HYPERREFLEXIVITY AND OPERATOR IDEALS

TIMUR OIKHBERG

Abstract. Suppose $(\mathfrak{B}, \boldsymbol{\beta})$ is an operator ideal, and $\mathcal{A}$ is a linear space of operators between Banach spaces X and Y. Modifying the classical notion of hyperreflexivity, we say that $\mathcal{A}$ is called $\mathfrak{B}$-hyperreflexive if there exists a constant C such that, for any $T \in B(X, Y)$ with $\alpha = \sup \boldsymbol{\beta}(qTi) < \infty$ (the supremum runs over all isometric embeddings i into X, and all quotient maps of Y, satisfying $q\mathcal{A}i = 0$), there exists $a \in \mathcal{A}$, for which $\boldsymbol{\beta}(T - a) \leqslant C\alpha$. In this paper, we give examples of $\mathfrak{B}$-hyperreflexive spaces, as well as of spaces failing this property. In the last section, we apply $\mathcal{S}_{\mathcal{E}}$-hyperreflexivity of operator algebras ($\mathcal{S}_{\mathcal{E}}$ is a regular symmetrically normed operator ideal) to constructing operator spaces with prescribed families of completely bounded maps.

1. Introduction

1.1. **Main definitions.** In the early 1970s, the notions of hyperreflexivity of operator algebras (and, more generally, of subspaces of $B(H)$, where H is a Hilbert space) was introduced by W. Arveson [4], in order to compute the distance to nest algebras. Later, the notion of hyperreflexivity was expanded to subspaces of $B(X, Y)$, where X and Y are Banach spaces.

In this paper, we consider hyperreflexivity of spaces of operators with respect to operator ideals. More precisely: suppose $\mathcal{X}$ is a class of Banach spaces, stable under taking subspaces and quotients. Suppose $\mathfrak{B}$ is a maximal Banach operator ideal, defined for members of $\mathcal{X}$. That is, for any $X, Y \in \mathcal{X}$, $\mathfrak{B}(X, Y)$ is a subset of $B(X, Y)$, equipped with the norm $\boldsymbol{\beta}(\cdot)$. $(\mathfrak{B}(X, Y), \boldsymbol{\beta})$ is a Banach space. The ideal property means that, for any $X_0, X, Y, Y_0 \in \mathcal{X}$, and every $T \in \mathfrak{B}(X, Y)$, $T_X \in B(X_0, X)$, and $T_Y \in B(Y, Y_0)$, we have $\boldsymbol{\beta}(T_Y T T_X) \leqslant \|T_Y\| \boldsymbol{\beta}(T) \|T_X\|$. Maximality of $\mathfrak{B}$ means that, for every $T \in \mathfrak{B}(X, Y)$, $\boldsymbol{\beta}(T) = \sup \boldsymbol{\beta}(qTi)$, where $i : E \to X$ is an injection, $q : Y \to F$ is a quotient, and the spaces $E, F \in \mathcal{X}$ are finite dimensional. For further information about operator ideals, see [13, 14, 37, 47].

We say that a maximal Banach ideal $\mathfrak{B}$ is *nice* if there exists a sequence of positive scalars $\beta_n \nearrow \infty$, so that for any n-dimensional $E \in \mathcal{X}$ and any $T \in B(E, Y)$, $\boldsymbol{\beta}(T) \geqslant \beta_n$ whenever $\|Te\| \geqslant 1$ for any $e \in E$. By Dvoretzky Theorem, this is equivalent to the existence of a sequence of positive scalars $\beta_n' \nearrow \infty$ s.t. $\boldsymbol{\beta}(T) \geqslant \beta_n'$ whenever $T \in B(\ell_2^n, Y)$ is such that $\|Te\| \geqslant 1$ for any $e \in \ell_2^n$.

1991 *Mathematics Subject Classification.* 46A32, 46B28, 47L05, 46L07, 47L25.

The author was partially supported by the NSF grant DMS-0500957.

A number of well-known operator ideals turns out to be nice. Recall that, for $1 \leqslant q \leqslant p \leqslant \infty$, the (p,q)-summing norm of $T \in B(X,Y)$ (denoted by π_{pq}) is defined as the smallest positive constant c such that, for every $x_1, \ldots, x_n \in X$,

$$\big(\sum_i \|Tx_i\|^p\big)^{1/p} \leqslant c \sup_{x^* \in X^*, \|x^*\| \leqslant 1} \big(\sum_i |x^*(x_i)|^q\big)^{1/q}.$$

The ideal of (p,q)-summing operators is denoted by Π_{pq}. If $p = q$, we speak of p-summing operators. The norm and the ideal are then denoted by π_p and Π_p, respectively. It turns out that (see e.g. p. 207 of [14]) Π_{pq} is nice if $p \in [1, \infty)$, and $1/q - 1/p < 1/2$. In particular, the ideal Π_p is nice for $p \in [1, \infty)$. The ideal I_p of p-integral maps is nice for $p \in [1, \infty)$ (to see this, compare the p-summing and p-integral norms). The ideal D_p of p-dominated maps ($p \in [1, \infty)$) is nice by 17.4.3 and 17.4.7 of [37]. On the other hand, the ideal Γ_p of p-factorable operators is maximal (see e.g. Chapter 9 of [14]), but not nice.

Suppose $\mathcal{A}$ is a non-empty absolutely convex subset of $B(X,Y)$, closed in the weak operator topology (that is, in the topology determined by the family of semi-norms $p_{x,y^*}(T) = y^*(Tx)$, with $x \in X$ and $y^* \in Y^*$). Denote the Minkowski gauge functional, associated with $\mathcal{A}$, by $\rho_{\mathcal{A}}$ (or simply by ρ, if there is no possibility of confusion). Suppose $\mathfrak{B}$ is an operator ideal For $T \in B(X,Y)$ denote by $d_{\mathcal{A},\mathfrak{B}}(T)$ the infimum of all $\lambda > 0$ with the property that, for any $\gamma \geqslant 1$, $\boldsymbol{\beta}(uTv) \leqslant \lambda\gamma$ whenever the contractions $v : E \to X$ and $u : Y \to F$ ($E, F \in \mathcal{X}$ are finite dimensional) satisfy $\boldsymbol{\beta}(uav) \leqslant \gamma$ for every $a \in \mathcal{A}$. For $C > 0$, we say that $\mathcal{A}$ is *$C - \mathfrak{B}$-Azoff-Shehada hyperreflexive* (*$C - \mathfrak{B}$-ASHR*, for short) if, for any $T \in B(X,Y)$ with $d_{\mathcal{A},\mathfrak{B}}(T)$ finite, and any $\varepsilon > 0$, we can write $T = a + b$, with $a \in \mathbb{F}\mathcal{A}$, $b \in \mathfrak{B}(X,Y)$, and $\rho(a) + \boldsymbol{\beta}(b) \leqslant (C + \varepsilon) d_{\mathcal{A},\mathfrak{B}}(T)$.

Throughout the paper, we work with maximal Banach ideals. In this case, the condition that E and F are finite dimensional is redundant.

If $\mathcal{A} \hookrightarrow B(X,Y)$ (throughout the paper, we use the notation "$Z_1 \hookrightarrow Z_2$" to mean "Z_1 is a closed linear subspace of a Banach space Z_2") is $C - \mathfrak{B}$-ASHR (or $\mathfrak{B}$-ASHR), we simply say that $\mathcal{A}$ is *$C - \mathfrak{B}$-hyperreflexive* (resp. *$\mathfrak{B}$-hyperreflexive*). The space $\mathcal{A}$ is $C - \mathfrak{B}$-hyperreflexive iff we have

$$\mathrm{dist}_{\mathfrak{B}}(T, \mathcal{A}) := \inf_{a \in \mathcal{A}} \boldsymbol{\beta}(T - a) \leqslant C d_{\mathcal{A},\mathfrak{B}}(T),$$

where

$$d_{\mathcal{A},\mathfrak{B}}(T) = \sup \boldsymbol{\beta}(uTv), \tag{1.1}$$

with the sup taken over all finite rank contractions u and v with $u\mathcal{A}v = 0$. Note that $d_{\mathcal{A},\mathfrak{B}}(T) = \sup \boldsymbol{\beta}(qTi)$, where the supremum runs over all injections $i : E \to X$ and quotient maps $q : Y \to F$, for which $E, F \in \mathcal{X}$ are finite dimensional, and $q\mathcal{A}i = 0$. Indeed, for u and v as in (1.1), take $E = \operatorname{ran} v$, and $G = \ker u$. Denote by v' the astriction of v to E, and by u' the operator on $F = Y/G$, defined by $u'(qy) = uy$.

Let i be the embedding of E into X, and let $q : Y \to F$ be the quotient map. Then $q\mathcal{A}i = 0$, and $u'qTiv' = uTv$ for any $T \in B(X,Y)$. As u' and v' are contractions, we have $\boldsymbol{\beta}(uTv) \leqslant \boldsymbol{\beta}(qTi)$.

The simplest example of a $\mathfrak{B}$-ASHR subset of $B(X,Y)$ ($\mathfrak{B}$ is a maximal Banach ideal) is $\{0\}$. Other examples and counterexamples are found in Sections 2 and 3.

To indicate the connection with the classical definition of hyperreflexivity (see e.g. [12, 30]), recall that a subspace $\mathcal{A}$ of $B(X,Y)$ is *C-hyperreflexive* if, for every $T \in B(X,Y)$,

$$\inf_{a\in\mathcal{A}} \|T - a\| \leqslant C \sup_{x\in X, \|x\|=1} \mathrm{d}(Tx, \mathcal{A}x) \tag{1.2}$$

(d stands for the distance in Y). A subspace $\mathcal{A}$ is called *hyperreflexive* if it is C-hyperreflexive for some C. Suppose $\mathfrak{B}$ is the ideal of bounded linear maps. Denote by i_x the canonical embedding of $\mathbb{C}x$ into X, and by q_x the quotient map $Y \to Y/\overline{\mathcal{A}x}$. Then the right hand side of (1.2) is nothing but $C \sup_{x\in X} \|q_x T i_x\|$, while the left hand side equals $\mathrm{dist}_{\mathfrak{B}}(T, \mathcal{A})$. On the other hand, suppose $i : E \to X$ and $q : Y \to Y/G$ are an embedding and a quotient map, respectively, such that $q\mathcal{A}i = 0$. Pick $\varepsilon > 0$, and find a norm one $x \in E$ s.t. $\|qTix\| > (1-\varepsilon)\|qTix\|$. Then $\mathcal{A}x \subset G$, hence (in the above notation)

$$\|q_x T i_x\| \geqslant \|qTix\| > (1-\varepsilon)\|qTix\|.$$

Thus, the left hand side of (1.2) becomes left hand side of (1.2) when we take $\mathfrak{B}$ to be the ideal of bounded operators.

Over the last thirty years, a lot of information about hyperreflexivity has been accumulated (see e.g. [11]). For instance, by [44], that any one-dimensional subspace of $B(X,Y)$ is 3-hyperreflexive (1-hyperreflexive if X and Y are Hilbert spaces, see [3]). More generally, by [30], any reflexive finite dimensional space of operators is hyperreflexive. In the case of $\mathfrak{B}$-hyperreflexivity, this is false. By Theorem 2.12 and Proposition 2.14, for many pairs of infinite dimensional Banach spaces (X,Y), and many nice ideals $\mathfrak{B}$, there exists $T \in B(X,Y)$ s.t. $\mathbb{F}T$ is not $\mathfrak{B}$-hyperreflexive (here, $\mathbb{F}$ stands for the field of scalars, either $\mathbb{R}$ or $\mathbb{C}$). Moreover, by Theorem 2.15, for many nice ideals $\mathfrak{B}$ (such as Γ_p for $1 < p < \infty$, and Π_q for $1 \leqslant q < 2$) there exists a Banach space X s.t. $\mathbb{F}I_X$ is not $\mathfrak{B}$-hyperreflexive. However, by Theorem 2.1, $\mathbb{F}I_X$ is Π_{q2}-hyperreflexive (for $2 \leqslant q \leqslant \infty$) for any Banach space X. Moreover, if $\mathfrak{B}$ is a nice operator ideal, and X is a $\mathcal{L}_p$ space ($1 \leqslant p \leqslant \infty$), then $\mathbb{F}I_X$ is $\mathfrak{B}$-hyperreflexive (Theorem 2.9).

In Section 3, we consider spaces of operators between complex Hilbert spaces. In particular, Theorem 3.1 shows that any von Neumann algebra is $\mathcal{S}_{\mathcal{E}}$-hyperreflexive, when $\mathcal{E}$ is a reflexive symmetric sequence space $\mathcal{E}$ (the question of whether every von Neumann algebra is hyperreflexive is open). By Theorem 3.3, the linear span of a non-compact operator is $\mathcal{S}_{\mathcal{E}}$-hyperreflexive. On the other hand, the nest algebras with infinite nests, and the algebras of analytic Toeplitz or Laurent operators, are hyperreflexive, but not $\mathcal{S}_{\mathcal{E}}$-hyperreflexive for most symmetric sequence spaces $\mathcal{E}$ (Theorems 3.4, 3.5).

In Section 4, we apply $\mathfrak{B}$-hyperreflexivity to the problem of constructing operator spaces with prescribed sets of c.b. maps. Theorem 4.1 shows: suppose $\mathcal{E}$ is a symmetric sequence space satisfying certain properties, $\mathcal{S}_{\mathcal{E}}$ is the corresponding operator ideal, H is a separable Hilbert space, and $\mathcal{A}$ is an absolutely convex $\mathcal{S}_{\mathcal{E}}$-ASHR subspace of the unit ball of $B(H)$, containing the identity I_H. The H can be equipped with an operator space structure X, s.t. $CB(X) = \mathbb{C}\mathcal{A} + \mathcal{S}_{\mathcal{E}}$.

Remark 1.1. (1) The term "Azoff-Shehada hyperreflexivity" is inspired by the works of E. Azoff and H. Shehada (see e.g. [5]) on reflexivity of convex sets.

(2) By changing the above definitions slightly, we define $d'_{\mathcal{A},\mathfrak{B}}(T)$ ($\mathcal{A} \subset B(X,Y)$, $T \in B(X,Y)$ as the infimum of all $\lambda > 0$ with the property that, for any $\gamma > 0$ (as opposed to "$\gamma \geqslant 1$" in the definition of $d_{\mathcal{A},\mathfrak{B}}(T)$), $\boldsymbol{\beta}(qTi) \leqslant \lambda\gamma$ whenever the injection $i : E \to X$ and the quotient map $q : Y \to F$ ($E, F \in \mathcal{X}$ are finite dimensional) satisfy $\boldsymbol{\beta}(qai) \leqslant \gamma$ for every $a \in \mathcal{A}$. Clearly, $d'_{\mathcal{A},\mathfrak{B}}(T) \geqslant d_{\mathcal{A},\mathfrak{B}}(T)$. For any $C > 1$, we may have $d'_{\mathcal{A},\mathfrak{B}}(T) \geqslant C d_{\mathcal{A},\mathfrak{B}}(T)$ (consider, for instance, the case of $X = Y = \mathbb{F}$, $T = I_{\mathbb{F}}$, and $\mathcal{A} = \{\lambda I_{\mathbb{F}} \,|\, |\lambda| \leqslant C^{-1}\}$.

We say that $\mathcal{A}$ is $(C - \mathfrak{B})'$-Azoff-Shehada hyperreflexive if for any $T \in B(X,Y)$ with $d'_{\mathcal{A},\mathfrak{B}}(T)$ finite, and any $\varepsilon > 0$, we can write $T = a+b$, with $a \in \mathbb{F}\mathcal{A}$, $b \in \mathfrak{B}(X,Y)$, and $\rho(a) + \boldsymbol{\beta}(b) \leqslant (C+\varepsilon) d'_{\mathcal{A},\mathfrak{B}}(T)$. It is not clear whether the notion of $(C-\mathfrak{B})'$-ASHR is strictly weaker than of $C - \mathfrak{B}$-ASHR. In this paper, we mostly consider $\mathfrak{B}$-hyperreflexivity of spaces of operators, in which case, the distinction between the two versions of versions of ASHR is irrelevant. We selected our definition of ASHR over the possible alternatives for the sake of applications to the theory of operator spaces (Section 4).

Remark 1.2. In this paper, all the ideals are assumed to be normed. Many proofs will also go through for quasi-normed ideals. Indeed, suppose first that $(\mathfrak{B}, \boldsymbol{\beta})$ is a normed ideal. If T is a rank n contraction, then it factors contractively through an n-dimensional space, and therefore, by the existence of an Auerbach basis, it can be represented as a sum of rank 1 contractions $T_1, \ldots, T_m$, with $m \leqslant n$. By 6.1.5 of [37], $\boldsymbol{\beta}(T_k) = \|T_k\|$ for each k, hence $\boldsymbol{\beta}(T) \leqslant n$. Therefore, $\boldsymbol{\beta}(T) \leqslant \|T\| \mathrm{rank}\, T$ for any finite rank operator T.

Now suppose the ideal $(\mathfrak{B}, \boldsymbol{\beta})$ is quasi-normed. By Section 6.2 of [37] (and passing to an equivalent ideal norm if necessary), we can assume the existence of $r \in (0,1]$ such that $\boldsymbol{\beta}(\sum_{k=1}^m S_k)^r \leqslant \sum_{k=1}^m S_k \boldsymbol{\beta}(S_k)^r$ for any $S_1, \ldots, S_m \in \mathfrak{B}$. Then $\boldsymbol{\beta}(T) \leqslant \|T\|(\mathrm{rank}\, T)^{1/r}$. The proofs where finding an upper estimate for $\boldsymbol{\beta}(T)$ (for a finite rank T) is important can be done in this setting, too. Thus, Theorems 2.1, 2.12, 2.15, and Corollary 2.2 still hold in the quasi-normed setting. On the other hand, we cannot establish Lemma 2.11 without assuming that the ideal involved is normed. We do not know if Theorems 2.9 and 2.10 remain valid for quasi-normed ideals.

1.2. **Preliminaries.** We some observations, to be used throughout the paper. First, we prove that $\mathfrak{B}$-hyperreflexivity, and $\mathfrak{B}$-ASHR, are stable under isomorphisms.

Proposition 1.3. *Suppose X_1, X_2, Y_1, and Y_2 are Banach spaces, $(\mathfrak{B}, \boldsymbol{\beta})$ is a maximal ideal, $U : X_1 \to X_2$ and $V : Y_1 \to Y_2$ are isomorphisms, and $\mathcal{A} \hookrightarrow B(X_1, Y_1)$ is $C - \mathfrak{B}$-hyperreflexive. Then $\mathcal{A}' = V\mathcal{A}U^{-1}$ is a $\|V\|\|V^{-1}\|C\|U\|\|U^{-1}\| - \mathfrak{B}$-hyperreflexive subspace of $B(X_2, Y_2)$.*

Proof. Consider $T \in B(X_2, Y_2)$, such that $\boldsymbol{\beta}(A'TB') < \lambda < 1$ whenever the contractions A' and B' satisfy $A'(V\mathcal{A}U^{-1})B' = 0$. We have to show that $\mathrm{dist}_{\mathfrak{B}}(T, \mathcal{A}') < \|V\|\|V^{-1}\|C\|U\|\|U^{-1}\|$. To this end, consider $S = V^{-1}TU$, and note that $\boldsymbol{\beta}(ASB) < \lambda\|U\|\|V^{-1}\|$ whenever $B : E \to Y_1$ and $A : X_1 \to F$ are contractions with $A\mathcal{A}B = 0$. Indeed, $B' = UB/\|U\|$ and $A' = AV^{-1}/\|V^{-1}\|$ are also contractions, and $A'\mathcal{A}'B' = 0$. Moreover, $ASB = \|U\|\|V^{-1}\|A'TB'$. Thus, $\boldsymbol{\beta}(ASB) < \lambda\|U\|\|V^{-1}\|$, hence there exists $a \in \mathcal{A}$ s.t. $\boldsymbol{\beta}(S - a) < C\|U\|\|V^{-1}\|$. Let $a' = VaU^{-1}$. Then

$$\boldsymbol{\beta}(T - a') \leqslant \|V\|\|U^{-1}\|\boldsymbol{\beta}(S - a) < C\|U\|\|V^{-1}\|\|V\|\|U^{-1}\|$$

(here, we use the identity $T = VSU^{-1}$). ∎

A similar statement holds for Azoff-Shehada hyperreflexivity.

Proposition 1.4. *Suppose X_1, X_2, Y_1, and Y_2 are Banach spaces, $(\mathfrak{B}, \boldsymbol{\beta})$ is a maximal ideal, $U : X_1 \to X_2$ and $V : Y_1 \to Y_2$ are isomorphisms, and $\mathcal{A} \subset B(X_1, Y_1)$ is $\mathfrak{B}$-Azoff-Shehada hyperreflexive. Then $\mathcal{A}' = V\mathcal{A}U^{-1} \subset B(X_2, Y_2)$ is $\mathfrak{B}$-Azoff-Shehada hyperreflexive.*

The proof of this proposition is similar to that of Proposition 1.3. We do not compute the $\mathfrak{B}$-Azoff-Shehada hyperreflexivity constant explicitly, since we never need it.

Similarly, we show that "deformations" of sets of operators preserve their $\mathfrak{B}$-Azoff-Shehada hyperreflexivity.

