previous proposition for sequence spaces was established in [6], where the following was proved: suppose $1 \leqslant p \leqslant \infty$, and $a = (a_i)$ is a bounded sequence, and $D_a : \ell^{p'} \to \ell^p$ maps δ_i' to $a_i\delta_i$ (here, (δ_i') and (δ_i) are the canonical bases in $\ell^{p'}$ and ℓ^p, respectively). Then $\pi_p(D_a) = \|a\|_p$. In particular, for $p = 1$, $\|D_a\| = \pi_1(D_a)$. In contrast to this, the c.b. norm and the completely 1-summing norm of Schur multipliers from $\mathcal{S}^\infty$ or $B(\ell^2)$ into $\mathcal{S}^1$ are not equivalent. Indeed, by [20], there exist matrices ϕ and ψ s.t. $|\psi_{ij}| \leqslant \phi_{ij}$ for any pair (i,j), $\phi \in \mathcal{S}^1$, while $\psi \notin \mathcal{S}^1$. By Proposition 5.5, $\mathbf{S}_\psi$ is not completely 1-summing. However, by [18], $\mathbf{S}_\psi$ is a completely bounded map from $\mathcal{S}^\infty$ (or $B(\ell^2)$) to $\mathcal{S}^1$.

To prove the results of this section, we state an immediate consequence of [16], Theorem 5.9.

Lemma 5.7. *Consider an operator $u : M_n \to \mathcal{S}_n^p$ ($1 \leqslant p \leqslant \infty$). Then $\pi_p^o(u) \leqslant 1$ if and only if there exist non-negative a and b in the unit ball of $\mathcal{S}_n^{2p}$, such that $\|(u(x^{(ij)}))\|_{\mathcal{S}_N^p[\mathcal{S}_n^p]} \leqslant \|(ax^{(ij)}b)\|_{\mathcal{S}_N^p[\mathcal{S}_n^p]}$ whenever $x^{(ij)}$ ($1 \leqslant i,j \leqslant N$) are elements of M_n.*

Next we prove Theorem 5.1. The implication (3) ⇒ (2) follows immediately from Theorem 2.1. Furthermore, (3) ⇒ (1) for $p = 2$ is immediate.

Proof of Theorem 5.1, (2) ⇒ (3), $p = 2$. It suffices to show the finite dimensional version. That is, consider an $n \times n$ matrix $\phi = (\phi_{k\ell})$, such that $|\phi_{k\ell}| \leqslant a_k b_\ell$, where $(a_k), (b_\ell)$ are non-negative numbers, with $\sum_k a_k^4 = \sum_\ell b_\ell^4 = 1$. By Lemma 5.7, we have to show that

$$\|(\mathbf{S}_\phi(x^{(ij)}))\|_{\mathcal{S}_N^2[\mathcal{S}_n^2]} \leqslant \|(ax^{(ij)}b)\|_{\mathcal{S}_N^2[\mathcal{S}_n^2]},$$

where $a = \operatorname{diag}((a_i))$ and $b = \operatorname{diag}((b_j))$. However,

$$\|(\mathbf{S}_\phi(x^{(ij)}))\|^2_{\mathcal{S}_N^2[\mathcal{S}_n^2]} = \sum_{i,j} \|\mathbf{S}_\phi(x^{(ij)})\|^2_{\mathcal{S}_n^2} \text{ and } \|(ax^{(ij)}b)\|^2_{\mathcal{S}_N^2[\mathcal{S}_n^2]} = \sum_{i,j} \|ax^{(ij)}b\|^2_{\mathcal{S}_n^2},$$

hence it suffices to show that $\|\mathbf{S}_\phi x\|_{\mathcal{S}_n^2} \leqslant \|axb\|_{\mathcal{S}_n^2}$ for any $n \times n$ matrix x. Writing $x = (x_{k\ell})$ and $\phi = (\phi_{k\ell})$, we see that

$$\|\mathbf{S}_\phi x\|_{\mathcal{S}_n^2}^2 = \sum_{k,\ell=1}^n |\phi_{k\ell}|^2 |x_{k\ell}|^2 \leqslant \sum_{k,\ell=1}^n a_k^2 b_\ell^2 |x_{k\ell}|^2 = \|axb\|_{\mathcal{S}_n^2}^2,$$

which completes the proof. ■

Proof of Theorem 5.1, (1) ⇒ (2). We handle the finite dimensional case first. Suppose an $n \times n$ matrix ϕ is such that $\pi_p^o(\mathbf{S}_\phi) \leqslant 1$ ($\mathbf{S}_\phi$ is viewed as a map from M_n to $\mathcal{S}_n^p$). We shall show that there exist $a = (a_i)_{i=1}^n$ and $b = (b_j)_{j=1}^n$ in the unit ball of ℓ_n^{2p}, with non-negative entries, such that $|\phi_{ij}| \leqslant a_i b_j$.