Proposition 1.5. *Suppose $\mathcal{A}$ and $\mathcal{A}'$ are non-empty absolutely convex subsets of $B(X, Y)$, and $\mathcal{A}$ is $C - \mathfrak{B}$-Azoff-Shehada hyperreflexive. Suppose, furthermore, that $C_1\mathcal{A} \subset \mathcal{A}' \subset C_2\mathcal{A}$ $(0 < C_1 < C_2)$. Then $\mathcal{A}'$ is $C \max\{1, C_1^{-1}\} \max\{1, C_2\} - \mathfrak{B}$-ASHR.*

Proof. Suppose $T \in B(X, Y)$ is such that, for any $\gamma \geqslant 1$, and any pair of finite rank contractions u, v with $\boldsymbol{\beta}(uav) \leqslant \gamma$ for any $a \in \mathcal{A}'$, we have $\boldsymbol{\beta}(uTv) \leqslant \gamma$. Fix $\gamma' \geqslant 1$, and suppose the finite rank contractions u and v are such that $\boldsymbol{\beta}(uav) \leqslant \gamma'$ for any $a \in \mathcal{A}$. Then $\boldsymbol{\beta}(uav) \leqslant \max\{1, C_2\}\gamma'$ for any $a \in \mathcal{A}'$, and therefore,

$\boldsymbol{\beta}(uTv) \leqslant \max\{1, C_2\}\gamma'$. Thus, for every $\varepsilon > 0$ we can write $T = a + b$, with $\rho_{\mathcal{A}}(a) + \boldsymbol{\beta}(b) < C \max\{1, C_2\} + \varepsilon$. However, $\rho_{\mathcal{A}'}(a) \leqslant \rho_{\mathcal{A}}(a)/C_1$, hence

$$\rho_{\mathcal{A}'}(a) + \boldsymbol{\beta}(b) < \max\{1, C_1^{-1}\}(\rho_{\mathcal{A}}(a) + \boldsymbol{\beta}(b)) \leqslant \max\{1, C_1^{-1}\}(C \max\{1, C_2\} + \varepsilon).$$

ε can be arbitrarily small, hence we are done. ■

We also need to mention a connection between $\mathfrak{B}$-hyperreflexivity of subspaces, and $\mathfrak{B}$-Azoff-Shehada hyperreflexivity of their unit balls. Clearly, if $\mathcal{A}$ is a WOT closed subspace of $B(X, Y)$, then its closed unit ball $\mathrm{Ba}(\mathcal{A})$ is also WOT closed.

Proposition 1.6. *Suppose* $(\mathfrak{B}, \boldsymbol{\beta})$ *is a maximal ideal, and let* $\mathcal{A}$ *be a subspace of* $B(X, Y)$*, closed in the weak operator topology, is* $C - \mathfrak{B}$*-hyperreflexive. Then* $\mathrm{Ba}(\mathcal{A})$ *is* $(2C + 1) - \mathfrak{B}$*-Azoff-Shehada hyperreflexive.*

Proof. Suppose $\mathcal{A}$ is $C - \mathfrak{B}$-hyperreflexive, $T \in B(X, Y)$, and $d_{\mathrm{Ba}(\mathcal{A}),\mathfrak{B}}(T) < 1$. We shall show that, for any $\varepsilon > 0$, we can write $T = a + b$, with $a \in \mathcal{A}$, $b \in B(X, Y)$, $\|b\| < C$, and $\|a\| \leqslant C + 1$.

Indeed, $\boldsymbol{\beta}(qTi) \leqslant d_{\mathrm{Ba}(\mathcal{A}),\mathfrak{B}}(T) < 1$ whenever $q\mathcal{A}i = 0$. As $\mathcal{A}$ is $C - \mathfrak{B}$-hyperreflexive, there exist $a \in \mathcal{A}$ and $b \in B(X, Y)$ s.t. $T = a + b$, and $\boldsymbol{\beta}(b) < C$. We shall how that $\|a\| \leqslant C + 1$. Indeed, otherwise there exist norm 1 $x \in X$ and $y^* \in Y^*$ s.t. $y^*(ax) > C + 1$. Denote by i the embedding of $\mathbb{F}x$ into X, and by q the quotient map $Y \to Y/\ker y^*$. Then qai is a rank 1 map, hence $\|qai\| = \boldsymbol{\beta}(qai) \geqslant C + 1$. Similarly,

$$\boldsymbol{\beta}(qa_0 i) = \|qa_0 i\| \leqslant \|q\|\|a_0\|\|i\| \leqslant 1$$

for any $a_0 \in \mathrm{Ba}(\mathcal{A})$. But, by the triangle inequality,

$$\boldsymbol{\beta}(qTi) \geqslant \boldsymbol{\beta}(qai) - \boldsymbol{\beta}(qbi) > C + 1 - \boldsymbol{\beta}(b) > 1,$$

a contradiction. Thus, $\|a\| \leqslant C + 1$. ■

Finally, we introduce several Banach space definitions, with an eye for stating a version of the "principle of small perturbations".

A family of finite dimensional subspaces $(F_n)_{n \in \mathbb{N}}$ of a Banach space X is said to be a *finite dimensional decomposition* (*FDD*, for short) if $X = \overline{\mathrm{span}_{n \in \mathbb{N}} F_n}$, and $\sup_n \|P_n\| < \infty$, where P_n is the projection onto $\mathrm{span}[F_i \,|\, i \leqslant n]$, with $\ker P_n = \overline{\mathrm{span}[F_i \,|\, i > n]}$. The number $\sup_n \|P_n\|$ is called the *FDD constant* of (F_n). We say that a sequence (F_n) of subspaces of X is an *FDD sequence* if it is an FDD of $\overline{\mathrm{span}_{n \in \mathbb{N}} F_n}$.

A sequence $(x_n)_{n \in \mathbb{N}}$ in a Banach space X is called a *basis* (respectively, a basic sequence) if $(\mathbb{F}x_n)_{n \in \mathbb{N}}$ is an FDD (resp. FDD sequence) in X. The FDD constant in this case is called the *basic constant.* A basis (or a basic sequence) (x_i) is called C*-unconditional* if, for any eventually null sequence $(\alpha_i)_{i \in \mathbb{N}}$ of scalars, and for any sequence $(\omega_i)_{i \in \mathbb{N}}$ with $|\omega_i| = 1$ for every i, we have $\|\sum_i \alpha_i \omega_i x_i\| \leqslant C \|\sum_i \alpha_i x_i\|$. The infimum of all such C's is called the *unconditionality constant* of (x_n). The basis (x_n) is called *normalized* if $\|x_n\| = 1$ for each n.

We denote by $\nu_1(T)$ the nuclear norm of an operator T. It is known that the set of nuclear operators is a Banach ideal, but not a maximal ideal. For any Banach ideal $(\mathfrak{B}, \boldsymbol{\beta})$, and any operator T, we have $\|T\| \leqslant \boldsymbol{\beta}(T) \leqslant \nu_1(T)$.

Lemma 1.7. *Suppose (F_n) is a C-FDD sequence in a Banach space X. Suppose, furthermore, that the operators $T \in B(X)$ and $T_n \in B(F_n)$ ($n \in \mathbb{N}$) satisfy $\sum_n \dim F_n \cdot \nu_1(T|_{F_n} - T_n) < \infty$. Then there exists $\tilde{T} \in B(X)$ such that $\tilde{T}|_{F_n} = T_n$, and $\nu_1(\tilde{T} - T) \leqslant 2C \sum_n \dim F_n \cdot \nu_1(T|_{F_n} - T_n)$.*

Proof. Set $Y = \overline{\mathrm{span}[F_1, F_2, \ldots]}$, and let $P_n : Y \to \mathrm{span}[F_1, \ldots F_n]$ be a projection with $\ker P_n = \overline{\mathrm{span}[F_i \,|\, i > n]}$. Let $Q_1 = P_1$, and $Q_n = P_n - P_{n-1}$ for $n \geqslant 2$. Then $\|Q_n\| \leqslant 2C$ for each n. Clearly, Q_n extends to a projection $\tilde{Q}_n$ from X onto F_n, with $\|\tilde{Q}_n\| \leqslant \|Q_n\| \dim F_n$. Then $\tilde{T} = T - \sum_{n=1}^{\infty} (T - T_n)\tilde{Q}_n$ has the desired properties. ∎

2. The Banach space case

2.1. **The Banach space case: positive results.** In this subsection, we give examples of $\mathfrak{B}$-hyperreflexive spaces of operators.

Theorem 2.1. *If X is an infinite dimensional Banach space, and $2 \leqslant q \leqslant \infty$, then $\mathbb{F}I_X$ is $C - \Pi_{q2}$-hyperreflexive, where C is an absolute constant (independent of X and q).*

It is easy to see that, for any linear operator u, $\pi_{\infty 2}(u) = \|u\|$. By [30], every 1-dimensional space is hyperreflexive. Thus, only the case of $2 \leqslant q < \infty$ needs to be considered.

Before proving the theorem, we state and prove its corollary.

Corollary 2.2. *Suppose P and Q are infinite rank projections in Banach spaces X and Y, respectively. Suppose, furthermore, that $A \in B(X, Y)$ is such that QAP is invertible as a linear map from $\mathrm{ran}\, P$ to $\mathrm{ran}\, Q$, and $A(\ker P) \subset \ker Q$. Then, for $2 \leqslant q \leqslant \infty$, $\mathbb{F}A$ is Π_{q2}-hyperreflexive.*

Proof. We shall use $C_0, C_1, \ldots$ to denote constants, depending only on P, Q, and A. Let $X_2 = \ker P$ and $Y_2 = \ker Q$. Any $T \in B(X, Y)$ can be written as $T = QTP + (I - Q)TP + QT(I - P) + (I - Q)T(I - P)$. Suppose $d_{\mathbb{F}A, \Pi_{q2}}(T) < 1$. Viewing QAP as an invertible operator from $\mathrm{ran}\, P$ to $\mathrm{ran}\, Q$, we conclude that $d_{\mathbb{F}QAP, \Pi_{q2}}(QTP) < 1$. By Theorem 2.1 and Proposition 1.3, there exists $\lambda \in \mathbb{F}$ s.t. $\pi_{q2}(QTP - \lambda QAP) < C_0$. We shall show that $\pi_{q2}(T - \lambda A) < C_1$. Without loss of generality, we can assume that $\lambda = 0$.

Note first that

$$\pi_{q2}((I-Q)TP) \leqslant \|I-Q\|\|P\|\pi_{q2}\Big(\frac{I-Q}{\|I-Q\|}T\frac{P}{\|P\|}\Big)$$
$$\leqslant \|I-Q\|\|P\|d_{\mathbb{F}A,\Pi_{q2}}(T) < \|I-Q\|\|P\|$$

(we use the fact that $(I-Q)AP = 0$). Similarly, $\pi_{q2}(QT(I-P)) < \|Q\|\|I-P\|$.

To handle $(I-Q)T(I-P)$, pick $c \in (0, \pi_{q2}((I-Q)T(I-P)))$, and prove that $c \leqslant C_2$. Indeed, if its not so, then there exists $w : \ell_2^n \to X$ s.t. $\pi_{q2}((I-Q)T(I-P)w) > c$. Let $v_1 = (I-P)w/\|I-P\|$. Then $\pi_{q2}((I-Q)Tv_1) > c/\|I-P\|$. We estimate the left hand side from above.

By p. 207 of [14], $\lim_n \pi_{q2}(I_{\ell_2^n}) = \infty$, hence, by Dvoretzky Theorem, QTP is strictly singular. Perturbing T slightly (cf. Lemma 1.7), we can assume the existence of an n-dimensional subspace $F \hookrightarrow \operatorname{ran} P$ s.t. $T|_F = 0$, and $d(\ell_2^n, F) < 2$. Find a contraction $v_2 : \ell_2^n \to F$ s.t. $\|v_2\|^{-1} < 2$. Then $v = (v_1 + v_2)/2$ is a contraction. Let $q : Y \to Y/A(\operatorname{ran} v)$ be a quotient map. We shall show that $\pi_{q2}(qTv) > C_3 c$. To this end, observe that, for any $\eta \in \ell_2^n$, $Av_1\eta = Av_1 \circ (v_2^{-1}A^{-1}) \circ Av_2\eta$, hence

$$\|Av_1\eta\| \leqslant C_4\|Av_2\eta\|, \text{ with } C_4 = 2\|A\|^2\|(A|_{\operatorname{ran} P})^{-1}\|.$$

Therefore, for any $\xi \in \ell_2^n$,

$$2\|qTv\xi\| = \inf_{\eta\in\ell_2^n} \|T(v_1+v_2)\xi - A(v_1+v_2)\eta\|$$
$$\geqslant \max\Big\{\frac{1}{\|P\|}\|Av_2\eta\|, \frac{1}{\|I-P\|}\|Tv_1\xi - Av_1\eta\|\Big\}$$
$$\geqslant \frac{1}{2\|P\|}\sup_t \max\{t, \|Tv_1\xi\| - C_4 t\}$$
$$\geqslant \frac{1}{2\|P\|}\Big(\frac{C_4}{C_4+1}t + \frac{1}{C_4+1}(\|Tv_1\xi\| - C_4 t)\Big) \geqslant 2C_3\|Tv_1\xi\|.$$

Thus, by definition of Π_{q2}, $\pi_{q2}(qTv) \geqslant C_3\pi_{q2}(Tv_1) > C_3 c$. But $qAv = 0$, hence $\pi_{q2}(qTv) \leqslant 1$, which yields the desired estimate for c. ■

To prove Theorem 2.1, we need a series of lemmas. The first one is a part of the Banach space lore.

Lemma 2.3. *Suppose E is a finite dimensional subspace of an infinite dimensional Banach space X, and $\varepsilon > 0$. Then X contains a finite codimensional subspace Y, so that $\|e+y\| \geqslant \max\{(1+\varepsilon)^{-1}\|e\|, (2+\varepsilon)^{-1}\|y\|\}$ for every $e \in E$ and $y \in Y$.*

The next result is very easy to verify.

Lemma 2.4. *Suppose (x_n) is a normalized basic sequence in a Banach space X, with a basic constant c. Suppose, furthermore, that $T \in B(X)$ is such that, for every $n \in \mathbb{N}$, $\|Tx_n\| < \gamma_n$, where $\sum_k \gamma_k < \infty$. Then, for any $x \in \operatorname{span}[x_k \,|\, k \geqslant n]$, we have $\|Tx\| \leqslant 2c\|x\|\sum_{k=n}^\infty \gamma_k$.*

Proof. Write $x = \sum_{k=n}^{\infty} \alpha_k x_k$. Then, for any $m \geqslant n$, $\| \sum_{k=n}^{m} \alpha_k x_k \| \leqslant c\|x\|$, and therefore,

$$|\alpha_m| = \| \sum_{k=n}^{m} \alpha_k x_k - \sum_{k=n}^{m-1} \alpha_k x_k \| \leqslant 2c\|x\|.$$

We conclude the proof by observing that $\|Tx\| \leqslant \sum_{k=n}^{\infty} |\alpha_k|\gamma_k$. ∎

The following result seems to be well-known, too. We sketch the proof for the sake of completeness.

Lemma 2.5. *Suppose X is an infinite dimensional complex Banach space, and (γ_n) is a sequence of positive numbers. Then for every $T \in B(X)$ there exists $\lambda \in \mathbb{C}$ and a normalized basic sequence (x_n) in X, such that $\lim_n (T - \lambda)x_n = 0$, and, for each $n \in \mathbb{N}$, $\| \sum_{k=1}^{n} \alpha_k x_k \| \leqslant (1 + \gamma_n) \| \sum_{k=1}^{\infty} \alpha_k x_k \|$ whenever $\alpha_1, \alpha_2, \ldots$ is an eventually null sequence of scalars.*

Sketch of the proof. By Lemma 2.3, it suffices to prove that, for some $\lambda \in \mathbb{C}$, $T - \lambda$ is not an isomorphism on any finite codimensional subspace of X. Indeed, otherwise the generalized Fredholm index $i(T - \lambda)$ is defined for every λ. By the continuity of the generalized Fredholm index (see Section 4.4 of [1]), $i(T - \lambda)$ is independent of λ. But $T - \lambda$ is an isomorphism on X for $|\lambda| > \|T\|$, hence $T - \lambda$ is a Fredholm operator of index 0 for each $\lambda \in \mathbb{C}$. Thus, $\sigma_{ess}(T) = \emptyset$, a contradiction. ∎

Corollary 2.6. *Suppose X is an infinite dimensional complex Banach space. Then for every $T \in B(X)$ there exists $\lambda \in \mathbb{C}$ so that, for every finite dimensional subspace E of X, $\varepsilon > 0$, and $n \in \mathbb{N}$, there exists $\lambda \in \mathbb{C}$, and a subspace F of X, $(1 + \varepsilon)$-isomorphic to ℓ_2^n, and such that (i) $\|(T - \lambda)f\| \leqslant \varepsilon\|f\|$ for any $f \in F$, and (ii) for any $e \in E$ and $f \in F$, $\|e + f\| \geqslant \max\{(1 + \varepsilon)^{-1}\|e\|, (2 + \varepsilon)^{-1}\|f\|\}$.*

Proof. By Lemma 2.5, there exist $\lambda \in \mathbb{C}$ and a sequence (x_n) in X, with basic constant less than 2, such that $\|(T - \lambda)x_n\| < 1/(5 \cdot 4^{n+1})$. By Lemma 2.4, $\|(T - \lambda)y\| < 4^{-n}$ for every $y \in Y_n = \mathrm{span}[x_i \,|\, i > n]$. This yields an infinite dimensional $Y \hookrightarrow X$ s.t. $\|(T - \lambda)y\| < \varepsilon$ for every $y \in Y$. An application of Lemma 2.3 completes the proof. ∎

Proof of Theorem 2.1, the complex case. Suppose $d_{\mathbb{C}I_X, \Pi_{q2}}(T) < 1$, and show that $\pi_{q2}(T - \lambda I_X) \leqslant 36$, with $\lambda \in \mathbb{C}$ from Corollary 2.6. Clearly, it suffices to assume that $\lambda = 0$, and prove that $\pi_{q2}(T) \leqslant 36$.

Pick $c \in (0, \pi_{q2}(T))$. By Chapter 11 of [47], there exists a contraction $u : \ell_2^n \to X$ s.t. $\pi_{q2}(Tu) > c$. Perturbing T slightly and applying Corollary 2.6, we prove the existence of an n-dimensional $F \hookrightarrow X$, s.t. $d(F, \ell_2^n) < 2$, $T|_F = 0$, and $\|e + f\| \geqslant \max\{\|e\|, \|f\|\}/3$ for any $f \in F$ and $e \in E = \mathrm{span}[\mathrm{ran}\, u, \mathrm{ran}\, Tu]$. Find $u_1 : \ell_2^n \to F$ s.t. $\|u_1\| = 1$, $\|u_1^{-1}\| < 2$. Then $v = (u + u_1)/6$ is a contraction.

Denote by q the quotient map $X \to X/\operatorname{ran} v$. Then, for every $\xi \in \ell_2^n$,

$$\|qTv\xi\| \geqslant \|Tu\xi\|/36. \tag{2.1}$$

Indeed, any element of $\operatorname{ran} v$ can be written uniquely as $e+f$, with $e \in E$, $f \in F$, and $\|f\| \geqslant \|e\|$. Then

$$\|Tv\xi - (e+f)\| = \|(Tv\xi - e) + f\| \leqslant \frac{1}{3}\max\{\|Tv\xi - e\|, \|f\|\}.$$

If $\|f\| \leqslant \|Tv\xi\|/2$, then $\|Tv\xi - e\| \geqslant \|Tv\xi\| - \|e\| \geqslant \|Tv\xi\|/2$, and $\|Tv\xi - (e+f)\| \geqslant \|Tv\xi\|/6$. If $\|f\| \geqslant \|Tv\xi\|/2$, the same inequality also holds. Therefore, for any $\xi \in \ell_2^n$, $\|qTv\xi\| \geqslant \|Tv\xi\|/6$. Moreover, $Tv\xi = Tu\xi/6$, which yields (2.1).

By definition of the $(q,2)$-summing norm, $\pi_{q2}(qTv) \geqslant c/36$. However, $qv = 0$, hence $\pi_{q2}(qTv) \leqslant d_{\mathbb{C}I_X, \Pi_{q2}}(T) < 1$, and we are done. ■

We next handle the real case of Theorem 2.1. For a real Banach space X define its *complexification* X_c as a complex Banach space, isomorphic to $X \oplus X$ as a real space. Multiplication by $i = \sqrt{-1}$ is defined as $i(x,y) = (-y,x)$. The norm is defined by

$$\|(x,y)\|_{X_c} = \max_{\phi \in [0,2\pi]} \|\cos\phi \cdot x + \sin\phi \cdot y\|_X.$$

Consequently,

$$\max\{\|x\|, \|y\|\} \leqslant \|(x,y)\|_{X_c} \leqslant \|x\| + \|y\|. \tag{2.2}$$

It is well known (see e.g. Section 1.1 of [1]) that X_c is indeed a complex Banach space. Henceforth, we identify $(x,y) \in X_c$ with $x + iy$. For $T \in B(X,Y)$, its complexification $T_c \in B(X_c, Y_c)$ is defined by setting $T_c(x+iy) = Tx + iTy$.

Since we are working with direct sums of Banach spaces, we need:

Lemma 2.7. *Suppose X_1 and X_2 are infinite dimensional Banach spaces. Denote by P_1 and P_2 the canonical projections from $X_1 \oplus_\infty X_2$ onto X_1 and X_2, respectively. Then every infinite dimensional subspace Y of $X_1 \oplus_\infty X_2$ contains an infinite dimensional subspaces Z such that either $P_1|_Z$ or $P_2|_Z$ is invertible.*

Proof. Suppose that, for any infinite dimensional $Z \hookrightarrow Y$, $P_1|_Z$ is not invertible. Then there exists a normalized basic sequence (x_n), with the basic constant less than 2, and such that $\|P_1 x_n\| < 2^{-(n+2)}$ for each n. We shall show that P_2 is an isometry on $Z = \operatorname{span}[x_n \,|\, n \in \mathbb{N}]$. Indeed, by Lemma 2.4, $\|P_1 x\| < \|x\|$ for every $x \in Z$. However, $\|x\| = \max\{\|P_1 x\|, \|P_2 x\|\}$, hence $\|P_2 x\| = \|x\|$ for any $x \in Z$. ■

Lemma 2.8. *Suppose X_c is the complexification of a real Banach space, $T \in B(X)$, and a norm one vector $z = x + iy \in X_c$ (with $x, y \in X$) satisfies $\|T_c z - iz\| < 1/20$ (here, T_c is the complexification of T). Then, for any $a, b \in \mathbb{R}$, $\|ax + by\| \geqslant C\max\{|a|, |b|\}$, where $C = 1/(4(\|T\| + 4))$.*

Proof. By (2.2), $1/2 \leqslant \max\{\|x\|, \|y\|\} \leqslant 1$. By definitions of X_c and T_c, $T_c z = Tx + iTy$, and $iz = -y + ix$, hence $T_c z - iz = (Tx + y) + i(Ty - x)$. By (2.2) again,

$$\max\{\|Tx + y\|, \|Ty - x\|\} < 1/20.$$

Show first that $\|x + by\| \geqslant C$ for each $b \in \mathbb{R}$. Indeed, suppose $\|x + by\| < \gamma < 1$. Then $|b|\|y\| \leqslant \|x + by\| + \|x\| < 2$, hence $|b| \leqslant 4$. Therefore,

$$\|T(x + by) - (-y + bx)\| \leqslant \|Tx + y\| + |b|\|Ty - x\| < \frac{1}{4},$$

hence

$$\| - y + bx\| \leqslant \|T\|\|x + by\| + \|T(x + by) - (-y + bx)\| < \|T\|\gamma + \frac{1}{4}.$$

Therefore,

$$\begin{aligned} \frac{1}{2} &\leqslant (1 + b^2)\|y\| = \|b(x + by) - (-y + bx)\| \\ &\leqslant |b|\|x + by\| + \| - y + bx\| < (4 + \|T\|)\gamma + \frac{1}{4}, \end{aligned}$$

and therefore, $\gamma \geqslant 1/(4(\|T\| + 4))$. This show that $\|ax + by\| \geqslant C|a|$. Similarly, we show that $\|ax + by\| \geqslant C|b|$. ■

Proof of Theorem 2.1, the real case. Consider $T \in B(X)$, with $d_{\mathbb{R}I_X,\Pi_2}(T) < 1$. By Lemma 2.5, for any $T \in B(X)$ there exists $\lambda = \alpha + i\beta \in \mathbb{C}$ $(\alpha, \beta \in \mathbb{R})$ s.t. the restriction of $T_c - \lambda I_{X_c}$ to an infinite dimensional subspace Z of X_c is compact. We shall show that $\beta = 0$. This would imply the existence of infinite dimensional $Y \hookrightarrow X$ s.t. $(T - \alpha I)|_Y$ is compact. Indeed, we can view Z as a subspace of $X \oplus X$, and denote by P_1 and P_2 the projections onto the first and the second copies of X (the "real" and "imaginary" parts of X_c, respectively). By Lemma 2.7, we can assume the existence of $c > 0$, and of an infinite dimensional $Z_0 \hookrightarrow Z$ s.t. $\|P_1 z\| \geqslant c\|z\|$ for each $z \in Z_0$. Find a normalized basic sequence $(z_n)_{n\in\mathbb{N}}$ in Z_0, s.t. $\|(T - \alpha I)z_n\| < 4^{-n}$. Let $y_n = P_1 z_n$. Note that $\|y_n\| \geqslant c$ for each n, and $(y_n)_{n\in\mathbb{N}}$ is a basic sequence in Y. Moreover, by (2.2),

$$\|(T - \alpha I_X)y_n\| \leqslant \|(T_c - \alpha I_{X_c})z_n\| \leqslant 4^{-n}c^{-1}\|y_n\|$$

for each n. Thus, $(T - \alpha I_X)|_Y$ is compact, where $Y = \mathrm{span}[P_1 z_n \,|\, n \in \mathbb{N}]$. The proof can then be completed as in the complex case.

By considering $T - \alpha I_X$ instead of T, we can assume that $\alpha = 0$. We need to establish that, if the restriction of $(T_c - i\beta I_{X_c})$ to an infinite dimensional subspace of X_c is compact, and $d_{\mathbb{R}I_X,\Pi_2}(T)$ is finite, then $\beta = 0$.

Indeed, otherwise we can assume that $\beta = 1$. We are going to find (inductively) a normalized sequence $z_n = x_n + iy_n \in X_c$ $(x_n, y_n \in X)$ s.t. $\|T_c z_n - iz_n\| < 1/(8 \cdot 4^{n+1}(\|T\|+4))$, and the spaces $F_n = \mathrm{span}[x_n, y_n]$ form an FDD sequence with constant less than 2.

First note that, by Lemma 2.5, there exists a normalized sequence $z'_n \in X$, which has basic constant less than 2, and such that $\|T_c z'_n - i z'_n\| < 1/(4^{n+4}(\|T\|+4))$ for each n. Let $z_1 = z'_1$. Suppose we have already selected $z_k \in \mathrm{span}[z'_m \,|\, m \geqslant k]$ for $1 \leqslant k \leqslant n$. To pick $z_{n+1} \in \mathrm{span}[z'_m \,|\, m > n]$, find a finite collection $\mathcal{F}$ of norm 1 linear functionals in X^*, such that $\max_{f\in\mathcal{F}} f(x) > (1 + 4^{-(n+1)})^{-1}\|x\|$ for any $x \in \mathrm{span}[F_1, \ldots, F_n]$. Consider

$$Y = \Big(\cap_{f\in\mathcal{F}} \ker f\Big) \bigcap \Big(\cap_{f\in\mathcal{F}} \ker (T^* f)\Big) \hookrightarrow X.$$

As $\dim(X/Y) < \infty$, $Y \cap P_1(\mathrm{span}[z'_m \,|\, m > n])$ is non-trivial (as before, P_1 is the "real part" projection from X_c to X, that is, $P_1(x + iy) = x$). Below we prove that any norm 1 $z_{n+1} = x_{n+1} + i y_{n+1} \in \mathrm{span}[z'_m \,|\, m > n]$, with $x_{n+1} \in Y$, has the desired properties. By Lemma 2.4, $\|T_c z_{n+1} - i z_{n+1}\| < 1/(8 \cdot 4^{n+1}(\|T\|+4))$. To show that (F_n) is an FDD sequence, prove first that

$$\|w + w'\| \geqslant (1 - 2 \cdot 4^{-(n+1)})\|w\| \tag{2.3}$$

for any $w' \in F_{n+1}]$, and $w \in \mathrm{span}[F_1, \ldots, F_n]$. Clearly, we can assume $\|w\| = 1$. If $\|w'\| \geqslant 2$, there is nothing to prove. Otherwise, write $w' = a x_{n+1} + b y_{n+1}$, where $a, b \in \mathbb{R}$, and (by Lemma 2.8) $\max\{|a|, |b|\} \leqslant 8(\|T\|+4)$. By (2.2), $\|T x_{n+1} - y_{n+1}\| < 1/(8 \cdot 4^{n+1}(\|T\|+4))$. Note that, for any $f \in \mathcal{F}$, $f(x_{n+1}) = f(T x_{n+1}) = 0$, hence

$$f(w') = a f(x_{n+1}) + b\big(f(T x_{n+1}) + f(y_{n+1} - T x_{n+1})\big) = b f(y_{n+1} - T x_{n+1}),$$

and therefore, $|f(w')| \leqslant |b| \|y_{n+1} - T x_{n+1}\| < 4^{-(n+1)}$. (2.3) now follows from

$$\|w + w'\| \geqslant \max_{f\in\mathcal{F}} |f(w + w')| \geqslant \max_{f\in\mathcal{F}} |f(w)| - \max_{f\in\mathcal{F}} |f(w')| > 1 - 2 \cdot 4^{-(n+1)}.$$

Next consider a sequence $(w_k)_{k=1}^N$, with $w_k \in F_k$. By (2.3), $\prod_{j=m+1}^N (1 - 2 \cdot 4^{-j})^{-1} \|\sum_{k=1}^N w_k\| \geqslant \|\sum_{k=1}^m w_k\|$ for $1 \leqslant m < N$. However, $\ln((1-\varepsilon)^{-1}) < 1 + 1.2\varepsilon$ for $\varepsilon \in (0, 1/8]$, hence

$$\ln\Big(\prod_{j=2}^\infty (1 - 2 \cdot 4^{-j})^{-1}\Big) < 2.4 \sum_{j=2}^\infty 4^{-j} = 0.2 < \ln 1.25.$$

Therefore, $\|\sum_{k=1}^m w_k\| \leqslant 1.25 \|\sum_{k=1}^N w_k\|$, hence $(F_n)_{n\in\mathbb{N}}$ is an FDD sequence, with constant not exceeding 1.25.

Define $T_n \in B(F_n)$ by setting $T_n x_n = y_n$, $T_n y_n = -x_n$. To find an upper estimate for $\nu_1(T|_{F_n} - T_n)$, define projections $P_1, P_2 \in B(F_n)$ by setting $P_1 x_n = x_n$, $P_1 y_n = 0$, $P_2 y_n = y_n$, and $P_2 x_n = 0$. By the Lemma 2.8, $\|P_1\|, \|P_2\| \leqslant C = 4(\|T\|+4)$. As

$P_1 + P_2 = I_{F_n}$, we have

$$\begin{aligned}\nu_1(T|_{F_n} - T_n) &\leqslant \nu_1((T|_{F_n} - T_n)P_1) + \nu_1((T|_{F_n} - T_n)P_2) \\ &\leqslant \|P_1\|\nu_1((T - T_n)|_{\operatorname{ran} P_1}) + \|P_2\|\nu_1((T - T_n)|_{\operatorname{ran} P_2}) \\ &\leqslant C\Big(\|(T - T_n)\frac{x_n}{\|x_n\|}\| + \|(T - T_n)\frac{y_n}{\|y_n\|}\|\Big) \\ &\leqslant 2C\big(\|(T - T_n)x_n\| + \|(T - T_n)y_n\|\big) \\ &= 2C\big(\|Tx_n + y_n\| + \|Ty_n - x_n\|\big) \leqslant 4C\|T_c z_n - iz_n\| < 4^{-n}.\end{aligned}$$

Applying Lemma 1.7, we can find $\tilde{T} \in B(X)$ s.t. $\nu_1(T - \tilde{T}) < 2$, and $\tilde{T}|_{F_n} = T_n$ for each n. Let $Y = \operatorname{span}[F_n \,|\, n \in \mathbb{N}]$, and denote by S the operator $\tilde{T}|_Y$ (viewed as acting into Y). Clearly, $S^2 = -I_Y$. Find a $2n$-dimensional subspace of $E \hookrightarrow Y$, for which there exists a contraction $u : \ell_2^{2n} \to E$ with $\|u^{-1}\| < 2$. By Lemma 3.4 of [46], there exists an n-dimensional $G \hookrightarrow \ell_2^{2n}$ s.t. $\|P_{G^\perp} u^{-1} S u \xi\| \geqslant \|\xi\|$ for each $\xi \in G$ ($P_{G^\perp}$ stands for the orthogonal projection onto $G^\perp$). That is, $\|u^{-1} S u \xi + \eta\| \geqslant \|\xi\|$ for any $\xi, \eta \in G$. Therefore, $\|S u \xi + u\eta\| \geqslant \|\xi\|/2$ for such η and ξ.

Denote by q the quotient of X by $u(G)$. Then, for any $\xi \in G$, $\|q\tilde{T}u\xi\| \geqslant \|\xi\|/2$. Therefore, by Proposition 19.1 of [47],

$$d_{\mathbb{R}I_X, \Pi_{q2}}(\tilde{T}) \geqslant \pi_{q2}(q\tilde{T}u) \geqslant \frac{1}{2}\pi_{q2}(I_{\ell_2^n}) \geqslant \frac{n^{1/q}}{4},$$

hence, for every $n \in \mathbb{N}$,

$$d_{\mathbb{R}I_X, \Pi_{q2}}(T) \geqslant d_{\mathbb{R}I_X, \Pi_{q2}}(\tilde{T}) - \pi_{q2}(T - \tilde{T}) \geqslant \frac{n^{1/q}}{4} - 2,$$

which yields a contradiction. ■

As seen from the proof above, the ideals Π_{q2} are special, due to their connection to the Hilbert spaces (and due to Dvoretzky Theorem). For an arbitrary nice ideal $\mathfrak{B}$, $\mathbb{F}I_X$ can be shown to be $\mathfrak{B}$-hyperreflexive if X has some "structure."

Theorem 2.9. *If X is an $\mathcal{L}_{p,\kappa}$-space ($1 \leqslant p \leqslant \infty$, $\kappa \geqslant 1$), and $\mathfrak{B}$ is a maximal ideal, then $\mathbb{F}I_X$ is $C - \mathfrak{B}$-hyperreflexive, with C dependent only on κ.*

We say that a basis (e_i) is a Banach space E is *self-repeating* if there exists $C \geqslant 1$ s.t., for any $N \in \mathbb{N}$, and for any infinite $\mathcal{S} \subset \mathbb{N}$, there exist $i_1, \ldots, i_N$ in $\mathcal{S}$, so that $\kappa^{-1}\|\sum_{j=1}^N \alpha_j e_j\| \leqslant \|\sum_{j=1}^N \alpha_j e_{i_j}\| \leqslant \kappa\|\sum_{j=1}^N \alpha_j e_j\|$ for all $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$. It is easy to observe that every subsymmetric basis is self-repeating.

Theorem 2.10. *Suppose X is a Banach space with a self-repeating unconditional basis, and $\mathfrak{B}$ is a maximal ideal. Then $\mathbb{F}I_X$ is $\mathfrak{B}$-hyperreflexive.*

Note however that simply the existence of an unconditional basis does not entail the $\mathfrak{B}$-hyperreflexivity of $\mathbb{F}I_X$ (see Theorem 2.15).

We start the proof of the two preceding theorems by learning to “diagonalize” operators. If X is a Banach space with an unconditional basis $(e_i)_{i\in\mathcal{I}}$, we call $T \in B(X)$ *diagonal* if, for every $i \in \mathcal{I}$, $Te_i = t_i e_i$ $(t_i \in \mathbb{F})$.

Lemma 2.11. *Suppose $\mathfrak{B}$ is a maximal ideal, X is a Banach space with a C-unconditional basis $(e_i)_{i=1}^N$ ($N \in \mathbb{N} \cup \{\infty\}$), and $T \in B(X)$ is such that $d_{\mathbb{F}I_X,\mathfrak{B}}(T)$ is finite. Then there exists a diagonal $T_1 \in B(X)$ s.t. $\boldsymbol{\beta}(T - T_1) \leqslant 4C^2\boldsymbol{\beta}(T)$, and $d_{\mathbb{F}I_X,\mathfrak{B}}(T_1) \leqslant (4C^2 + 1)\boldsymbol{\beta}(T)$.*

Proof. By homogeneity, we can assume that $d_{\mathbb{F}I_X,\mathfrak{B}}(T) < 1$. We shall only consider the case of infinite dimensional X (the finite dimensional one is tackled similarly).

Denote by f_i the functional biorthogonal to e_i, let $t_i = f_i(Te_i)$, and denote by T_1 the linear operator on X, defined by $T_1e_i = t_ie_i$. We shall show that $\boldsymbol{\beta}(T-T_1) \leqslant 4C^2$.

For $\mathcal{F} \subset \mathbb{N}$, define the projection $P_\mathcal{F}$ by setting

$$P_\mathcal{F} e_i = \begin{cases} e_i & i \in \mathcal{F} \\ 0 & i \notin \mathcal{F} \end{cases} .$$

For simplicity, we denote $P_{\{1,\ldots,n\}}$ by Q_n. Observe that

$$Q_nT_1Q_n = Q_nTQ_n - 2^{2-n} \sum_{\mathcal{F}\subset\{1,\ldots,n\}} (Q_n - P_\mathcal{F})TP_\mathcal{F}.$$

However, $(Q_n - P_\mathcal{F})P_\mathcal{F} = 0$, hence

$$\boldsymbol{\beta}((Q_n - P_\mathcal{F})TP_\mathcal{F}) \leqslant \|Q_n - P_\mathcal{F}\|\|P_\mathcal{F}\| < C^2,$$

and therefore, $\boldsymbol{\beta}(Q_nTQ_n - Q_nT_1Q_n) < 4C^2$ for each n.

It remains to show that $\boldsymbol{\beta}(T - T_1) \leqslant 4C^2$. To this end, it suffices to prove that an operator $S \in B(X)$ satisfies $\boldsymbol{\beta}(S) \leqslant 1$ whenever $\boldsymbol{\beta}(Q_nSQ_n) < 1$ for each n. Indeed, for every $\varepsilon \in (0, \boldsymbol{\beta}(S))$ there exist finite rank contractions A and B such that $\boldsymbol{\beta}(ASB) > \boldsymbol{\beta}(S) - \varepsilon$. Note that $\lim_m \nu_1(Q_mB - B) = 0$, hence, by a small perturbation argument, we can assume that $Q_mB = B$. By a small perturbation once again, we can assume that, for some $n \geqslant m$, $Q_nSQ_m = SQ_m$. Thus,

$$1 > \boldsymbol{\beta}(Q_nSQ_n) \geqslant \boldsymbol{\beta}(AQ_nSQ_nB) = \boldsymbol{\beta}(ASB) > \boldsymbol{\beta}(S) - \varepsilon.$$

Since ε is arbitrary, we get the desired estimate. ∎

Proof of Theorem 2.9. In this proof, $C_0, C_1, \ldots$ stand for constants, which depend only on κ (we do not keep track on the rate of growth of our constants C_i as $\kappa \to \infty$, but it appears to be polynomial).

By [25], X contains an increasing net of finite dimensional subspaces $(X_i)_{i\in\mathcal{I}}$ (ordered by inclusion), s.t. $d(X_i, \ell_p^{K_i}) < C_0$ (K_i is even), $X = \overline{\cup_{i\in\mathcal{I}} X_i}$, and, for each i, there exists a projection P_i from X onto X_i of norm less than C_0. Then, for each i, there exists a normalized basis $(e_j^{(i)})_{j=1}^{K_i}$ in X_i, such that, for any sequence of scalars $(\alpha_j)_{j=1}^{K_i}$,

$$C_0^{-1}\|\sum_{j=1}^{K_i}\alpha_j e_j^{(i)}\| \leqslant \|\sum_{j=1}^{K_i}\alpha_j\delta_j\| \leqslant C_0\|\sum_{j=1}^{K_i}\alpha_j e_j^{(i)}\| \tag{2.4}$$

($(\delta_j)_{j=1}^{K_i}$ is the canonical basis in $\ell_p^{K_i}$).

Suppose $T \in B(X)$ is such that $\boldsymbol{\beta}(ATB) < 1$ whenever A and B are finite rank contractions with $AB = 0$. For each i consider $T_i = P_iTP_i$ as an operator on X_i. For any pair of finite rank contractions $B : E \to X_i$ and $A : X_i \to F$, $\boldsymbol{\beta}(AT_iB) = \boldsymbol{\beta}(AP_iT_iB) < C_0$ whenever $AB = 0$ (we view B as taking E to X, and AP_i as mapping X to F). By Lemma 2.11, there exists an operator $S_i \in B(X)$, diagonal with respect to $(e_j^{(i)})_{j=1}^{K_i}$, such that $\boldsymbol{\beta}(T_i|_{X_i} - S_i) < C_1$. We shall show that there exists $s^{(i)} \in \mathbb{F}$ s.t.

$$\boldsymbol{\beta}(S_i - s^{(i)}I_{X_i}) < C_2. \tag{2.5}$$

Consider the real case first. We have: $S_ie_j^{(i)} = s_je_j^{(i)}$. By changing the enumeration, we can assume that $s_1 \geqslant s_2 \geqslant \ldots \geqslant s_{K_i}$. Pick $s = s^{(i)} \in [s_{K_i/2+1}, s_{K_i/2}]$ (in other words, $s^{(i)}$ is a median of the sequence (s_j)), and show that (2.5) holds for this value of $s^{(i)}$. To achieve this, consider the maps $B : \ell_p^{K_i/2} \to X_i$ and $A : X_i \to \ell_p^{K_i/2}$, defined as follows:

$$B\delta_j = \frac{1}{2C_0}(e_j + e_{K_i-j}),\ \ Ae_j = \frac{\delta_j}{2C_0},\ \ Ae_{K_i-j} = -\frac{\delta_j}{2C_0}\ \ (1 \leqslant j \leqslant K_i/2) \tag{2.6}$$

($(\delta_j)_{j=1}^{K_i/2}$ is the canonical basis for $\ell_p^{K_i/2}$). Clearly, $\max\{\|A\|, \|B\|\} \leqslant 1$, and $AB = 0$. Therefore,

$$\boldsymbol{\beta}(AS_iB) \leqslant \boldsymbol{\beta}(AT_iB) + \boldsymbol{\beta}(T_i - S_i) < C_0 + C_1.$$

On the other hand, $4C_0^2AS_iB\delta_j = (s_j - s_{K_i-j})\delta_j$. Denote by Λ, Λ_1, and Λ_2 the diagonal operators on $\ell_p^{K_i/2}$, whose diagonal entries are, respectively, $(s_j - s_{K_i-j})$, $(s_j - s)$, and $(s - s_{K_i-j})$. Then $\Lambda = 4C_0^2AS_iB$, hence $\boldsymbol{\beta}(\Lambda) \leqslant 4C_0^2\boldsymbol{\beta}(AS_iB) \leqslant C_3$. Moreover, there exist (diagonal) contractions $U_1, U_2 \in B(\ell_p^{K_i/2})$ s.t. $\Lambda_r = U_r\Lambda$ ($r = 1, 2$). Therefore, $\boldsymbol{\beta}(\Lambda_r) \leqslant \boldsymbol{\beta}(\Lambda)$. Finally, $S_i - sI_{X_i} = W_1\Lambda_1V_1 + W_2\Lambda_2V_2$, with

$$V_1 : X_i \to \ell_p^{K_i/2} : e_j \mapsto \begin{cases} \delta_j & j \leqslant K_i/2 \\ 0 & j > K_i/2 \end{cases},\quad W_1 : \ell_p^{K_i/2} \to X_i : \delta_j \mapsto e_j,$$

and V_2, W_2 defined similarly. By (2.4), $\|V_r\|, \|W_r\| \leqslant C_0$. This implies

$$\boldsymbol{\beta}(S_i - sI_{X_i}) \leqslant \sum_{r=1}^{2}\|W_r\|\|V_r\|\boldsymbol{\beta}(\Lambda_r) < C_2,$$

hence (2.5) holds.