Suppose $\mathbf{S}_\phi : M_n \to \mathcal{S}_n^p$ satisfies $\pi_p^o(\mathbf{S}_\phi) \leqslant 1$. By Lemma 5.7, there exist c and d in the unit ball of $\mathcal{S}_n^{2p}$ s.t. $\|\mathbf{S}_\phi x\|_p \leqslant \|cxd\|_p$ for any $n \times n$ matrix x. If U and V are $n \times n$ matrices with ± 1 on the main diagonal, and zeroes elsewhere, we have $\|\mathbf{S}_\phi x\|_p = \|\mathbf{S}_\phi(UxV)\|_p \leqslant \|cUxVd\|_p$, and

$$\|\phi \diamond x\|_p^p \leqslant \operatorname{Tr}\Big(\big((cUxVd)(d^*Vx^*Uc^*) \big)^{p/2} \Big) = \operatorname{Tr}\Big(\big(d^*Vx^*Uc^*cUxVd \big)^{p/2} \Big). \tag{5.1}$$

The last inequality follows from the fact that any continuous function on a compact interval can be approximated by a polynomial arbitrarily well, hence $\operatorname{Tr}(F(AB)) = \operatorname{Tr}(F(BA))$ whenever AB and BA are non-negative matrices, and F is continuous on $[0, \infty)$ (in our case, $F(t) = t^{p/2}$). By a special case of Lieb's concavity (see e.g. Theorem 8.10 of [21]), the function $T \mapsto \operatorname{Tr} T^\alpha$ is concave on the set of non-negative matrices for $0 < \alpha \leqslant 1$.

Note that, for any matrix y, $\operatorname{Ave}(UyU)$ is a matrix with the same entries as y on the main diagonal, and zeroes elsewhere (here, we average over all possible choices of signs in U). Averaging with respect to U in (5.1), we obtain a diagonal operator a in the unit ball of $\mathcal{S}_n^{2p}$ with non-negative entries, such that $a^2 = \operatorname{Ave}_U(Uc^*cU)$, and

$$\|\phi \diamond x\|_p^p \leqslant \operatorname{Tr}\Big(\big(d^*Vx^*a^2xVd \big)^{p/2} \Big) = \operatorname{Tr}\Big(\big(x^*a^2xVdd^*V \big)^{p/2} \Big)$$

for any V as above. There exists a diagonal operator b in the unit ball of $\mathcal{S}_n^{2p}$ with non-negative entries, such that $b^2 = \operatorname{Ave}_V(Vd^*dV)$. Then

$$\|\phi \diamond x\|_p^p \leqslant \operatorname{Tr}\Big(\big(x^*a^2xb^2 \big)^{p/2} \Big) = \operatorname{Tr}\Big(\big((bx^*a)(axb) \big)^{p/2} \Big) = \|axb\|_p^p$$

for any matrix x.

Now consider the infinite dimensional case. By truncating to the upper left corner of size $n \times n$, we prove that, for every n, there exist $a^{(n)} = (a_1^{(n)}, \ldots, a_n^{(n)}, 0, \ldots)$ and $b^{(n)} = (b_1^{(n)}, \ldots, b_n^{(n)}, 0, \ldots)$ in the unit ball of ℓ^{2p}, with non-negative entries, such that $|\phi_{ij}| \leqslant a_i^{(n)} b_j^{(n)}$ for $1 \leqslant i, j \leqslant n$. Then there exist $a = (a_i)$, $b = (b_j)$, and $n_1 < n_2 < \ldots$ s.t. $\lim_k a_i^{(n_k)} = a_i$ for every i, and $\lim_k b_j^{(n_k)} = b_j$ for every j. Clearly, a and b are still in the unit ball of ℓ^{2p}, and $|\phi_{ij}| \leqslant a_i b_j$ for all i and j. ■