The complex case is slightly more involved. There, we write $s_j = \alpha_i + i\beta_j$. Let α and β be medians of (α_j) and (β_j), respectively. As above, it suffices to show that the diagonal operators $(\alpha_j - \alpha)$ and $(\beta_j - \beta)$, acting on X_i, have $\boldsymbol{\beta}$-norms bounded by a constant $C_2/2$. Now suppose $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_{K_i}$. With A and B as in (2.6), we show that the

$$\boldsymbol{\beta}\big(\mathrm{diag}\,(\alpha_j - \alpha_{K_i-j} + i(\beta_j - \beta_{K_i-j}))_{j=1}^{K_i/2}\big) < C_5.$$

Here, $\mathrm{diag}\,(t_j)_{j=1}^{K_i/2}$ is viewed as a diagonal operator on $\ell_p^{K_i/2}$. If $|t_j| \geqslant |t_j'|$ for $1 \leqslant j \leqslant K_i/2$, then $\boldsymbol{\beta}(\mathrm{diag}\,(t_j)) \geqslant \boldsymbol{\beta}(\mathrm{diag}\,(t_j'))$. Therefore, $\boldsymbol{\beta}(\mathrm{diag}\,(\alpha_j - \alpha_{K_i-j})_{j=1}^{K_i/2}) < C_5$. As in the real case, we conclude that $\boldsymbol{\beta}(\mathrm{diag}\,(\alpha_j - \alpha)_{j=1}^{K_i/2}) < C_5$. Similarly, we show that $\boldsymbol{\beta}(\mathrm{diag}\,(\beta_j - \beta)_{j=1}^{K_i/2}) < C_5$. Thus, (2.5) holds, with $s^{(i)} = \alpha + i\beta$, and $C_2 = 2C_5$.

The inequality (2.5), and the reasoning preceding it, imply that

$$\boldsymbol{\beta}((P_iTP_i - s^{(i)}P_i)|_{X_i}) \leqslant \boldsymbol{\beta}(P_iTP_i|_{X_i} - S_i) + \boldsymbol{\beta}(S_i - s^{(i)}I_{X_i}) < C_1 + C_2 = C_6.$$

But $(I_X - P_i)I_XP_i = 0$, hence

$$\boldsymbol{\beta}((I_X - P_i)TP_i) \leqslant \|I_X - P_i\|\|P_i\|C < (C_0 + 1)C_0C.$$

Therefore,

$$\boldsymbol{\beta}((T - s^{(i)}I_X)|_{X_i}) \leqslant \boldsymbol{\beta}((P_iT - s^{(i)}I_X)|_{X_i}) + \boldsymbol{\beta}((I_X - P_i)TP_i) < C_7.$$

Passing to a subnet of $\mathcal{I}$ if necessary, we can assume that $\lim_i s^{(i)} = s$ exists. We shall show that $\boldsymbol{\beta}(T - sI_X) < C_7$. Indeed, suppose $j \succ i$. Then

$$\boldsymbol{\beta}((T - s^{(j)}I_X)|_{X_i}) \leqslant \boldsymbol{\beta}((T - s^{(j)}I_X)|_{X_j}) < C_7.$$

Taking the limit, we see that, for each i, $\boldsymbol{\beta}((T - sI_X)|_{X_i}) \leqslant C_7$. Therefore,

$$\boldsymbol{\beta}(T - sI_X) = \sup_i \boldsymbol{\beta}((T - sI_X)|_{X_i}) \leqslant C_7,$$

and we are done. ∎

Proof of Theorem 2.10. Suppose (e_i) is a self-repeating unconditional basis in a Banach space X. By renorming, we can assume that this basis is 1-unconditional, and normalized. Consider $T \in B(X)$ with $d_{\mathbb{F}I_X,\mathfrak{B}}(T) < 1$, and show that $\boldsymbol{\beta}(T - sI_X) \leqslant C_0$ for some $s \in \mathbb{F}$ (in this proof, C_0 and C_1 denote constants, depending only on the "self-repeating constant" κ). By Lemma 2.11, there exists a diagonal operator S s.t. $\boldsymbol{\beta}(S - T) < C_0$. We use the notation $S = \mathrm{diag}\,(s_1, s_2, \ldots)$ (that is, $Se_i = s_ie_i$). Let s be a cluster point of the sequence (s_i), and show that $\boldsymbol{\beta}(S - sI_X) \leqslant C_1$.

Without loss of generality, assume $s = 0$. Denote by Q_n the n-th basis projection ($Q_ne_i = e_i$ if $i \leqslant n$, $Q_ne_i = 0$ if $i > n$). As in the proof of the previous theorem, it suffices to show that $\boldsymbol{\beta}(SQ_n - sQ_n) < C_1$ for each n. By a small perturbation method, and using the definition of the self-repeating basis, we can assume the existence of $i_n > \ldots > i_1 > n$, s.t. $\kappa^{-1}\|\sum_{j=1}^n \alpha_je_j\| \leqslant \|\sum_{j=1}^n \alpha_je_{i_j}\| \leqslant \kappa\|\sum_{j=1}^n \alpha_je_j\|$ for all $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$, and $Se_{i_j} = 0$ for $1 \leqslant j \leqslant n$. Let $E = \mathrm{span}[e_1, \ldots, e_n]$, and consider $B : E \to X : e_j \to (e_j + e_{i_j})/(2\kappa)$. Define $A : X \to E$ by setting

$Ae_j = e_j/(2\kappa)$, $Ae_{i_j} = -e_j/(2\kappa)$, $Ae_k = 0$ if $k \notin \{1, \ldots, n, i_1, \ldots, i_n\}$. Clearly, A and B are contractions, and $AB = 0$. Moreover, $ASBe_j = (s_j - s)/(4\kappa^2)e_j$. Identifying E with $\operatorname{ran} Q_n$, we obtain: $Q_n S = ABQ_n/(4\kappa^2)$. Thus, $\boldsymbol{\beta}(Q_n S) \leqslant 4\kappa^2 \boldsymbol{\beta}(ASB) \leq 4\kappa^2$, and we are done. ■

2.2. **The Banach space case: counterexamples.** It is known (see e.g. [30]) that every 1-dimensional space is hyperreflexive (more generally, every reflexive finite dimensional space of operators is hyperreflexive). In this subsection, we give examples of 1-dimensional spaces which are not $\mathfrak{B}$-hyperreflexive.

Theorem 2.12. *Suppose X and Y are Banach spaces, and $\mathfrak{B}$ is a maximal Banach ideal.*

1. *Suppose the norms $\|\cdot\|$ and $\boldsymbol{\beta}(\cdot)$ are not equivalent on $F(X,Y)$. Then there exists an approximable $T \in B(X,Y)$, for which $\mathbb{F}T$ is not $\mathfrak{B}$-hyperreflexive.*
2. *For every $C > 0$ there exists $T \in B(X)$ such that $\mathbb{F}T$ is not C-$\mathfrak{B}$-hyperreflexive.*

Remark 2.13. G. Pisier (see e.g. [39, 40]) constructed an example of a Banach space X_P on which every approximable operator is nuclear. We do not know whether there exists $T \in B(X_P)$ for which $\mathbb{F}T$ is not $\mathfrak{B}$-hyperreflexive.

Proof. (1) For $n \in \mathbb{N}$ let $\phi(n) = \sup_{\operatorname{rank} u = n} \boldsymbol{\beta}(u)/\|u\|$ (note that $1 \leqslant \phi(n) \leqslant n$, and $\lim_n \phi(n) = \infty$). Find a sequence of 4-tuples $(\alpha_k, n_k, u_k, F_k)_{k \in \mathbb{N}}$, where, for each k, $\alpha_k > 0$, $n_k \in \mathbb{N}$, $u_k \in B(X,Y)$ is a contraction of rank not exceeding n_k, F_k is a finite dimensional subspace of X, and the following collection of inequalities is satisfied:

$$(2.7) \qquad \begin{array}{ll} (1) & \boldsymbol{\beta}(u_k) \geqslant \boldsymbol{\beta}(u_k|_{F_k}) > \phi(n_k)/2. \\ (2) & \alpha_{k+1} < 2^{-6(k+1)} \min_{j \leqslant k} \alpha_j (\max\{\dim F_j, 4n_j\})^{-1}. \\ (3) & \phi(n_{k+1}) > 2 \cdot 10^k \alpha_{k+1}^{-1} (\sum_{j=1}^k n_j + 10^k). \end{array}$$

Indeed, the selection of such 4 tuples can be done inductively. First let $\alpha_1 = 2^{-6}$, $n_1 = 10$, and pick u_1 and F_1 to satisfy (2.7(1)). If the first k 4-tuples have already been selected, find $\alpha_{k+1} > 0$ for which (2.7(2)) holds. Then find n_{k+1} to satisfy (2.7(3)). Finally, we pick u_{k+1} and F_{k+1} satisfying (2.7(1)).

Consider $T = \sum_{k=1}^\infty \alpha_k u_k$ (the sum converges in $B(X,Y)$, since $0 < \alpha_k \leqslant 2^{-6k}$). We claim that $\mathbb{F}T$ is not $\mathfrak{B}$-hyperreflexive. To this end, show first that $T \notin \mathfrak{B}(X,Y)$. Indeed, for each $k > 1$,

$$(2.8) \qquad \boldsymbol{\beta}(T) \geqslant \boldsymbol{\beta}(T|_{F_k}) \geqslant \alpha_k \boldsymbol{\beta}(u_k|_{F_k}) - \sum_{j=1}^{k-1} \alpha_j \boldsymbol{\beta}(u_j|_{F_k}) - \sum_{j=k+1}^{\infty} \alpha_j \boldsymbol{\beta}(u_j|_{F_k}).$$

But, for $j < k$, $\boldsymbol{\beta}(u_j|_{F_k}) \leqslant \boldsymbol{\beta}(u_j) \leqslant \phi(n_j) \leqslant n_j$, hence $\sum_{j=1}^{k-1} \alpha_j \boldsymbol{\beta}(u_j|_{F_k}) \leqslant \sum_{j=1}^{k-1} n_j$. On the other hand, for $j > k$, $\boldsymbol{\beta}(u_j|_{F_k}) \leqslant \dim F_k$. By (2.7(2)), $\alpha_j \dim F_k < 2^{-6j} \alpha_k$

Finally, by (2.7(3)), $\alpha_k\boldsymbol{\beta}(u_k|_{F_k}) > \alpha_k\phi(n_k)/2$. Therefore, (2.8) and (2.7(3)) imply:

$$\boldsymbol{\beta}(T) \geqslant \boldsymbol{\beta}(T|_{F_k}) \geqslant 10^{2k} + 10^k \sum_{j=1}^{k-1} n_j - \sum_{j=1}^{k-1} n_j - \alpha_k \sum_{j=k+1}^{\infty} 2^{-6j} > 10^k.$$

But $k > 1$ can be arbitrarily large, hence $T \notin \mathfrak{B}(X,Y)$.

Therefore, for $k > 1$, $\mathrm{dist}_{\mathfrak{B}}(\alpha_k u_k, \mathbb{F}T) = \alpha_k\boldsymbol{\beta}(u_k) > \alpha_k\phi(n_k)/2 > 10^{2k}$. Next we estimate $d_{\mathbb{F}T,\mathfrak{B}}(\alpha_k u_k)$ from above. To this end, suppose A and B are contractions, and $ATB = 0$. Then

$$-\alpha_k A u_k B = \sum_{j=1}^{k-1} \alpha_j A u_j B + \sum_{j=k+1}^{\infty} \alpha_j A u_j B,$$

and therefore,

$$\boldsymbol{\beta}(\alpha_k A u_k B) \leqslant \sum_{j=1}^{k-1} \alpha_j \boldsymbol{\beta}(u_j) + \boldsymbol{\beta}\Big(\sum_{j=k+1}^{\infty} \alpha_j A u_j B\Big). \tag{2.9}$$

But $\boldsymbol{\beta}(u_j) \leqslant \phi(n_j)$. Moreover, by (2.7(2)),

$$\Big\| \sum_{j=k+1}^{\infty} \alpha_j A u_j B \Big\| \leqslant \sum_{j=k+1}^{\infty} \alpha_j \leqslant \frac{\alpha_k}{4n_k} \sum_{j=k+1}^{\infty} 2^{-6j} \leqslant \frac{\alpha_k}{2^{6k+1} n_k},$$

and

$$\mathrm{rank}\,\Big(\sum_{j=k+1}^{\infty} \alpha_j A u_j B\Big) = \mathrm{rank}\,\Big(\sum_{j=1}^{k} \alpha_j A u_j B\Big) \leqslant \sum_{j=1}^{k} n_j < 2n_k.$$

Therefore,

$$\boldsymbol{\beta}\Big(\sum_{j=k+1}^{\infty} \alpha_j A u_j B\Big) \leqslant \Big\| \sum_{j=k+1}^{\infty} \alpha_j A u_j B \Big\| \mathrm{rank}\,\Big(\sum_{j=1}^{k} \alpha_j A u_j B\Big) \leqslant \frac{\alpha_k}{2^{6k}}.$$

Thus, by (2.9) and (2.7),

$$\boldsymbol{\beta}(\alpha_k A u_k B) \leqslant \sum_{j=1}^{k-1} n_j + \frac{\alpha_k}{2^{6k}} < \frac{\alpha_k \phi(n_k)}{10^{k-1}} + \frac{\alpha_k}{2^{6k}} < \frac{\alpha_k \phi(n_k)}{2^{k+1}}.$$

Therefore,

$$d_{\mathbb{F}T,\mathfrak{B}}(\alpha_k u_k) < \frac{\alpha_k \phi(n_k)}{2^{k+1}} < \frac{\boldsymbol{\beta}(\alpha_k u_k)}{2^k}.$$

If $\mathbb{F}T$ is $C - \mathfrak{B}$-hyperreflexive, then $C > 2^k$. To complete the proof of part (a), recall that k can be arbitrarily large.

(2) Pick a finite rank $S \in B(X)$ with $\boldsymbol{\beta}(S) > C$, and let $T = S - \lambda I_X$, where $\lambda \in (0, (\mathrm{rank}\,(S))^{-1})$. Then $T \notin \mathfrak{B}(X)$, hence $\mathrm{dist}_{\mathfrak{B}}(S, \mathbb{F}T) = \boldsymbol{\beta}(S)$. On the other hand, suppose the contractions A and B satisfy $ATB = 0$. Then $ASB = \lambda AB$, and $\boldsymbol{\beta}(ASB) = \lambda\boldsymbol{\beta}(AB) \leqslant \lambda \mathrm{rank}\,(ASB) \leqslant \lambda \mathrm{rank}\,(S) < 1$. ■

Proposition 2.14. *Suppose $\mathfrak{B}$ is a nice ideal, and a pair X, Y of infinite dimensional Banach spaces satisfies one of the following conditions:*

(1) *$X = Y$ has the Bounded Approximation property.*
(2) *There exists $C > 0$ such that, for every n, X contains a C-complemented C-isomorphic copy of ℓ_2^n.*
(3) *X has a complemented infinite dimensional K-convex subspace.*

Then the norms $\boldsymbol{\beta}(\cdot)$ and $\|\cdot\|$ are not equivalent, and, by Theorem 2.12(1), there exists $T \in B(X,Y)$ so that $\mathbb{F}T$ is not $\mathfrak{B}$-hyperreflexive.

Proof. (1) By definition of the Bounded Approximation property, there exists a constant C with the property that for any finite dimensional subspace E of X, there exists a finite rank $u \in B(X)$, s.t. $u|_E = I_E$, and $\|u\| \leqslant C$. The ideal $(\mathfrak{B}, \boldsymbol{\beta})$ is nice, hence for every $\lambda > 0$ there exists a finite dimensional $E \hookrightarrow X$ s.t. $\boldsymbol{\beta}(I_E) > C\lambda$. Therefore, $\boldsymbol{\beta}(u)/\|u\|$ can be as large as possible, for finite rank $u \in B(X)$.

To prove (2), fix n, and select $E_n \hookrightarrow X$, C-isomorphic to ℓ_2^n, for which there exists a projection $P_n : X \to E_n$ $\|P_n\| \leqslant C$. Find $F_n \hookrightarrow Y$ s.t. $d(F_n, \ell_2^n) < 2$. Then there exists $u_n : E_n \to F_n$ of norm less than $2C$, s.t. u_n^{-1} is a contraction. Then $\|u_n P_n\| \leqslant 2C$, while $\boldsymbol{\beta}(u_n P_n) \geqslant \phi(n) \nearrow \infty$. Finally, (2) implies (3), by [38]. ■

Theorem 2.15. *Suppose $(\mathfrak{B}, \boldsymbol{\beta})$ is a maximal Banach ideal, such that either*

$$\lim_{p \searrow 2} \lim_{n \to \infty} \frac{\boldsymbol{\beta}(I_{\ell_p^n})}{\boldsymbol{\beta}(I_{\ell_2^n})} = \infty,$$

or

$$\lim_{p \nearrow 2} \lim_{n \to \infty} \frac{\boldsymbol{\beta}(I_{\ell_p^n})}{\boldsymbol{\beta}(I_{\ell_2^n})} = \infty.$$

Then there exists a Banach space X with an unconditional basis such that $\mathbb{F}I_X$ is not $\mathfrak{B}$-hyperreflexive.

Remark 2.16. Examples of ideals covered by the previous theorem include I_p ($1 \leqslant p < 2$), and Π_{pq} ($q \in [1,2)$, $1/p - 1/q + 1/2 \geqslant 0$), and Γ_p ($1 < p < \infty$). In particular, the case of Π_q ($q \in [1,2)$) is covered.

The cases of I_p and Γ_p follow from Chapter 22 of [37]. To deal with Π_{pq}, observe that, by p. 207 of [14], $\pi_{pq}(I_{\ell_2^n}) = n^{1/p-1/q+1/2}$. Moreover, $\pi_{pq}(I_{\ell_u^n}) \geqslant n^{1/p-1/q+1/u'}$ (here, $u > 2$, and $1/u + 1/u' = 1$). To establish the last inequality, suppose $e_1, \ldots, e_n$ is the canonical basis in ℓ_u^n. Then

$$\sup_{x^* \in X^*, \|x^*\| \leqslant 1} \Big(\sum_i |x^*(e_i)|^q \Big)^{1/q} = n^{1/q-1/u'},$$

and $(\sum_i \|e_i\|^p)^{1/p} = n^{1/p}$.

Proof. We only deal with the case of

$$\lim_{p\searrow 2}\lim_{n\to\infty}\frac{\boldsymbol{\beta}(I_{\ell_p^n})}{\boldsymbol{\beta}(I_{\ell_2^n})}=\infty,$$

as the second one is handled similarly. We work with the space

$$X=\Big(\sum_{i=1}^{\infty}\ell_{p_i}^{n_i}\Big)_2,$$

with $p_i\searrow 2$, and $n_i\nearrow\infty$. More precisely:

(1) n_i's increase very fast: for each k,

$$c_k=\boldsymbol{\beta}(I_{\ell_{p_k}^{n_k}})\geqslant 100^{k+1}\sum_{i=1}^{k-1}n_i,\ \ 6^{-1}\cdot 100^{-(k+1)}c_k>d_k=\boldsymbol{\beta}(I_{\ell_2^{2n_k}})$$

(this is possible, since, for $m\in\mathbb{N}$ and $j\leqslant 2m$, $\boldsymbol{\beta}(I_{\ell_2^m})\leqslant\boldsymbol{\beta}(I_{\ell_2^{2m}})\leqslant 2\boldsymbol{\beta}(I_{\ell_2^m})$).

(2) p_k's approach 2 "even faster": for each k, $5^{2n_k(1/2-1/p_{k+1})}<2$.

The space X was introduced by W. Johnson in [20] (see also pp. 112-113 of [27]). He proved that all subspaces of X have the Bounded Approximation Property, although X is not isomorphic to a Hilbert space.

Denote by P_i the natural contractive projection onto $X_i=\ell_{p_i}^{n_i}$. Then $\boldsymbol{\beta}(P_i)\geqslant c_i$. To complete the proof, we have to show that $d_{\mathbb{F}I_X,\mathfrak{B}}(P_i)\leqslant\sum_{k=1}^{i-1}n_k+6d_i$. Assuming the above inequality is true, we would have, for each i

$$\frac{\boldsymbol{\beta}(P_i)}{d_{\mathbb{F}I_X,\mathfrak{B}}(P_i)}\geqslant\frac{c_i}{c_i/100^{i+1}+c_i/100^{i+1}}>100^i,$$

which, by definition, means that $\mathbb{F}I_X$ is not $\mathfrak{B}$-hyperreflexive.

Fix i, and show that, for any subspace E of X, $\boldsymbol{\beta}(q_EP_ii_E)\leqslant\sum_{k=1}^{i-1}n_k+6d_i$, where i_E is the injection of E into X, and $q_E:X\to X/E$ is the quotient map. To achieve this, let $Q=I_X-\sum_{k=1}^{i}P_k$. Clearly, $-q_EP_ii_E=q_E(\sum_{k=1}^{i-1}P_k)i_E+q_EQi_E$, hence

$$\boldsymbol{\beta}(q_EP_ii_E)\leqslant\boldsymbol{\beta}(q_E(\sum_{k=1}^{i-1}P_k)i_E)+\boldsymbol{\beta}(q_EQi_E).\tag{2.10}$$

But

$$\boldsymbol{\beta}(q_E(\sum_{k=1}^{i-1}P_k)i_E)\leqslant\operatorname{rank}(q_E(\sum_{k=1}^{i-1}P_k)i_E)\leqslant\sum_{k=1}^{i-1}n_k.$$

To estimate $\boldsymbol{\beta}(q_EQi_E)$, note that $q_EQi_E=-q_E(\sum_{k=1}^{i}P_k)i_E$, hence $\operatorname{rank}(q_EQi_E)\leqslant 2n_i$. Let $Y=\operatorname{ran}Q$, and consider $F=Q(E)$ and $Z=\ker(q_EQ)$ as subspaces of Y. Then $q_EQi_E=(q_EQ)(Qi_E)$ factors canonically (contractively) via q_Zi_F (where $i_F:F\to Y$ and $q_Z:Y\to Y/Z$ are an injection and a quotient map, respectively), and $\dim F/(F\cap Z)\leqslant 2n_i$. By p. 112 of [27] or [20], $G=q_Z(F)$ is 3-isomorphic to

G_1/G_2, where G_1 is a subspace of Y, of dimension not exceeding 5^{2n_i}. But then (cf. [23], or [21]) G_1 is 2-isomorphic to a Hilbert space. Therefore, $d(G, \ell_2^{\dim G}) \leqslant 6$, and

$$\boldsymbol{\beta}(q_E Q i_E) \leqslant \boldsymbol{\beta}(I_G) \leqslant 6\boldsymbol{\beta}(I_{\ell_2^{\dim G}}) \leqslant 6d_i,$$

Going back to (2.10), we obtain the result. ∎

Remark 2.17. In [22], D. Larson proposed the following generalization of hyperreflexivity from algebras of operators to general Banach algebra. We say that a subspace $\mathcal{A}$ of a Banach algebra $\mathcal{U}$ is *C-Larson hyperreflexive* (*C-LHR*, for short) if, for any $u \in \mathcal{U}$, we have

$$\inf_{a \in \mathcal{A}} \|u - a\| \leqslant C \sup\{\|vuw\| \mid v, w \in \mathcal{U},\ v\mathcal{A}w = 0,\ \max\{\|v\|, \|w\|\} \leqslant 1\}.$$

We say that $\mathcal{A}$ is *Larson hyperreflexive* (*LHR*) if it is C-Larson hyperreflexive for some C. The motivation for this definition comes from the following observation: a subspace of $\mathcal{U} = B(X)$ (X being a Banach space) is C-Larson hyperreflexive iff it is C-hyperreflexive in the usual sense.

As noted above, every one-dimensional subspace of $B(X)$ is hyperreflexive. This is not the case if we consider Larson hyperreflexivity.

Corollary 2.18. *There exists a Banach algebra $\mathcal{U}$ with the identity I, such that $\mathbb{C}I$ is not Larson hyperreflexive.*

Proof. Find an infinite dimensional Banach space X and a maximal Banach ideal $\mathfrak{B}$ s.t. $\mathbb{C}I_X$ is not $\mathfrak{B}$-hyperreflexive (the existence of such X and $\mathfrak{B}$ was established in Theorem 2.15). Define $\mathcal{U}$ as $\mathbb{C}I_X \oplus_1 \mathfrak{B}(X)$ (for $\lambda \in \mathbb{C}$ and $T \in \mathfrak{B}(X)$, we set $\|\lambda I_X \oplus_1 T\|_{\mathcal{U}} = |\lambda| + \boldsymbol{\beta}(T)$). We can view $\mathcal{U}$ as a subalgebra of $B(X)$, with the usual operator multiplication, but with a different norm. It remains to show that $\mathbb{C}I$ is not Larson hyperreflexive. Indeed, for any $n \in \mathbb{N}$ there exists $T \in \mathfrak{B}(X)$ s.t. $\boldsymbol{\beta}(T) > n$, and $\boldsymbol{\beta}(vTw) < 1$ whenever v and w are contractions satisfying $vw = 0$. Then

$$\inf_{\lambda \in \mathbb{C}} \|T - \lambda I_X\|_{\mathcal{U}} = \boldsymbol{\beta}(T) > n.$$

On the other hand, if $v, w \in \mathcal{U}$ are such that $\max\{\|v\|_{\mathcal{U}}, \|w\|_{\mathcal{U}}\} \leqslant 1$, then v and w belong to the unit ball of $B(X)$. Therefore,

$$\begin{aligned} &\sup\{\|vTw\|_{\mathcal{U}} \mid v, w \in \mathcal{U},\ vw = 0,\ \max\{\|v\|_{\mathcal{U}}, \|w\|_{\mathcal{U}}\} \leqslant 1\} \\ &\quad \leqslant \sup\{\boldsymbol{\beta}(vTw) \mid v, w \in B(X),\ vw = 0,\ \max\{\|v\|_{B(X)}, \|w\|_{B(X)}\} \leqslant 1\} < 1. \end{aligned}$$

As n can be arbitrarily large, $\mathbb{C}I$ is not Larson hyperreflexive. ∎

Remark 2.19. One can strengthen the statements of Theorems 2.12 and 2.15 by considering the ampliations of the operators involved. For $T \in B(X, Y)$ and $n \in \mathbb{N} \cup \{\infty\}$, the *$n$-th ampliation* of T is defined as $T^{(n)} = I_{\ell_2^n} \otimes T \in B(\ell_2^n(X), \ell_2^n(Y))$. For $\mathcal{A} \subset B(X, Y)$, set $\mathcal{A}^{(n)} = \{T^{(n)} \mid T \in \mathcal{A}\}$. The proofs of the theorems yield, respectively:

(1) Suppose X and Y are Banach spaces, and $\mathfrak{B}$ is a maximal Banach ideal, for which the norms $\|\cdot\|$ and $\boldsymbol{\beta}(\cdot)$ are not equivalent on $F(X,Y)$. Then there exists an approximable $T \in B(X,Y)$, for which $\mathbb{F}T^{(n)}$ is not $\mathfrak{B}$-hyperreflexive for any $n \in \mathbb{N}$.

(2) Suppose $(\mathfrak{B}, \boldsymbol{\beta})$ is a maximal Banach ideal, and either $\lim_{p\searrow 2} \lim_{n\to\infty} \frac{\boldsymbol{\beta}(I_{\ell_p^n})}{\boldsymbol{\beta}(I_{\ell_2^n})} = \infty$, or $\lim_{p\nearrow 2} \lim_{n\to\infty} \frac{\boldsymbol{\beta}(I_{\ell_p^n})}{\boldsymbol{\beta}(I_{\ell_2^n})} = \infty$. Then there exists a Banach space X with an unconditional basis such that $\mathbb{F}I_{\ell_2^n(X)}$ is not $\mathfrak{B}$-hyperreflexive for any $n \in \mathbb{N}$.

This contrasts sharply with the classical setting: by [24], if $\mathcal{A}$ is an n-dimensional subspace of of $B(X,Y)$, then $\mathcal{A}^{(2n)}$ is reflexive, and therefore, by [30], hyperreflexive.

3. The Hilbert space case

3.1. **The Hilbert space case: introduction.** In this section, we deal with the class of Hilbert spaces, and the separable symmetrically normed ideals (see [17, 45] for general information on the topic). To describe these ideals, suppose $\mathcal{E}$ is a *symmetric sequence space*, that is, a Banach space of sequences of complex scalars, such that, for any sequence $(\alpha_i)_{i\in\mathbb{N}}$ of scalars, any sequence $(\omega_i)_{i\in\mathbb{N}}$ of scalars with $|\omega_i| = 1$, and any permutation π of $\mathbb{N}$, we have $\|(\alpha_i)_{i\in\mathbb{N}}\|_{\mathcal{E}} = \|(\omega_i \alpha_{\pi(i)})_{i\in\mathbb{N}}\|_{\mathcal{E}}$. Henceforth, we assume that $\|(1,0,0,\ldots)\|_{\mathcal{E}} = 1$. Then $\mathcal{S}_{\mathcal{E}}$ consists of all compact operators T with $(s_j(T))_{j\in\mathbb{N}} \in \mathcal{S}_{\mathcal{E}}$ ($s_1(T) \geqslant s_2(T) \geqslant \ldots \geqslant 0$ are the singular values of T), with the norm $\|T\|_{\mathcal{E}} = \|(s_j(T))_{j\in\mathbb{N}}\|_{\mathcal{E}}$.

Suppose a symmetric normed space $\mathcal{E}$ coincides with $\mathcal{E}_0 = \mathrm{span}[e_n \,|\, n \in \mathbb{N}] \hookrightarrow \mathcal{E}$, where $e_n = (0,\ldots,0,1,0,\ldots)$ (1 occupies n-th position). It is known (Section III.6 of [17]) that $\mathcal{S}_{\mathcal{E}}$ is a separable symmetrically normed ideal. Conversely, if $\mathcal{S}$ is a separable symmetrically normed ideal, then $\mathcal{S} = \mathcal{S}_{\mathcal{E}}$, for a symmetrically normed space $\mathcal{E}$ satisfying $\mathcal{E} = \mathcal{E}_0$.

As an analogue of maximality, we require the space $\mathcal{E}$ to be *mononormalizing* (the term comes from p. 88 of [17]; in [45], such spaces are called *regular*). That is, we require that two conditions be satisfied: (i) if $\lim_n \|(x_1,\ldots,x_n,0,0,\ldots)\|_{\mathcal{E}} = C < \infty$, then $(x_i)_{i\in\mathbb{N}} \in \mathcal{E}$, and $\|(x_i)_{i\in\mathbb{N}}\|_{\mathcal{E}} = C$; and (ii) if $(x_i)_{i\in\mathbb{N}} \in \mathcal{E}$, then $\lim_n \|(x_n, x_{n+1},\ldots)\|_{\mathcal{E}} = 0$. Note that these two conditions guarantee $\mathcal{E} = \mathcal{E}_0$. As before, we say that the the symmetric sequence space $\mathcal{E}$, and the corresponding ideal $\mathcal{S}_{\mathcal{E}}$, are *nice*, if $\mathcal{E}$ is mononormalizing, and $\lim_n \|I_{\ell_2^n}\|_{\mathcal{E}} = \infty$, or equivalently, $\lim_n c_{\mathcal{E}}(n) = \infty$, where $c_{\mathcal{E}}(n) = \|(1,\ldots,1,0,\ldots)\|_{\mathcal{E}}$ (n 1's followed by 0's). In other words, a symmetric mononormalizing sequence space $\mathcal{E}$ is nice if the formal identity map $I : \mathcal{E} \to c_0$ is not an isomorphism.

It is easy to see that, for every $\mathcal{A} \hookrightarrow B(H,K)$ (H and K are Hilbert spaces) and $T \in B(H,K)$, $d_{\mathcal{A},\mathcal{E}}(T) = \sup \|P^{\perp}_{\mathcal{A}(\mathrm{ran}\, Q)} T Q\|_{\mathcal{E}}$, where the supremum is taken over all orthogonal projections Q, and $P^{\perp}_{\mathcal{A}(\mathrm{ran}\, Q)}$ is the orthogonal projection with kernel $\mathcal{A}(\mathrm{ran}\, Q)$. In particular, if $\mathcal{A} \hookrightarrow B(H)$ is a unital operator algebra, then

$d_{\mathcal{A},\mathcal{E}}(T) = \sup \|Q^\perp T Q\|_\mathcal{E}$, with the supremum taken over all orthogonal projections Q onto invariant subspaces of $\mathcal{A}$.

3.2. **The Hilbert space case: main results.** As in the Banach space setting, the differences between the classical hyperreflexivity and $\mathcal{S}_\mathcal{E}$-hyperreflexivity are numerous. For instance, it is unknown whether every von Neumann algebra is hyperreflexive (this question is equivalent to the famous Kadison Similarity Problem, see e.g. Section 27 of [41]). However, we have:

Theorem 3.1. *Suppose $\mathcal{E}$ is a nice sequence space.*

(1) *If $\mathcal{E}$ is reflexive, or equals ℓ_1, then any von Neumann algebra is $8 - \mathcal{S}_\mathcal{E}$-hyperreflexive.*
(2) *If H and K are Hilbert spaces, then $B(K) \otimes I_H$ is $4 - \mathcal{S}_\mathcal{E}$-hyperreflexive.*

Remark 3.2. E. Christensen [8] (see also Section 9 of [11]) proved that any injective von Neumann algebra is hyperreflexive. See [9] for more examples of hyperreflexive von Neumann algebras, and for the connections to other open problems.

By Theorem 3.1, $\mathbb{C}I_H$ is $4 - \mathcal{S}_\mathcal{E}$-hyperreflexive for any Hilbert space H. By Theorem 2.12, not every 1-dimensional subspace of $B(H)$ is $\mathcal{S}_\mathcal{E}$-hyperreflexive. However, many such subspaces are:

Theorem 3.3. *If $\mathcal{E}$ is a nice sequence space, and $A \in B(H,K)$ is not compact, then $\mathbb{C}A$ is $4(2 + \|A\| \|A\|_{ess}^{-1}) - \mathcal{S}_\mathcal{E}$-hyperreflexive.*

On the other hand, it is known (see e.g. [4, 11]) that any nest algebra is hyperreflexive. The algebras of analytic Toeplitz and analytic Laurent operators are hyperreflexive ([10] and [43], respectively). By contrast, we have:

Theorem 3.4. *Suppose $\mathcal{E}$ is a nice sequence space. Then a nest algebra $\mathcal{A}$ is $\mathcal{S}_\mathcal{E}$-hyperreflexive if and only if the corresponding nest contains finitely many projections.*

The algebra $\mathcal{L}$ of analytic Laurent operators is the algebra of multiplication operators M_ϕ ($\phi \in H_\infty$), acting on $L_2(\mathbb{T})$, where the unit circle $\mathbb{T}$ is equipped with the normalized Haar measure. One can view $\mathcal{L}$ as the weak*-closed subalgebra of $B(L_2)$ generated by the bilateral shift.

Now denote by $P^{(H)}$ the orthogonal projection from $L_2(\mathbb{T})$ onto the Hardy space H_2. The algebra $\mathcal{T}$ consists of analytic Toeplitz operators $T_\phi = P^{(H)} M_\phi|_{H_2}$ (as before, $\phi \in H_\infty$). Clearly, both $\mathcal{L}$ and $\mathcal{T}$ are algebraically isomorphic, and weak* isometric, to H^∞.

Theorem 3.5. *Suppose $\mathcal{A}$ is either the algebra of analytic Toeplitz operators, or the algebra of analytic Laurent operators, $n \in \mathbb{N}$, and $\mathcal{E}$ is a nice symmetric sequence space. Then $\mathcal{A}^{(n)}$ is not $\mathcal{S}_\mathcal{E}$-hyperreflexive.*

Remark 3.6. Here, $\mathcal{A}^{(n)}$ denotes the n-th ampliation of $\mathcal{A}$ (defined in Remark 2.19). As shown in Proposition 3.16, $\mathcal{A}^{(\infty)}$ is $\mathcal{E}$-hyperreflexive whenever $\mathcal{E}$ is nice, and $\mathcal{A}$ is a unital weak* closed subalgebra of $B(H)$ (H is a Hilbert space).

Proof of Theorem 3.1. (1) We use the standard connection between derivations and hyperreflexivity. Consider a von Neumann sub-algebra N of $B(H)$ (H is a Hilbert space). Suppose $T \in B(H)$ is such that, for any N-invariant orthogonal projection P, we have $\|(I-P)TP\|_{\mathcal{E}} \leqslant 1$. We shall show that there exists $a \in N$ s.t. $\|T-a\|_{\mathcal{E}} \leqslant 8$. To this end, introduce a map $\delta : N' \to \mathcal{S}_{\mathcal{E}} : b \mapsto bT - Tb$ (N' is the commutant of N). To check that δ is bounded, note that, for any $P \in N'$,

$$\begin{aligned}\|\delta P\|_{\mathcal{E}} &= \|PT - TP\|_{\mathcal{E}} = \|PT(I-P) - (I-P)TP\|_{\mathcal{E}} \\ &\leqslant \|PT(I-P)\|_{\mathcal{E}} + \|(I-P)TP\|_{\mathcal{E}} \leqslant 2.\end{aligned}$$

Therefore, $\|\delta U\|_{\mathcal{E}} \leqslant 4$ if U is self-adjoint unitary, which, in turn, implies that $\|\delta\| \leqslant 8$. By [19], there exists $S \in \mathcal{S}_{\mathcal{E}}$, s.t. $\|S\|_{\mathcal{E}} \leqslant 8$, and $\delta b = bS - Sb$ for any $b \in N'$. In particular, $bS - Sb = bT - Tb$ for any $b \in N'$, hence $b(T-S) = (T-S)b$, and, by the double commutant theorem, $T - S \in N$.

(2) It suffices to consider the case of H being finite dimensional, that is, $H = \ell_2^n$. Suppose $T \in \ell_2^n(K)$ satisfies $d_{B(K)^{(n)},\mathcal{E}}(T) < 1$, and show the existence of $A \in B(K)$ s.t. $\|A^{(n)} - T\|_{\mathcal{E}} < 4$. Denote by $P_1, \ldots, P_n$ the orthogonal projections from $\ell_2^n(K)$ onto the copies of K. Let $A_i = P_i T P_i$, viewed as acting on K, and write $T = \sum_{i=1,j}^n E_{ij} \otimes A_{ij}$, where $(E_{ij})_{i,j=1}^n$ are matrix units in $B(\ell_2^n)$. Let $n_1 = \lfloor n/2 \rfloor$, $n_2 = n - n_1$. Consider the Grassman manifold $\mathcal{G}$ of n_1-dimensional subspaces of ℓ_2^n, equipped with the rotation-invariant probability measure. By Lemma 4.6 of [34],

$$\int_{\mathcal{G}} ((I_{\ell_2^n} - P) \otimes I_K)(E_{ii} \otimes A_{ii})(P \otimes I_K)\, dP = \frac{n_1 n_2}{n^2 - 1}(E_{ii} - \frac{1}{n} I_{\ell_2^n}) \otimes A_{ii}$$

for each i, and

$$\int_{\mathcal{G}} ((I_{\ell_2^n} - P) \otimes I_K)(E_{ij} \otimes A_{ij})(P \otimes I_K)\, dP = \frac{n_1 n_2}{n^2 - 1} E_{ij} \otimes A_{ij}$$

when $i \neq j$. Let $A = \sum_{i=1}^n A_{ii}/n$. Then

$$\int_{\mathcal{G}} ((I_{\ell_2^n} - P) \otimes I_K) T (P \otimes I_K)\, dP = \frac{n_1 n_2}{n^2 - 1}(T - I_{\ell_2^n} \otimes A),$$

But $((I_{\ell_2^n} - P) \otimes I_K) B(K)^{(n)} (P \otimes I_K) = 0$ for each $P \in \mathcal{G}$, hence

$$\|((I_{\ell_2^n} - P) \otimes I_K) T (P \otimes I_K)\|_{\mathcal{E}} \leqslant d_{B(K)^{(n)},\mathcal{E}}(T) < 1.$$

Therefore, $\|T - A^{(n)}\|_{\mathcal{E}} < (n^2 - 1)/(n_1 n_2) \leqslant 4$. ■

Remark 3.7. Suppose the sequence space $\mathcal{E}$ in Theorem 3.1(1) is such that there exists a constant κ with the property that, for two disjointly supported vectors

$x, y \in \mathcal{E}$, we have $\|x+y\| \leqslant \kappa \max\{\|x\|, \|y\|\}$ (for instance, if $\mathcal{E} = \ell_p$, then $\kappa = 2^{1/p}$ works). Then, in the above proof, we have

$$\|PT(I-P) - (I-P)TP\|_{\mathcal{E}} \leqslant \kappa \max\{\|PT(I-P)\|_{\mathcal{E}}, \|(I-P)TP\|_{\mathcal{E}}\},$$

and we conclude that any von Neumann algebra is $4\kappa - \mathcal{S}_{\mathcal{E}}$-hyperreflexive.

Proof of Theorem 3.3. Suppose, without loss of generality, that $A \in B(H,K)$ satisfies $\|A\|_{ess} < 1$. Pick $c \in (0, \|A\|_{ess})$. Suppose $T_0 \in B(H,K)$ is such that $d_{\mathbb{C}A,\mathcal{E}}(T_0) < 1$. We shall show the existence of $\lambda \in \mathbb{C}$ for which $\|T_0 - \lambda A\|_{\mathcal{E}} \leqslant 8 + 4\|A\|c^{-1}$.

Use polar decomposition to find orthogonal projections $P \in B(H)$ and $Q \in B(K)$ s.t. $A = QAP + (I-Q)A(I-P)$, $\|(I-Q)A(I-P)h\| \leqslant 1$, and $\|QAP\xi\| \geqslant c\|\xi\|$ for any $\xi \in \operatorname{ran} P$. Observe that $QA(I-P) = (I-Q)AP = 0$, hence

$$\|T_0 - (QT_0P + (I-Q)T_0(I-P))\|_{\mathcal{E}} \leqslant \|QT_0(I-P)\|_{\mathcal{E}} + \|(I-Q)T_0P\|_{\mathcal{E}} \leqslant 2d_{\mathbb{C}A,\mathcal{E}}(T_0) < 2.$$

Let $T = QT_0P + (I-Q)T_0(I-P)$. It remains to prove that $\|T - \lambda A\|_{\mathcal{E}} \leqslant 6 + 4\|A\|c^{-1}$ for some $\lambda \in \mathbb{C}$.

By Theorem 3.1(2), $\mathbb{C}P \hookrightarrow B(\operatorname{ran} P)$ is $4 - \mathcal{S}_{\mathcal{E}}$-hyperreflexive. Moreover, QAP (viewed as an operator from $\operatorname{ran} P$ to $\operatorname{ran} Q$) has an inverse of norm not exceeding c^{-1}. Hence, by Proposition 1.3, $\mathbb{C}QAP$ is $4\|A\|c^{-1} - \mathcal{S}_{\mathcal{E}}$-hyperreflexive. If the contractions $v : H \to \operatorname{ran} P$ and $u : \operatorname{ran} Q \to K$ satisfy $u(QAP)v = 0$, then

$$\|uQTPv\|_{\mathcal{E}} = \|uQT_0Pv\|_{\mathcal{E}} \leqslant d_{\mathbb{C}A,\mathcal{E}}(T_0) < 1.$$

Therefore, there exists $\lambda \in \mathbb{C}$ s.t. $\|QTP - \lambda QAP\|_{\mathcal{E}} < 4\|A\|c^{-1}$. Passing from T to $T - \lambda A$ if necessary, we may assume that $\lambda = 0$.