Proof of Theorem 5.1, (2) ⇒ (3). Suppose an $n \times n$ matrix $\phi = (\phi_{ij})$ satisfies $|\phi_{ij}| \leqslant a_i b_j$, where a_i and b_j are non-negative numbers, and $\sum_i a_i^{2p} = \sum_j b_j^{2p} \leqslant 1$. Write $\phi_{ij} = \omega_{ij}|\phi_{ij}|$, with $|\omega_{ij}| = 1$. Let $\psi_{ij} = \omega_{ij}|\phi_{ij}|^{1-p/2}$, $a_i' = a_i^{p/2}$, $b_i' = b_i^{p/2}$, $a_i'' = a_i^{(2-p)/2}$, and $b_i'' = b_i^{(2-p)/2}$. Then $\sum a_i'^4 = \sum a_i^{2p} = 1$, and, similarly, $\sum b_i'^4 = 1$, and $\sum a_i''^{2r} = \sum b_i''^{2r} = 1$. Moreover, $|\psi_{ij}| \leqslant a_i' b_j'$. As proved before, $\pi_2^o(\mathbf{S}_\psi) \leqslant 1$ ($\mathbf{S}_\psi$ is viewed as a map from $\mathcal{S}^\infty$ or $B(\ell^2)$ to $\mathcal{S}^2$). Moreover, by Theorem 2.1, $\mathbf{M}_{\mathrm{diag}\,(a_i''),\mathrm{diag}\,(b_i'')}$ is a complete contraction from $\mathcal{S}^2$ to $\mathcal{S}^p$. ■

Proof of Theorem 5.2. Suppose $\mathbf{S}_\phi : \mathcal{S}^\infty \to \mathcal{S}^2$ is a complete contraction, and prove that $\pi_2^o(\mathbf{S}_\phi) \leqslant C$, for some universal constant C. Note that $\mathbf{S}_\phi^* : \mathcal{S}^2 \to \mathcal{S}^1$ is a completely contractive Schur multiplier, given by the same matrix ϕ. Therefore, $\mathbf{S}_\psi = \mathbf{S}_\phi^* \mathbf{S}_\phi$ is a complete contraction (here, $\psi = \phi \diamond \phi$, or in other words, $\psi_{ij} = \phi_{ij}^2$ for $i, j \in \mathbb{N}$). By [18], there exist sequences (α_i) and (β_j) of non-negative numbers, such that $\sum_i \alpha_i^2 = \sum_j \beta_j^2 \leqslant C^2$ (C is an absolute constant), such that $|\psi_{ij}| \leqslant \alpha_i \beta_j$ for all i, j. Let $a_i = \sqrt{\alpha_i}$, and $b_j = \sqrt{\beta_j}$. Then $\|(a_i)\|_4, \|(b_j)\|_4 \leqslant \sqrt{C}$, and $|\phi_{ij}| \leqslant a_i b_j$. By Theorem 5.1, $\pi_2^o(\mathbf{S}_\phi) \leqslant C$. ■

Proof of Theorem 5.4. Clearly, $\|M_\phi\| = \|\phi\|_p \leqslant \|M_\phi\|_{cb} \leqslant \pi_p^o(M_\phi)$, and $\|M_\phi\| \leqslant \pi_p(M_\phi)$. It remains to show that $\|\phi\|_p \geqslant \max\{\pi_p(M_\phi), \pi_p^o(M_\phi)\}$. Suppose $\|\phi\|_p = 1$, and, without loss of generality, $\phi > 0$ everywhere. Consider the probability measure $\nu = \phi^p \mu$, and the complete isometries

$$T_\infty : L^\infty(\mu) \to L^\infty(\nu) : f \mapsto f \ \text{ and } \ T_p : L^p(\nu) \to L^p(\mu) : g \mapsto \phi g.$$

Let J be the formal identity map from $L^\infty(\nu)$ to $L^p(\nu)$. Then $M_\phi = T_p J T_\infty$. By Pietsch Factorization Theorem, $\pi_p(M_\phi) \leqslant 1$. Finally, by Proposition 5.12 of [16], $\pi_p^o(J) \leqslant 1$, hence $\pi_p^o(M_\phi) \leqslant 1$ ■

Proof of Proposition 5.5. As usual, it suffices to consider the finite dimensional version of the problem. That is, suppose ϕ is an $n \times n$ matrix, determining a Schur multiplier $\mathbf{S}_\phi : \mathcal{S}_n^{p'} \to \mathcal{S}_n^p$ ($1/p + 1/p' = 1$). We have to show that $\pi_p^o(\mathbf{S}_\phi) = \|\phi\|_p$.

We can identify $\mathbf{S}_\phi$ with an element $u = \sum_{i,j=1}^n \phi_{ij} E_{ij} \otimes E_{ij} \in \mathcal{S}_n^p[\mathcal{S}_n^p]$, or with $u' = \sum_{i,j=1}^n \phi_{ij} E_{ii,jj} \in \mathcal{S}_{n^2}^p$. By Lemma 5.14 of [16], $\pi_p^o(\mathbf{S}_\phi) = \|u\| = \|u'\| = \|\phi\|_p$. ■

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