Next we estimate $\|(I-Q)T(I-P)\|_{\mathcal{E}}$. For every $\varepsilon > 0$ there exists an n-dimensional subspace E_1 of $\operatorname{ran}(I-P)$ s.t. $\|(I-Q)T(I-P)|_{E_1}\|_{\mathcal{E}} > \|(I-Q)T(I-P)\|_{\mathcal{E}} - \varepsilon$. As $\|A\|_{ess} < 1$, and QTP is compact, we can assume (by perturbing T slightly) the existence of an n-dimensional $E_2 \hookrightarrow \operatorname{ran} P$ s.t. $\|A|_{E_2}\| < 1$, and $T|_{E_2} = 0$. Consider the isometries $u_j : \ell_2^n \to E_j$ $(j = 1,2)$, and let $u = (u_1 + u_2)/\sqrt{2}$. Let $E = \operatorname{ran} u$, and let R be the orthogonal projection onto $A(E)^{\perp}$. We shall show that, for any $\xi \in \ell_2^n$, $\|RTu\xi\| \geqslant \|Tu_1\xi\|/2$. Indeed,

$$\begin{aligned} 2\|RTu\xi\|^2 &= 2\inf_{\eta \in \ell_2^n} \|Tu\xi + Au\eta\|^2 = \inf_{\eta \in \ell_2^n} \big(\|Tu_1\xi + Au_1\eta\|^2 + \|Au_2\eta\|^2\big) \\ &\geqslant \inf_{\eta \in \ell_2^n} \big((\|Tu_1\xi\| - \|Au_1\eta\|)^2 + \|Au_2\eta\|^2\big). \end{aligned}$$

However, $\|Au_2\eta\| \geqslant c\|\eta\|$, while $\|Au_1\eta\| \leqslant c\|\eta\|$. Therefore,

$$2\|RTu\xi\|^2 \geqslant \inf_t \big((\|Tu_1\xi\| - t)^2 + t^2\big) = \|Tu_1\xi\|^2/2.$$

We can, therefore, write $Tu_1 = SRTu$, where $\|S\| \leqslant 2$. Thus, $2\|RTu\|_{\mathcal{E}} \geqslant \|Tu_1\|_{\mathcal{E}}$. But $RAu = 0$, hence $\|RTu\|_{\mathcal{E}} \leqslant d_{\mathbb{C}A,\mathcal{E}}(T_0) + \|T - T_0\|_{\mathcal{E}} < 3$. Then $\|(I-Q)T(I-$

$P)\|_{\mathcal{E}} < \|Tu_1\|_{\mathcal{E}} + \varepsilon \leqslant 2\|RTu\|_{\mathcal{E}} + \varepsilon$. As $\varepsilon > 0$ is arbitrary, we conclude that $\|(I-Q)T(I-P)\|_{\mathcal{E}} \leqslant 6$, and

$$\|T\|_{\mathcal{E}} \leqslant \|QTP\|_{\mathcal{E}} + \|(I-Q)T(I-P)\|_{\mathcal{E}} \leqslant 6 + 4\|A\|c^{-1}.$$

Therefore, $\|T_0\|_{\mathcal{E}} \leqslant 8 + 4\|A\|c^{-1}$. The constant c can be arbitrarily close to $\|A\|_{ess}$, hence the desired estimate for $\mathrm{dist}_{\mathcal{E}}(T, \mathbb{C}A)$. ■

Proof of Theorem 3.4. The case of a finite nest is easy to handle. If a nest on a Hilbert space H is infinite, find a strictly increasing sequence of projections $0 = P_0 < P_1 < \ldots < P_n < P_{n+1} = I$ in the nest. For $1 \leqslant k \leqslant n$, pick unit vectors $\eta_k \in \mathrm{ran}\,(P_{k+1} - P_k)$, $\xi_k \in \mathrm{ran}\,(P_k - P_{k-1})$. Consider $T = \sum_{k=1}^n \xi_k \otimes \eta_k$. That is, for $h \in H$, $Th = \sum_{k=1}^n \langle h, \xi_k\rangle \eta_k$. Clearly, $\|T\| = 1$, and $\mathrm{rank}\, T = n$. We shall show that

$$\mathrm{dist}_{\mathcal{E}}(T, \mathcal{A}) \geqslant c_{\mathcal{E}}(n) = \|(1, \ldots, 1, 0, \ldots)\|_{\mathcal{E}} \tag{3.1}$$

(the sequence on the right contains n 1's, followed by 0's), and, for any projection P from the nest,

$$\mathrm{rank}\,((I-P)TP) \leqslant 2. \tag{3.2}$$

Assuming these two inequalities hold, we complete the proof by observing that

$$\|(I-P)TP\|_{\mathcal{E}} \leqslant \|T\|\mathrm{rank}\,((I-P)TP) \leqslant 2,$$

hence $\mathrm{dist}_{\mathcal{E}}(T, \mathcal{A})/d_{\mathcal{A},\mathcal{E}}(T) \geqslant c_{\mathcal{E}}(n)/2$, and letting n grow without a bound.

To handle (3.1), denote by P and Q the orthogonal projections onto $\mathrm{span}[\eta_k \,|\, 1 \leqslant k \leqslant n]$ and $\mathrm{span}[\xi_k \,|\, 1 \leqslant k \leqslant n]$, respectively. For $a \in \mathcal{A}$, $\|T - a\|_{\mathcal{E}} \leqslant \|PTQ - PaQ\|_{\mathcal{E}}$ (here, we identify $PB(H)Q$ with $n \times n$ matrices). We can view $PTQ - PaQ$ as an $n \times n$ matrix (the bases in the range space and in the domain space are (η_k) and (ξ_k), respectively). Under this identification, PaQ is a strictly upper triangular matrix, and PTQ is the identity matrix I. By the pinching inequality ((IV.52) of [6], or Theorem 1.19 of [45]), $\|PTQ - PaQ\|_{\mathcal{E}} \geqslant \|I\|_{\mathcal{E}} = c_{\mathcal{E}}(n)$, hence (3.1).

To deal with (3.2), pick P from the nest, and find $i \in \{1, \ldots, n\}$ s.t. $P_{i-1} \leqslant P \leqslant P_i$. Then, for $h \in h$,

$$(I-P)TPh = \sum_{k=1}^n \langle h, P\xi_k\rangle (I-P)\eta_k.$$

But $P\xi_k = 0$ for $k \geqslant i-1$, while $(I-P)\eta_k = 0$ for $i \geqslant k$. Hence, only two terms in the centered sum above do not vanish. ■

Proof of Theorem 3.5. For the sake of brevity, we use the notation $L_2^{[k]}$ and $H_2^{[k]}$ for $L_2 \otimes_2 \ell_2^k$, resp. $H_2 \otimes_2 \ell_2^k$. Define $f_j \in L^\infty$ ($j \in \mathbb{Z}$) by setting $f_j(z) = z^j$. Consider the case of the analytic Toeplitz algebra $\mathcal{T}$ first. By Halmos's generalization of Beurling's Theorem (see [18], or Section 3.1 of [35]), any invariant subspace for $\mathcal{T}$ consists of functions

$$\{z \mapsto U(z)f(z) \,|\, f \in H_2^{[m]}\},$$

where $z \mapsto U(z) \in B(\ell_2^m, \ell_2^n)$ is analytic on $\mathbb{T}$, $U(z)$ is an isometry for every z, and $m \leqslant n$. Denote by P_U the corresponding projection.

Let T be the left shift (of multiplicity n) on $H_2^{[n]}$ – that is, for $j \in \{0, 1, \ldots\}$ and $h \in \ell_2^n$,

$$T(f_j \otimes h) = \begin{cases} f_{j-1} \otimes h & j \geqslant 1 \\ 0 & j = 0 \end{cases} .$$

In other words, $T = P^{(H)} M_{f_{-1}}|_{H_2} \otimes I_{\ell_2}$. As in the proof of Theorem 3.4, we can show that $T - a \notin \mathcal{S}_{\mathcal{E}}$ for any $a \in \mathcal{T}$. It remains to show that, for any U as above, $\|(I - P_U)TP_U\|_{\mathcal{E}} \leqslant n$. To this end, denote by $(e_k)_{k=1}^m$ the canonical basis in ℓ_2^m. Then $(f_j \otimes Ue_k)$ $(j \geqslant 0,\ 1 \leqslant k \leqslant m)$ form an orthonormal basis in $P_U(H_2^{[n]})$. It is easy to see that, for $1 \leqslant k \leqslant m$, and $j \geqslant 1$, $g : z \mapsto f_j(z)U(z)e_k$ is an analytic function, and $g(0) = 0$. The function $Tg : z \mapsto f_{j-1}(z)U(z)e_k$ belongs to $\operatorname{ran} P_U$, hence $(I - P_U)T(f_j \otimes Ue_k) = 0$. Thus, $\operatorname{ran}((I - P_U)TP_U) \leqslant n$. As T is a contraction, we conclude that $\|(I - P_U)TP_U\|_{\mathcal{E}} \leqslant n$.

Now deal with the analytic Laurent algebra $\mathcal{L}$. In this case, by [18], or by Section 3.1 of [35]), we have invariant subspaces of two types. First, there are those consisting of functions

$$\{z \mapsto U(z)f(z) \mid f \in H_2^{[m]}\},$$

where $m \leqslant n$, and $U : \mathbb{T} \to B(\ell_2^m, \ell_2^n)$ is a measurable map, such that $U(z)$ is an isometry for any $z \in \mathbb{T}$. Denote by P_U the orthogonal projection onto such subspace. Then, there are invariant subspaces of the form

$$\{z \mapsto U(z)f(z) \mid f \in L_2^{[m]}\},$$

with U as above. The corresponding orthogonal projection will be denoted by Q_U. Consider the left shift $T = M_{f_{-1}} \otimes I_{\ell_2^n}$ (that is, for $h \in \ell_2^n$, $T(f_j \otimes h) = f_{j-1} \otimes h$. As before, $T - a \notin \mathcal{S}_{\mathcal{E}}$ for any $a \in \mathcal{L}$, and $\|(I - P_U)TP_U\|_{\mathcal{E}} \leqslant n$ for any U. Moreover, $\operatorname{ran} Q_U$ is invariant under T, hence $(I - Q_U)TQ_U = 0$. Thus, $\|(I - P)TP\|_{\mathcal{E}} \leqslant n$ for any $\mathcal{L}$-invariant orthogonal projection P, and we are done. ∎

Next we study optimal hyperreflexivity constants. By [28], any 1-dimensional space of operators on a Hilbert space is 1-hyperreflexive. The situation is different for $\mathcal{S}_{\mathcal{E}}$-hyperreflexivity.

Proposition 3.8. *Suppose H and K are Hilbert spaces, $\mathcal{S}_{\mathcal{E}}$ is a nice ideal, and $\mathcal{A}$ is a $C - \mathcal{E}$-hyperreflexive subspace of $B(H, K)$, which does not contain any non-zero elements of $\mathcal{S}_{\mathcal{E}}$. (1) Then $C \geqslant \sqrt{2}$. (2) If, in addition, $\mathcal{A}$ contains an orthogonal projection, then $C \geqslant 2$.*

Remark 3.9. The condition $\mathcal{A} \cap \mathcal{S}_{\mathcal{E}} = \emptyset$ is important. Indeed, denote by $(e_i)_{i=1}^\infty$ the canonical basis in ℓ_2, and let $\mathcal{A} = \{T \in B(\ell_2) \mid \langle Te_1, e_1 \rangle = 0\}$. In other words, $\mathcal{A}$ is the set of infinite matrices, with 0 in the upper left corner. Suppose $\mathcal{E}$ is a symmetric sequence space. Let $T = e_1 \otimes e_1 \in B(\ell_2)$ (that is, $Th = \langle h, e_1\rangle e_1$ for any $h \in \ell_2$). Clearly, $\mathrm{dist}_{\mathcal{E}}(T, \mathcal{A}) = 1$. Now denote by P the orthogonal projection onto $\mathrm{span}[e_1]$, and observe that $\|PTP\|_{\mathcal{E}} = 1$, while $P\mathcal{A}P = 0$.

Proof. (1) Fix $c \in (0, 1)$, and find $a \in \mathcal{A}$ and $e \in H$ with $\|a\| = 1 = \|e\|$, and $\|ae\| > c$. Let $f = ae/\|ae\|$, and consider the finite rank operator $T = e \otimes f$ (that is, $Th = \langle h, e\rangle f$ for $h \in H$). By assumption, $\mathcal{A} \cap \mathcal{S}_{\mathcal{E}} = \{0\}$, hence $\mathrm{dist}(T, \mathcal{A}) = \|T\|_{\mathcal{E}} = \|T\| = 1$ (here, we use the fact that T is a rank 1 operator). Furthermore, if P and Q are orthogonal projections, then $QTP = Pe \otimes Qf$ again has rank 1 (that is, for $h \in H$, $QTPh = \langle h, Pe\rangle Qf$). Thus, $\|QTP\|_{\mathcal{E}} = \|QTP\| = \|Pe\|\|Qf\|$. To complete the proof, it suffices to show that

$$\|Pe\|\|Qf\| \leqslant \frac{1}{\sqrt{2}\|ae\|}. \tag{3.3}$$

for any pair of orthogonal projections P and Q, satisfying $QaP = 0$.

Suppose P and Q are as above. Write $Pe = \lambda e + e'$, where λ is a scalar, and $e \perp e'$. Then

$$0 = \langle Pe, e - Pe\rangle = \langle \lambda e + e', (1 - \lambda)e - e'\rangle = \lambda - |\lambda|^2 - \|e'\|^2,$$

hence $\lambda \in [0, 1]$. Moreover, $\|e'\| = \sqrt{\lambda(1 - \lambda)}$, and $\|Pe\|^2 = \lambda^2 + \|e'\|^2 = \lambda$. If $\lambda \leqslant 1/2$, then the inequality (3.3) is satisfied. Otherwise, write $aPe = \mu f + ae'$, where $\mu = \lambda\|ae\|$. As $QaPe = 0$, for any $g \in \mathrm{ran}\, Q$ we have $\mu\langle g, f\rangle = -\langle g, ae'\rangle$. In particular,

$$\|Qf\|^2 = \langle Qf, Qf\rangle = \langle Qf, f\rangle = -\mu^{-1}\langle Qf, ae'\rangle \leqslant \mu^{-1}\|Qf\|\|ae'\|,$$

hence $\|Qf\| \leqslant \mu^{-1}\|e'\|$. But $\|e'\| = \sqrt{\lambda(1 - \lambda)}$, hence

$$\|Pe\|\|Qf\| \leqslant \sqrt{\lambda} \cdot \mu^{-1}\sqrt{\lambda(1 - \lambda)} = \frac{\sqrt{1 - \lambda}}{\|ae\|} \leqslant \frac{1}{\sqrt{2}\|ae\|}$$

(recall that here, $\lambda \geqslant 1/2$). Thus, (3.3) holds.

(2) Suppose $\mathcal{A}$ contains an orthogonal projection R. Find $e \in \mathrm{ran}\, R$,and consider $T = e \otimes e$. As in part (a), it suffices to show that $\|QTP\| \leqslant 1/2$ if Q and P are orthogonal projections with $QRP = 0$.

Denote by P_1 and P_2 the orthogonal projections onto $\overline{RP(H)}$ and $\overline{(I - R)P(H)}$, respectively. Then $RP_2 = 0$, and $QRP_1 = QR \circ RP_1 = 0$ (to see this, approximate the elements of $\mathrm{ran}\, P_1$ by those in $\mathrm{ran}\,(RP)$). In fact, $\mathrm{ran}\, P_1$ is a closed subspace of $\mathrm{ran}\, R$. Therefore, $RP_1 = P_1$, hence $QP_1 = 0$, that is, $I - P_1 \geqslant Q$.

On the other hand, $P_1 + P_2 \geqslant P$, hence $\|QTP\| \leqslant \|QT(P_1 + P_2)\|$. But e is orthogonal to $\mathrm{ran}\, P_2 \hookrightarrow \mathrm{ran}\,(I - R)$, hence $TP_2 = 0$, and therefore, $\|QT(P_1 + P_2)\| = \|QTP_1\| \leqslant \|(I - P_1)TP_1\|$. Thus, it suffices to prove that, for any orthogonal projection P, $\|(I - P)TP\| \leqslant 1/2$. To achieve this, note that

$$T = PTP + (I - P)TP + PT(I - P) + (I - P)T(I - P),$$

and therefore,

$$1 = \|T\|_2^2 = \|PTP\|_2^2 + \|(I-P)TP\|_2^2 + \|PT(I-P)\|_2^2 + \|(I-P)T(I-P)\|_2^2,$$

hence

$$2\|(I-P)TP\|_2^2 = 2\|PT(I-P)\|_2^2 = 1 - (\|PTP\|_2^2 + \|(I-P)T(I-P)\|_2^2). \tag{3.4}$$

Denoting the canonical trace on $B(H)$ by tr, we have:

$$1 = \mathrm{tr}T = \mathrm{tr}(PTP) + \mathrm{tr}((I-P)T(I-P)) = \|PTP\|_2 + \|(I-P)T(I-P)\|_2$$

(we use the fact that all the operators involved have rank 1). By the arithmetic-geometric mean inequality, $\|PTP\|_2^2 + \|(I-P)T(I-P)\|_2^2 \geqslant 1/2$, and (3.4) yields the desired estimate for $\|(I-P)TP\|$. ■

We now modify the construction of Theorem 2.12 to give examples of 1-dimensional subspaces of $B(H)$ which are not $\mathcal{S}_\mathcal{E}$-hyperreflexive.

Proposition 3.10. *Suppose $\mathcal{E}$ is a nice symmetric sequence space, and T is a compact operator in $B(H,K)$, with singular numbers $t_1 \geqslant t_2 \geqslant \ldots > 0$.*

(1) *If $T \notin \mathcal{S}_\mathcal{E}$, and*

$$\limsup_n \frac{\|(t_1, \ldots, t_n, 0, 0, \ldots)\|_\mathcal{E}}{\|(t_{n+1}, \ldots, t_{2n}, 0, 0, \ldots)\|_\mathcal{E}} = \infty,$$

then $\mathbb{C}T$ is not $\mathcal{E}$-hyperreflexive.

(2) *If $T \in \mathcal{S}_\mathcal{E}$, and*

$$\limsup_n \frac{\|(t_{n+1}, t_{n+2}, \ldots)\|_\mathcal{E}}{\|(t_{n+1}, \ldots, t_{2n}, 0, 0, \ldots)\|_\mathcal{E}} = \infty,$$

then $\mathbb{C}T$ is not $\mathcal{E}$-hyperreflexive.

Proof. Write $T = \sum_{i=1}^\infty t_i e_i \otimes f_i$, where (e_i) and (f_i) are orthonormal systems in H and K, respectively. That is, $Th = \sum_{i=1}^\infty t_i \langle h, e_i \rangle f_i$ for any $h \in H$. Fix $C > 0$, and show that $\mathbb{C}T$ is not $C - \mathcal{E}$-hyperreflexive. In proving both (a) and (b), we shall use the operator $T_n = \sum_{i=1}^n t_i e_i \otimes f_i$ $(n \in \mathbb{N})$, and positive numbers $\tau(n) = \|(t_1, \ldots, t_n, 0, 0, \ldots)\|_\mathcal{E}$, $\tau'(n) = \|(t_{n+1}, \ldots, t_{2n}, 0, 0, \ldots)\|_\mathcal{E}$.

(1) Find $n \in \mathbb{N}$ with $\tau(n) > C\tau'(n)$. As $T_n \in \mathcal{S}_\mathcal{E}$ and $T \notin \mathcal{S}_\mathcal{E}$, $\mathrm{dist}_\mathcal{E}(T_n, \mathbb{C}T) = \|T_n\|_\mathcal{E} = \tau(n)$. It remains to show that $\|uT_nv\|_\mathcal{E} \leqslant \tau'(n)$ whenever the contractions u and v satisfy $uTv = 0$. Indeed, for such u and v, $uT_nv = u(T_n - T)v$. Thus, the singular values of uT_nv (call them $s_1 \geqslant s_2 \geqslant \ldots \geqslant 0$) are dominated by those of $T_n - T$. That is, $s_j \leqslant t_{n+j}$ for every $j \in \mathbb{N}$. Moreover, $\mathrm{rank}\,(uT_nv) \leqslant \mathrm{rank}\,T_n = n$, hence $s_j = 0$ for $j > n$. Therefore,

$$\|uT_nv\|_\mathcal{E} = \|(s_1, \ldots, s_n, 0, \ldots)\|_\mathcal{E} \leqslant \|(t_{n+1}, \ldots, t_{2n}, 0, \ldots)\|_\mathcal{E} = \tau'(n).$$

(2) Without loss of generality, assume that $\tau(1) = \|T\|_\mathcal{E} = 1$. For $n \in \mathbb{N}$ let $\tau''(n) = \|(t_{n+1}, t_{n+2}, \ldots)\|_\mathcal{E}$. The space $\mathcal{E}$ is mononormalizing, hence $\lim_n \tau''(n) = 0$.

Find n for which $2C\tau'(n) < \tau''(n) < 1/2$. Then, by the triangle inequality, $\tau(n) \geqslant \|T\|_{\mathcal{E}} - \tau''(n) > 1/2$, hence

$$\begin{aligned}\mathrm{dist}_{\mathcal{E}}(T - T_n, \mathbb{C}T) &= \inf_{\lambda\in\mathbb{C}} \|(\lambda t_1, \ldots, \lambda t_n, (1-\lambda)t_{n+1}, (1-\lambda)t_{n+2}, \ldots)\|_{\mathcal{E}} \\ &\geqslant \inf_{\lambda\in\mathbb{C}} \max\{|\lambda|\tau(n), (1-|\lambda|)\tau''(n)\} \geqslant \tau''(n)/2\end{aligned}$$

(to see the last inequality, consider the cases of $|\lambda| \leqslant 1/2$ and $|\lambda| > 1/2$ separately). It remains to show that $\|u(T-T_n)v\|_{\mathcal{E}} \leqslant \tau'(n)$ whenever the contractions u and v satisfy $uTv = 0$. To this end, denote the singular values of $u(T-T_n)v$ by $s_1 \geqslant s_2 \geqslant \ldots \geqslant 0$. Then $s_j \leqslant t_{n+j}$ for any j. Moreover, $\mathrm{rank}\,(u(T-T_n)v) = \mathrm{rank}\,(uT_nv) \leqslant n$, hence $s_j = 0$ for $j > n$. This implies

$$\|u(T-T_n)v\|_{\mathcal{E}} = \|(s_1, \ldots, s_n, 0, 0, \ldots)\|_{\mathcal{E}} \leqslant \tau'(n),$$

and we are done. ■

Remark 3.11. Suppose $T \in B(H,K)$. It is easy to see that, for every $m \in \mathbb{N}$ and T as in Proposition 3.10, $T^{(m)}$ is not $\mathcal{S}_{\mathcal{E}}$-hyperreflexive. This contrasts with the results of [30], where it was shown that, for any n-dimensional subspace $S \hookrightarrow B(H,K)$, $S^{(\lfloor\sqrt{2n}\rfloor)}$ is hyperreflexive.

Corollary 3.12. *Consider* $1 \leqslant p < \infty$.

(1) *Suppose an operator* $T \in B(H,K)$ *(*H *and* K *are Hilbert spaces) belongs to* $\mathcal{S}_{p\infty}$*, but not to* $\mathcal{S}_p$*. Then* $\mathbb{C}T$ *is not* $\mathcal{S}_p$*-hyperreflexive.*

(2) *There exists an operator* $T \in \mathcal{S}_p$*, such that* $\mathbb{C}T$ *is not* $\mathcal{S}_p$*-hyperreflexive.*

Proof. (1) As in Proposition 3.10, write $T = \sum_{i=1}^\infty t_i e_i \otimes f_i$, with $\sum_{k=1}^\infty t_k^p = \infty$. Without loss of generality, assume that $\sup_{k\in\mathbb{N}} t_k k^{1/p} = 1$. In the notation of the previous proposition, $\sup_n \tau(n) = \infty$ (otherwise, $T \in \mathcal{S}_p$). On the other hand, for each n, $\tau'(n) \leqslant \big(\sum_{k=n+1}^{2n} |t_k|^p\big)^{1/p} < 1$. An application of Proposition 3.10(1) completes the proof.

(2) Define a sequence (t_k) by setting $t_1 = 1$, and $t_k = (j+1)^{-2}2^{-j/p}$ for $k \in [2^j+1, 2^{j+1}]$ $(j \geqslant 0)$. Then, in the notation of Proposition 3.10, $\tau'(2^j) = (j+1)^{-2}$, and

$$\tau''(2^j) = \big(\sum_{n=j}^\infty (j+1)^{-2p}\big)^{1/p} \geqslant c(j+1)^{1/p-2},$$

where c is a constant. We apply Proposition 3.10(2) to finish the proof. ■

3.3. **Some common constructions, and an example.** In this section, we show that many classical constructions of operator theory (such as direct sums and ampliations) preserve the $\mathcal{S}_{\mathcal{E}}$-hyperreflexivity. We finish the section by proving that the unit ball of the second of the James quasi-reflexive space, viewed as a subset of the diagonal operators on ℓ_2, is $\mathcal{S}_2$-ASHR.

Proposition 3.13. *Suppose $\mathcal{E}$ is a nice sequence space, $\mathcal{I}$ and $\mathcal{J}$ are finite sets, and $(H_i)_{i\in\mathcal{I}}$, $(K_j)_{j\in\mathcal{J}}$ are Hilbert spaces. Suppose, furthermore, that, for any $(i,j) \in \mathcal{I}\times\mathcal{J}$, $\mathcal{A}_{ij}$ is a non-empty WOT closed absolutely convex $C-\mathcal{E}$-ASHR subset of $B(H_i,H_j)$. Let $H = (\sum_{i\in\mathcal{I}} H_i)_2$, $K = (\sum_{j\in\mathcal{J}} K_j)_2$, and denote by $\mathcal{A}$ the set of all $T \in B(H,K)$ for which $Q_jTP_i \in \mathcal{A}_{ij}$ for any $(i,j)\in\mathcal{I}\times\mathcal{J}$ (P_i and Q_j stand for the orthogonal projections onto H_i and K_j, respectively). Then $\mathcal{A}$ is $C(1+|\mathcal{I}||\mathcal{J}|)-\mathcal{E}$-ASHR. Moreover, if $\mathcal{E}=\ell_p$ ($1\leqslant p<\infty$), then $\mathcal{A}$ is $C(1+(|\mathcal{I}||\mathcal{J}|)^\alpha)-\mathcal{E}$-ASHR, with $\alpha=\max\{1/p,1/2\}$.*

Proof. Suppose $T \in B(H,K)$ is such that $\|uTv\|_\mathcal{E} < \gamma$ whenever $\gamma \geqslant 1$, and the contractions $v\in B(H)$ and $u\in B(K)$ satisfy $\|uav\|_\mathcal{E}\leqslant\gamma$ for any $a\in\mathcal{A}$. Restricting ourselves to the case when $\operatorname{ran} v \subset H_i$ and $(\ker u)^\perp \subset K_j$, we conclude that, for each $\varepsilon>0$, we can write $Q_jTP_i = a_{ij}+b_{ij}$, with $\rho(a_{ij})+\|b_{ij}\|_\mathcal{E} < C+\varepsilon/(|\mathcal{I}||\mathcal{J}|)$, $Q_ja_{ij}P_i = a_{ij}$, and $Q_jb_{ij}P_i = b_{ij}$. Let $a=\sum_{i,j}a_{ij}$ and $b=\sum_{i,j}b_{ij}$. Then $\rho(a) = \max_{i,j}\rho(a_{ij})\leqslant C$, and $\|b\|_\mathcal{E}\leqslant\sum_{i,j}\|b_{ij}\|_\mathcal{E} < C|\mathcal{I}||\mathcal{J}|+\varepsilon$. In case of $\mathcal{S}_\mathcal{E}=\mathcal{S}_p$, the lemma below yields a better upper estimate on $\|b\|_\mathcal{E}$. ∎

Lemma 3.14. *Suppose $1\leqslant p\leqslant\infty$, and let $q=\min\{p,2\}$. Suppose, furthermore, that H and K are Hilbert spaces, and $(P_i)_{i\in\mathcal{I}}$ and $(Q_j)_{j\in\mathcal{J}}$ are families of mutually orthogonal projections, such that $I_H=\sum_i P_i$, and $I_K=\sum_j Q_j$. Then $\|T\|_p \leqslant \big(\sum_{i,j}\|Q_jTP_i\|_p^q\big)^{1/q}$ for any $T\in B(H,K)$.*

Proof. Denote by (E_{ij}) the matrix units in the space of $|\mathcal{I}|\times|\mathcal{J}|$ matrices. Consider the operator

$$\Phi : (\sum_{i\in\mathcal{I},j\in\mathcal{J}} \mathcal{S}_p(\operatorname{ran}P_i,\operatorname{ran}Q_j))_q \to \mathcal{S}_p(H,K) : (T_{ij})_{i,j}\mapsto\sum_{i,j}E_{ij}\otimes T_{ij}.$$

By complex interpolation of non-commutative L_p-spaces (see e.g. [42]), it suffices to show that Φ is a contraction when $p\in\{1,2,\infty\}$. For $p=1,2$, this is obvious. To tackle the case of $p=\infty$, fix $\varepsilon>0$, and consider unit vectors $\xi\in H$ and $\eta\in K$, s.t. $\langle T\xi,\eta\rangle > \|T\|-\varepsilon$. For $i\in\mathcal{I}$ and $j\in\mathcal{J}$, let $\xi_i=P_i\xi$, and $\eta_j=Q_j\eta$. Using the convention $0/0=0$, we observe that $\|Q_jTP_i\| \geqslant |\langle Q_jTP_i\xi_i,\eta_j\rangle|/(\|\xi_i\|\|\eta_j\|)$. Moreover, $\sum_{i,j}(\|\xi_i\|\|\eta_j\|)^2=1$, hence

$$\Big(\sum_{i,j}\|Q_jTP_i\|^2\Big)^{1/2} \geqslant \Big(\sum_{i,j}\frac{|\langle T\xi_i,\eta_j\rangle|^2}{\|\xi_i\|^2\|\eta_j\|^2}\Big)^{1/2} \geqslant \sum_{i,j}\|\xi_i\|\|\eta_j\|\cdot\frac{|\langle T\xi_i,\eta_j\rangle|}{\|\xi_i\|\|\eta_j\|}$$
$$= \sum_{i,j}|\langle T\xi_i,\eta_j\rangle| \geqslant \Big|\sum_{i,j}\langle T\xi_i,\eta_j\rangle\Big| = |\langle T\xi,\eta\rangle| > \|T\|-\varepsilon$$

(we used Buniakovsky-Cauchy-Schwartz Inequality here). ∎

Proposition 3.15. *Suppose $(H_i)_{i\in\mathcal{I}}$, $(K_i)_{i\in\mathcal{I}}$ are families of Hilbert spaces, and, for each $i\in\mathcal{I}$, $\mathcal{A}_i$ is an absolutely convex subset of $B(H_i,K_i)$, closed in the weak operator* *topology. Denote by H and K the ℓ_2 direct sums of (H_i) and (K_i), respectively, and let $\mathcal{A} = (\sum \mathcal{A}_i)_\infty$ be a "diagonal" subset of $B(H, K)$. Then:*

1. *If $\mathcal{E}$ is a nice sequence space, and $\mathcal{A}$ is $C - \mathcal{S}_\mathcal{E}$-Azoff-Shehada hyperreflexive, then $\mathcal{A}_i$ is $C - \mathcal{S}_\mathcal{E}$-Azoff-Shehada hyperreflexive for each i.*
2. *If $1 \leqslant p < \infty$, and, for each i, $\mathcal{A}_i$ is $C - \mathcal{S}_p$-hyperreflexive subspace of $B(X, Y)$, then $\mathcal{A}$ is $(5C + 4) - \mathcal{S}_p$-hyperreflexive.*

Proof. (1) Suppose $\mathcal{A}$ is $C - \mathcal{S}_\mathcal{E}$-hyperreflexive. Fix i, and consider $T \in B(H_i, K_i)$, with $d_{\mathcal{A},\mathcal{E}}(T) < 1$. Define $\tilde{T} \in B(H, K)$ by setting $\tilde{T}|_{H_i} = T$, $\tilde{T}|_{H_i^\perp} = 0$. Then $\|A\tilde{T}B\|_\mathcal{E} < 1$ whenever A and B are contractions. Therefore, there exist $a \in \mathbb{C}\mathcal{A}$ and $b \in \mathcal{S}_\mathcal{E}$ s.t. $\tilde{T} = a + b$, and $\rho(a) + \|b\|_\mathcal{E} \leqslant C$. Clearly, one can assume that $a \in \mathcal{A}_i$.

(2) Suppose $\mathcal{A}_i$ is $C - \mathcal{S}_p$-hyperreflexive for each i. Consider $T \in B(H, K)$, such that $d_{\mathcal{A},p}(T) < \infty$. We shall show that $\mathrm{dist}_p(T, \mathcal{A}) < 5C + 4$.

Show first that T is an $\mathcal{S}_p$-perturbation of a "block-diagonal" operator. More precisely, denote by P_i (Q_i) the orthogonal projection from H onto H_i (respectively, from K onto K_i). Let $\tilde{T} = \sum_i Q_i T P_i$. For any finite $S \subset \mathcal{I}$, let $P^{(S)} = \sum_{i \in S} P_i$, and $Q^{(S)} = \sum_{i \in S} Q_i$. Note that

$$\|T - \tilde{T}\|_p = \sup_S \|Q^{(S)}(T - \tilde{T})P^{(S)}\|_p.$$

Moreover,

$$Q^{(S)}(T - \tilde{T})P^{(S)} = 4\mathrm{Ave}Q^{(S\backslash F)}(T - \tilde{T})P^{(F)},$$

where the average is taken over all subsets F of S. But $Q^{(S\backslash F)}\mathcal{A}P^{(F)} = 0$, hence $\|Q^{(S\backslash F)}(T - \tilde{T})P^{(F)}\|_p < 1$, and therefore, $\|T - \tilde{T}\|_p < 4$.

Then we show that each of the T_i's is an $\mathcal{S}_p$-perturbation of a member of $\mathcal{A}_i$. Indeed, consider orthogonal projections $P \in B(H_i)$ and $Q \in B(K_i)$ s.t. $Q\mathcal{A}_i P = 0$. Identifying Q and P with orthogonal projections on H and K, we see that $\|QTP\|_\mathcal{E} \leqslant 1$. Thus, $d_i = \mathrm{dist}_p(T_i, \mathcal{A}_i) < C$. Fix $\varepsilon > 0$, and write $T_i = a_i + b_i$, with $a_i \in \mathcal{A}_i$, $b_i \in B(H_i, K_i)$, and $\|b_i\|_p < (1 + \varepsilon)d_i$. Let $a = \sum a_i$, and $\tilde{b} = \sum_i b_i$. Clearly, $a \in \mathcal{A}$. We show next that $\|\tilde{b}\|_p < 5(1 + \varepsilon)^2 C$.

Indeed, for each i we can find orthogonal projections $P_i' \in B(H_i)$ and $Q_i' \in B(K_i)$, s.t. $Q_i'\mathcal{A}_i P_i' = 0$, and

$$\|Q_i' T_i P_i'\|_p = \|Q_i' b_i P_i'\|_p > (1 + \varepsilon)^{-1} C^{-1} d_i.$$

Let $P = \sum_i P_i'$, and $Q = \sum_i Q_i'$. Then $Q\mathcal{A}P = 0$, and

$$\|Q\tilde{T}P\|_p = \big(\sum_i \|Q_i' T_i P_i'\|_p^p\big)^{1/p} > (1 - \varepsilon)C^{-1} d,$$

where $d = (\sum_i d_i^p)^{1/p}$. Therefore,

$$\|QTP\|_p \geqslant \|Q\tilde{T}P\|_p - \|T - \tilde{T}\|_p > (1 + \varepsilon)^{-1} C^{-1} d - 4.$$

But, as noted above, $Q\mathcal{A}P = 0$, hence $\|QTP\|_p < 1$, and therefore, $(1+\varepsilon)^{-1}C^{-1}d < 5$. This yields the desired estimate for $\|\tilde{b}\|_p$.

$$\|\tilde{b}\|_p^p = \sum_i \|b_i\|_p^p < (1+\varepsilon)^p d^p < 5^p(1+\varepsilon)^{2p}C^p d^p.$$

Now recall that $T = a + (\tilde{b} + (T - \tilde{T}))$, and

$$\|\tilde{b} + (T - \tilde{T})\|_p \leqslant \|\tilde{b}\|_p + \|T - \tilde{T}\|_p < 5C(1+\varepsilon)^2 + 4.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\mathrm{dist}_p(T, \mathcal{A}) \leqslant 5C + 4$. ∎

Next we deal with ampliations of sets of operators. It is known (see e.g. Exercise 15.1 of [11]) that $\mathcal{A}^{(\infty)}$ is hyperreflexive when $\mathcal{A}$ is a weak* closed subspace of $B(H, K)$. A similar statement holds in our setting, too.

Proposition 3.16. *(1) If $\mathcal{A}$ is a weak* closed subset of $B(H, K)$, then $\mathcal{A}^{(\infty)}$ is $5 - \mathcal{S}_\mathcal{E}$-ASHR.*

(2) If $\mathcal{A} \subset B(H, K)$ is $C - \mathcal{S}_\mathcal{E}$-ASHR, then $\mathcal{A}^{(n)}$ is $(9Cn + 8) - \mathcal{S}_\mathcal{E}$-ASHR.

(3) If $\mathcal{A}$ is $C - \mathcal{S}_p$-hyperreflexive subspace of $B(H, K)$ ($1 \leqslant p < \infty$), then $\mathcal{A}^{(n)}$ is $(25C + 24) - \mathcal{S}_p$-hyperreflexive.

Proof. (1) Suppose $T \in B(\ell_2(H), \ell_2(K))$ is such that $d_{\mathcal{A}^{(\infty)},\mathcal{E}}(T) < 1 - \sigma$, where $\sigma > 0$. We shall show that there exists $a \in \mathcal{A}$ s.t. $\|T - a^{(\infty)}\|_\mathcal{E} \leqslant 4$.

First we prove the existence of $a \in B(H, K)$ as above. To this end, let $X = H \oplus_2 K$, and denote by j the "natural" embedding of $B(\ell_2(H), \ell_2(K))$ into a "corner" of $B(\ell_2(X))$. We shall show that $j(T)$ is a $\mathcal{S}_\mathcal{E}$-perturbation of $j(a^{(\infty)})$, for some $a \in B(H, K)$. To this end, denote by $\mathcal{B}$ the von Neumann algebra $B(X) \otimes I_{\ell_2}$. Observe that $\|vj(T)u\|_\mathcal{E} < 1 - \sigma$ whenever $u : X_0 \to X$ and $v : Y \to Y_0$ are contractions, and $v\mathcal{B}u = 0$. Indeed, denote by P and Q the orthogonal projections onto $\overline{\mathrm{ran}\,(P_H u)}$ and $(\ker v \cap K)^\perp$, respectively. Clearly, $QaP = 0$ for any $a \in B(H, K)$, hence $\|vj(T)u\|_\mathcal{E} \leqslant d_{\mathcal{A}^{(\infty)},\mathcal{E}}(T)$. Therefore, by Theorem 3.1, there exists $a \in B(X)$ s.t. $\|j(T) - a^{(\infty)}\|_\mathcal{E} < 4(1 - \sigma)$.

In fact, this a must belong to $B(H, K)$. Indeed, let P_H and P_K denote the orthogonal projections from $H \oplus_2 K$ onto H and K, respectively, and let $P_H^\perp$ and $P_K^\perp$ be their orthogonal complements. If $a \notin B(H, K)$, then either $P_K^\perp a$ or $aP_H^\perp$ is non-zero. In the first case, $(P_K^\perp \otimes I_{\ell_2})(a^{(\infty)} - j(T)) = P_K^\perp a \otimes I_{\ell_2}$ is not compact, a contradiction. The second case is handled similarly.

Next we show that $\rho(a) \leqslant 1$ (that is, $a \in \mathcal{A}$). Indeed, otherwise there exist $C > 0$, a norm 1 $\phi \in B(H, K)_*$ s.t. $\phi(a) > C$, yet $\phi(b) \leqslant C$ for any $b \in \mathcal{A}$ (we are using the fact that $\mathcal{A}$ is weak* closed).

The canonical identification of $B(H, K)_*$ with trace class operators allows us to write $\phi = \sum_{i=1}^\infty \alpha_i^2 \xi_i \otimes \eta_i$ (that is, for any $b \in B(H, K)$, $\phi(b) = \sum_{i=1}^\infty \alpha_i^2 \langle b\xi_i, \eta_i \rangle$), where $(\xi_i)_{\in\mathbb{N}} \subset H$ and $(\eta_i)_{\in\mathbb{N}} \subset K$ are orthonormal systems, and the non-negative numbers $(\alpha_i)_{\in\mathbb{N}}$ satisfy $\sum_i \alpha_i^2 = 1$. Then $|\sum_{i=1}^\infty \alpha_i^2 \langle b\xi_i, \eta_i\rangle| \leqslant C$ for any $b \in \mathcal{A}$, yet $\sum_{i=1}^\infty \alpha_i^2 \langle a\xi_i, \eta_i\rangle > C$.

To proceed, we identify ℓ_2 with $\ell_2 \otimes_2 \ell_2$. Denote by $(\delta_i)_{i\in\mathbb{N}}$ the canonical orthonormal basis in ℓ_2. Select $m \in \mathbb{N}$ s.t. $Cc_{\mathcal{E}}(m) > 1$, where, as before, $c_{\mathcal{E}}(m) = \|(1, \ldots, 1, 0, \ldots)\|_{\mathcal{E}}$, with m 1's (such a selection is possible, since $c_{\mathcal{E}}(m) \nearrow \infty$, and $c_{\mathcal{E}}(1) = 1$). For $1 \leqslant s \leqslant m$, define $h_s = \sum_{i=1}^\infty \alpha_i \xi_i \otimes \delta_i \otimes \delta_{s+N}$, and $k_s = \sum_{i=1}^\infty \alpha_i \eta_i \otimes \delta_i \otimes \delta_{s+N}$, where N is selected to be so large that

$$\|(T - a^{(\infty)})|_{H\otimes\ell_2\otimes\mathrm{span}[\delta_j\,|\,j>N]}\|_{\mathcal{E}} < \sigma C.$$

Denote by P and Q the orthogonal projections onto $\mathrm{span}[h_s\,|\,1 \leqslant s \leqslant m]$ and $\mathrm{span}[k_s\,|\,1 \leqslant s \leqslant m]$, respectively. We complete the proof by estimating $\|QTP\|_{\mathcal{E}}$ from below, and $\|Q(b \otimes I_{\ell_2})P\|_{\mathcal{E}}$ $(b \in \mathcal{A})$ from above.

Note that, for any $b \in B(H, K)$, and $1 \leqslant s, t \leqslant m$,

$$\langle (b \otimes I_{\ell_2\otimes_2\ell_2})h_s, k_t\rangle = \begin{cases} \phi(b) = \sum_{i=1}^\infty \alpha_i^2 \langle b\xi_i, \eta_i\rangle & s = t \\ 0 & s \neq t \end{cases}.$$

Therefore, for any such b, $Q(b \otimes I_{\ell_2\otimes_2\ell_2})P$ is represented by a matrix with $\phi(b)$ on the diagonal, and 0's away from it. Thus, $\|Q(b \otimes I_{\ell_2\otimes_2\ell_2})P\|_{\mathcal{E}} = c_{\mathcal{E}}(m)|\phi(b)|$. In particular, $\|Q(b \otimes I_{\ell_2\otimes_2\ell_2})P\|_{\mathcal{E}} \leqslant c_{\mathcal{E}}(m)C$ for $b \in \mathcal{A}$. On the other hand,

$$\|QTP\|_{\mathcal{E}} \geqslant \|Q(a \otimes I_{\ell_2\otimes_2\ell_2})P\|_{\mathcal{E}} - \|T|_{\mathrm{ran}\,P}\|_{\mathcal{E}} > Cc_{\mathcal{E}}(m) - C\sigma \geqslant Cc_{\mathcal{E}}(m)(1 - \sigma).$$

However, let $\gamma = c_{\mathcal{E}}(m)C$. Then

$$\|QTP\|_{\mathcal{E}} \leqslant d_{\mathcal{A}^{(\infty)},\mathcal{E}}(T)\gamma < (1 - \sigma)\gamma,$$

a contradiction.

(2) Suppose that for $T \in B(\ell_2^n(H), \ell_2^n(K))$ there exists $\sigma \in (0, 1/2)$ s.t. $\|uTv\|_{\mathcal{E}} < (1-2\sigma)\gamma$ whenever the contractions u and v are such that $\|ua^{(n)}v\|_{\mathcal{E}} \leqslant \gamma$, (as before, $\gamma \geqslant 1$). As in part (1), we use Theorem 3.1 to show that there exists $a_0 \in B(H, K)$ s.t. $\|T - a_0^{(n)}\|_{\mathcal{E}} < 8(1 - \sigma)$. Consequently,

$$\|ua_0^{(n)}v\|_{\mathcal{E}} \leqslant \|uTv\|_{\mathcal{E}} + \|T - a_0^{(n)}\|_{\mathcal{E}} < 8(1 - \sigma) + (1 - 2\sigma)\gamma \leqslant 9(1 - \sigma)\gamma \tag{3.5}$$

whenever $\gamma \geqslant 1$, and the finite rank contractions u and v are such that $\|ua^{(n)}v\|_{\mathcal{E}} \leqslant \gamma$ for every $a \in \mathcal{A}$.

Moreover, fix $i \in \{1, \ldots, n\}$, and restrict our attention to the contractions u and v s.t. $\mathrm{ran}\,u \subset H \otimes e_i$, and $(\ker v)^\perp \subset K \otimes e_i$ ($e_1, \ldots, e_n$ is the canonical basis for ℓ_2^n). Identify $H \otimes e_i$ and $K \otimes e_i$ with H and K, respectively. Then, by (3.5), for every $\varepsilon > 0$ there exist $a \in \mathcal{A}$ and $b \in B(H, K)$ s.t. $a_0 = a + \tilde{b}$, and $\rho(a) + \|\tilde{b}\|_{\mathcal{E}} < 9C$.

Let $b = \tilde{b}^{(n)} + (T - a_0^{(n)})$. Note that

$$\|b\|_{\mathcal{S}_{\mathcal{E}}} \leqslant \|\tilde{b}^{(n)}\|_{\mathcal{E}} + \|T - a_0^{(n)}\|_{\mathcal{E}} \leqslant n\|\tilde{b}^{(n)}\|_{\mathcal{E}} + 8.$$

Then $T = a \otimes I_{\ell_2^n} + b$, with

$$\rho_{\mathcal{A}^{(n)}}(a^{(n)}) + \|b\|_{\mathcal{E}} \leqslant \rho_{\mathcal{A}}(a) + n\|\tilde{b}\|_{\mathcal{E}} + 8 \leqslant n(\rho_{\mathcal{A}}(a) + \|\tilde{b}\|_{\mathcal{E}}) + 8 < 9Cn + 8.$$

Thus, $\mathcal{A}^{(n)}$ is $(9Cn+8)-\mathcal{S}_{\mathcal{E}}$-ASHR.

(3) Consider $T \in B(\ell_2^n(H), \ell_2^n(K))$ with $d_{\mathcal{A}^{(n)},p}(T) < 1$. By Proposition 3.15, there exist $a_1, \ldots, a_n \in \mathcal{A}$ s.t. $\|T_0 - T\|_p < 5C+4$, where $T_0 = \sum_{i=1}^n E_{ii} \otimes a_i$. Here, as before, (E_{ij}) denote the matrix units in $B(\ell_2^n)$. Let $n_1 = \lfloor n/2 \rfloor$, and $n_2 = n - n_1$. Using Lemma 4.6 of [34] as in the proof of Theorem 3.1(3) (and keeping the same notation), we get:

$$\int_{\mathcal{G}} ((I_{\ell_2^n} - P) \otimes I_H) T_0 (P \otimes I_H)\, dP = \frac{n_1 n_2}{n^2-1}(T_0 - I_{\ell_2^n} \otimes a),$$

where $a = (\sum_{i=1}^n a_i)/n \in \mathcal{A}$. But $((I_{\ell_2^n} - P) \otimes I_H)\mathcal{A}^{(n)}(P \otimes I_H) = 0$ for each $P \in \mathcal{G}$, hence

$$\|((I_{\ell_2^n} - P) \otimes I_H) T_0 (P \otimes I_H)\|_p \leqslant d_{\mathcal{A}^{(n)},p}(T) + \|T_0 - T\|_p < 5C+5,$$

hence $\|T_0 - a^{(n)}\|_p < 20C + 20$. ∎

Remark 3.17. If $\mathcal{A}$ is a weak* closed subspace of $B(H,K)$, and $\mathcal{E}$ is a nice sequence space, the proof of Proposition 3.16(1) shows that $\mathcal{A}^{(\infty)}$ is $4-\mathcal{S}_{\mathcal{E}}$-hyperreflexive.

Recall that one can turn a space of operators into an operator algebra, by embedding it into an off-diagonal corner. More precisely, consider $\mathcal{A} \hookrightarrow B(H_1, H_2)$ (H_1, H_2 are Hilbert spaces). Let $H = H_2 \oplus_2 H_1$, and define $\tilde{\mathcal{A}} \hookrightarrow B(H)$ as consisting of $\begin{pmatrix} \lambda I_{H_2} & A \\ 0 & \mu I_{H_1} \end{pmatrix}$, with $\lambda, \mu \in \mathbb{C}$, and $A \in \mathcal{A}$. Clearly, $\tilde{\mathcal{A}}$ is a unital operator algebra.

Proposition 3.18. *Suppose $\mathcal{E}$ is a nice sequence space, and $\mathcal{A}$, $\tilde{\mathcal{A}}$ are as above. Then:*

(1) If $\mathcal{A}$ is $C-\mathcal{S}_{\mathcal{E}}$-hyperreflexive, then $\tilde{\mathcal{A}}$ is $(9+C)-\mathcal{S}_{\mathcal{E}}$-hyperreflexive.

(2) If $\tilde{\mathcal{A}}$ is $C-\mathcal{S}_{\mathcal{E}}$-hyperreflexive, then $\mathcal{A}$ is $C-\mathcal{S}_{\mathcal{E}}$-hyperreflexive.

Proof. Denote by P_s the orthogonal projection from H onto H_s ($s = 1, 2$).

(1) Suppose $\mathcal{A}$ is $C-\mathcal{S}_{\mathcal{E}}$-hyperreflexive, and $T \in B(H)$ is such that $d_{\tilde{\mathcal{A}},\mathcal{E}}(T) < 1$. We shall show that $\mathrm{dist}_{\mathcal{E}}(T, \tilde{\mathcal{A}}) < 9 + C$. Clearly, $d_{P_r \tilde{\mathcal{A}} P_s, \mathcal{E}}(P_r T P_s) < 1$ whenever $r, s \in \{0, 1\}$. This allows us to consider each of the four blocks of $B(H)$ separately.

By definition of $\tilde{\mathcal{A}}$, $P_1 \tilde{\mathcal{A}} P_2 = 0$, hence $\|P_1 T P_2\|_{\mathcal{E}} \leqslant d_{\tilde{\mathcal{A}},\mathcal{E}}(T) < 1$. $P_2 \tilde{\mathcal{A}} P_1 = \mathcal{A}$ is $C-\mathcal{S}_{\mathcal{E}}$-hyperreflexive, hence there exists $A \in \mathcal{A}$ satisfying $\|P_2 T P_1 - A\|_{\mathcal{E}} < C$. By Theorem 3.1, $P_1 \tilde{\mathcal{A}} P_1 = \mathbb{C} I_{H_1}$ is $4-\mathcal{S}_{\mathcal{E}}$-hyperreflexive, hence there exists $\lambda_1 \in \mathbb{C}$ with $\|P_1 T P_1 - \lambda_1 P_1\|_{\mathcal{E}} < 4$. Similarly, there exists $\lambda_2 \in \mathbb{C}$ with $\|P_2 T P_2 - \lambda_2 P_2\|_{\mathcal{E}} < 4$. Letting $\tilde{A} = \begin{pmatrix} \lambda_2 I_{H_2} & A \\ 0 & \lambda_1 I_{H_1} \end{pmatrix}$, we conclude that $\|T - \tilde{A}\|_{\mathcal{E}} < 9 + C$.

(2) Suppose $\tilde{\mathcal{A}}$ is $C-\mathcal{S}_{\mathcal{E}}$-hyperreflexive. We have to prove that $\mathrm{dist}_{\mathcal{E}}(T, \mathcal{A}) < C$ whenever $T \in B(H_1, H_2)$ satisfies $d_{\mathcal{A},\mathcal{E}}(T) < 1$. To this end, consider $\tilde{T} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} \in B(H)$. By Proposition 56.4 of [7], $K \hookrightarrow H$ is an invariant subspace for

$\tilde{\mathcal{A}}$ iff $K = K_2 \oplus_2 K_1$, where $K_s \hookrightarrow H_s$ ($s = 1, 2$), and $\mathcal{A}K_1 \subset K_2$. Denoting the orthogonal projections onto K, K_1, and K_2 by Q, Q_1, and Q_2, respectively, we see that

$$d_{\tilde{\mathcal{A}},\mathcal{E}}(\tilde{T}) = \sup_Q \|(I-Q)\tilde{T}Q\|_{\mathcal{E}} = \sup_{Q_1,Q_2} \|(I-Q_2)\tilde{T}Q_1\|_{\mathcal{E}} < 1$$

(the suprema run over all the projections Q, Q_1, and Q_2, arising from invariant subspaces in a manner described above). Therefore, there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ and $A \in \mathcal{A}$ s.t. $\|\tilde{T} - \tilde{A}\|_{\mathcal{E}} < C$ for $\tilde{A} = \begin{pmatrix} \lambda_2 I_{H_2} & A \\ 0 & \lambda_1 I_{H_1} \end{pmatrix}$. Then $\|T - A\|_{\mathcal{E}} \leqslant \|P_2(\tilde{T} - \tilde{A})P_1\|_{\mathcal{E}} < C$, which is what we need. ■

To finish this section, we consider the James quasi-reflexive space, and its second dual. For $x = (x_i)_{i=1}^\infty \in \ell_\infty$ define

$$\|x\|_J = \sup_{i_1 < i_2 < \ldots < i_{2n}} \max\Big\{ \big(\sum_{j=1}^n |x_{i_{2j+1}} - x_{i_{2j}}|^2 \big)^{1/2}, \big(\sum_{j=1}^n |x_{i_{2j-1}} - x_{i_{2j}}|^2 \big)^{1/2} \Big\}$$

(we identify i_{2n+1} with i_1). The *James space* J is the completion of c_{00} (the space of all eventually null sequences) with respect to the norm $\|\cdot\|_J$ It is known (see e.g. Section 1.d of [26], or [16]) that $\dim J^{**}/J = 1$. Most remarkably, it turns out that J^{**} is isomorphic to J (in fact, for a certain equivalent norm on J, J and J^{**} are isometric).

By [2], J^{**} can be renormed to be a unital Banach algebra: for $x \in \ell_\infty$, set

$$\|x\|' = \sup\{\|xy\|_J \,|\, y \in c_{00}, \|y\|_J \leqslant 1\}$$

(here xy refers to the pointwise product of the two sequences). Denote by $\mathcal{B}$ the set of all $x \in \ell_\infty$ for which $\|x\|' < \infty$. It is shown in [2] that $\mathcal{B} = \operatorname{span}[J, \mathbf{1}]$, where $\mathbf{1} = (1, 1, \ldots) \in \ell_\infty$, and moreover, $\mathcal{B}$ is isomorphic to J^{**}.

We can view $\mathcal{B}$ as a subset of the diagonal of $B(\ell_2)$. More precisely, denote the canonical basis of ℓ_2 by $(e_n)_{n\in\mathbb{N}}$, and define $\pi : \mathcal{B} \to B(\ell_2)$ by setting $\pi(b)e_n = b_n e_n$, for each n. Let $\mathcal{A} = \pi(\mathrm{Ba}\mathcal{B})$. In this notation, we have:

Proposition 3.19. *The set $\mathcal{A}$ is $15 - \mathcal{S}_2$-Azoff-Shehada hyperreflexive.*

Remark 3.20. We do not know whether $\mathcal{A}$ is $\mathcal{S}_{\mathcal{E}}$-ASHR for other ideals $\mathcal{S}_{\mathcal{E}}$.

Proof. Recall some estimates for the norms of elements of $\mathcal{B}$. Note first that, for any $x \in \mathcal{B}$, $\|x\|' \geqslant \|x\|_\infty$. Indeed, fix $n \in \mathbb{N}$, and consider $y = (0, \ldots, 0, 1, 0, 0, \ldots) \in c_{00}$ (1 in the n-th position). Then $\|y\|_J = 1$, and $\|xy\|_J = |x_n|$.

Less trivially, by [2], $\|x\|' \geqslant \|x\|_J/2$ for any $x \in \ell_\infty$, and $\|x\|' \leqslant 2\|x\|_J$ for any $x \in \mathcal{J}$.

Now suppose $T \in B(\ell_2)$ satisfies $d_{\mathcal{A},2}(T) < 1$. Let $T_0 = \sum_n \langle Te_n, e_n\rangle e_n$, and $T_1 = T - T_0$. For $S \subset \mathbb{N}$, we have $P_S a P_{\mathbb{N}\backslash S} = 0$ for each $a \in \mathcal{A}$, hence $\|P_S T P_{\mathbb{N}\backslash S}\|_2 \leqslant$

1. But $P_S T_1 P_{\mathbb{N}\setminus S} = P_S T P_{\mathbb{N}\setminus S}$, hence $\|T_1\|_2 \leqslant 4$. If u and v are contractions, and $\|uav\|_2 \leqslant \gamma$ ($\gamma \geqslant 1$) for any $a \in \mathcal{A}$, then

$$\|uT_0v\|_2 \leqslant \|uTv\|_2 + \|T_1\|_2 \leqslant 5\gamma. \tag{3.6}$$

For convenience, let $t_n = \langle Te_n, e_n\rangle$. Abusing the notation somewhat, we identify T_0 with $(t_n)_{n\in\mathbb{N}} \in \ell_\infty$, and write things like $\|T_0\|'$.

Now consider $i_1 < i_2 < \ldots < i_{2n}$. For $1 \leqslant j \leqslant n$, let $\xi_j = (e_{i_{2j-1}} - e_{i_{2j}})/\sqrt{2}$, and $\eta_j = (e_{i_{2j-1}} + e_{i_{2j}})/\sqrt{2}$. Let P and Q be orthogonal projections onto span$[\xi_j \mid 1 \leqslant j \leqslant n]$ and span$[\eta_j \mid 1 \leqslant j \leqslant n]$, respectively. If $S \in B(\ell_2)$ is a diagonal operator (that is, $Se_n = s_n e_n$ for each n), then $QSP\xi_j = 2^{-1}(s_{2j-1} - s_{2j})e_j$, for $1 \leqslant j \leqslant n$. Thus,

$$\|QSP\|_2^2 = \sum_{j=1}^{n} |s_{2j-1} - s_{2j}|^2/4.$$

In particular, $\|Q\pi(b)P\|_2 \leqslant \|b\|_J/2$ for any $b \in \mathcal{B}$. Therefore, $\|QaP\|_2 \leqslant 1$ for any $a \in \mathcal{A}$ (as noted above, $\|b\|_J \leqslant 2\|b\|'$). By (3.6), $\big(\sum_{j=1}^n |t_{i_{2j-1}} - t_{i_{2j}}|^2\big)^{1/2} \leqslant 5$.

Similarly, we show that $(\sum_{j=1}^n |t_{i_{2j+1}} - t_{i_{2j}}|^2)^{1/2} \leqslant 5$.

Moreover, for $n \in \mathbb{N}$, consider the projection R_n onto span$[e_n]$. As noted in the beginning of the proof, $\|R_naR_n\|_2 = \|R_naR_n\| \leqslant 1$ for each $a \in \mathcal{A}$. Therefore, $|t_n| = \|R_nTR_n\|_2 < 1$. If t is a cluster point of the sequence (t_n), then $|t| \leqslant 1$. Moreover,

$$\|T_0 - t\mathbf{1}\|_J = \sup_{i_1<i_2<\ldots<i_{2n}} \max\Big\{\Big(\sum_{j=1}^{n} |t_{i_{2j+1}} - t_{i_{2j}}|^2\Big)^{1/2}, \Big(\sum_{j=1}^{n} |t_{i_{2j-1}} - t_{i_{2j}}|^2\Big)^{1/2}\Big\} \leqslant 5.$$

Therefore, $\|T_0 - t\mathbf{1}\|' \leqslant 2\|T_0 - t\mathbf{1}\|_J \leqslant 10$, and $\|T_0\|' \leqslant \|T_0 - t\mathbf{1}\|_J + |t| \leqslant 11$. Then we conclude that $\rho_{\mathcal{A}}(T_0) + \|T_1\|_2 \leqslant 15$. ∎

4. Applications to operator spaces

In this section, we apply $\mathcal{S}_{\mathcal{E}}$-AS-hyperreflexivity to constructing Hilbertian operator spaces with prescribed families of c.b. maps (see e.g. [31, 32, 33, 34] for other work in this direction). The reader is referred to e.g. [15, 36, 41] for general information on operator spaces.

Theorem 4.1. *Suppose H is a separable operator space, $\mathcal{E}$ is a nice sequence space such that the formal identity $\ell_2 \to \mathcal{E}$ is contractive, and $\mathcal{A}$ is a subset of the unit ball of $B(H)$ such that: (i) $\mathcal{A}$ is $C - \mathcal{S}_{\mathcal{E}}$-ASHR, (ii) $\mathcal{A}$ contains the identity I_H, and (iii) if $a, b \in \mathcal{A}$, then $ab \in \mathcal{A}$. Moreover, if $\lambda, \mu \in \mathbb{C}$ satisfy $|\lambda| + |\mu| \leqslant 1$, then $\lambda a + \mu b \in \mathcal{A}$. Then there exists an operator space X, isometric to H, such that:*

1. *If $b \in B(H)$ belongs to $\mathcal{S}_{\mathcal{E}}$, then $\|b\|_{cb} \leqslant \|b\|_{\mathcal{E}}$. If $a \in \mathcal{A}$, then $\|a\|_{cb} \leqslant \rho(a)$.*
2. *For every $\varepsilon > 0$, every $T \in CB(X)$ can be written as $T = a + b$, with $a \in \mathbb{C}\mathcal{A}$, $b \in \mathcal{S}_{\mathcal{E}}$, and $\rho(a) + \|b\|_{\mathcal{E}} < 4C\|T\|_{cb} + \varepsilon$.*

This theorem, together with the results of the previous sections, provides a way of producing operator spaces with prescribed families of completely bounded maps (and, consequently, with interesting properties). In [31], we used ampliations (as in Proposition 3.16(1)), while in [32], we worked with the Banach algebra arising from the James space (as in Proposition 3.19). Here we provide one more example.

Corollary 4.2. *Suppose $(H_i)_{i=1}^n$ are separable Hilbert spaces, $\mathcal{E}$ is a sequence space as in the statement of Theorem 4.1, and the $C - \mathcal{S}_\mathcal{E}$-hyperreflexive subspaces $\mathcal{A}_{ij}$ of $B(H_i, H_j)$ $(1 \leqslant i, j \leqslant n)$ are such that (i) $\mathcal{A}_{ii}$ is a unital operator algebras for any i, and (ii) $\mathcal{A}_{kj}\mathcal{A}_{ij} \subset \mathcal{A}_{ik}$ whenever $1 \leqslant i, j, k \leqslant n$. Then there exists an operator space X, isometric to $(\sum_i H_i)_2$, such that $T \in B(X)$ is completely bounded if and only if there exists $a \in B(H)$ such that $T - a \in \mathcal{S}_\mathcal{E}$, and $P_j a P_i \in \mathcal{A}_{ij}$ for any pair (i, j) (P_i denotes the orthogonal projection from H onto H_i). In particular, H_i's are completely complemented subspaces of X, and $T \in B(H_i, H_j)$ is completely bounded if and only if it is a $\mathcal{S}_\mathcal{E}$-perturbation of an element of $\mathcal{A}_{ij}$.*

Proof. Let $H = (\sum_i H_i)_2$, and denote by $\mathcal{A}$ the set of all $T \in B(H)$ s.t. $P_j T P_i \in \mathcal{A}_{ij}$ whenever $(1 \leqslant i, j \leqslant n$. By Proposition 3.13, $\mathcal{A}$ is $\mathcal{S}_\mathcal{E}$-hyperreflexive. Therefore, by Propositions 1.6 and 1.5, the unit ball of $\mathcal{A}$ (call it $\mathcal{A}^1$) is $\mathcal{S}_\mathcal{E}$-hyperreflexive. An application of Theorem 4.1 completes the proof. ∎

In particular, suppose $n = 2$, $\mathcal{A}_{ii} = \mathbb{C}I_{H_i}$ $(i = 1, 2)$, $\mathcal{A}_{12}\mathcal{A}_{21} \subset \mathbb{C}I_{H_1}$, and $\mathcal{A}_{21}\mathcal{A}_{12} \subset \mathbb{C}I_{H_2}$ (this can be achieved if, say, $\mathcal{A}_{21} = \{0\}$). Then there exist operator spaces H_1 and H_2 s.t. $T \in B(H_i, H_j)$ is completely bounded iff it is a $\mathcal{S}_\mathcal{E}$-perturbation of an element of $\mathcal{A}_{ij}$. This complements the results of [29].

The proof of Theorem 4.1 uses some ideas of [31]. To define the space X, pick a sequence $(n_i)_{i=1}^\infty \subset \mathbb{N}$, in which every positive integer occurs infinitely many times. By [34], there exists a family $(E_i)_{i=1}^\infty$ of finite dimensional operator spaces such that: (i) E_i is isometric to $\ell_2^{n_i}$, and (ii) for any operator $u : E_i^* \to E_j$, we have $\|u\|_1/(4+2^{-i}) \leqslant \|u\|_{cb} \leqslant \|u\|_1$ if $i = j$, $\|u\|_{cb} = \|u\|_2$ if $i \neq j$. Denote by $\mathcal{E}'$ the dual space of $\mathcal{E}$. Find a sequence of operators $u_i : H \to \ell_2^{n_i}$ such that $\|u_i\|_{\mathcal{E}'} = 1$ and, for any $\varepsilon > 0$, $n \in \mathbb{N}$, and $u : H \to \ell_2^n$, there exists $i \in \mathbb{N}$ for which $n_i = n$ and $\|u_i - u\|_1 < \varepsilon$. On the Banach space level, we identify the range of u_i with E_i described above. Denote by $\mathcal{K}_0$ the space of compact operators on ℓ_2 with finitely many non-zero entries. We define the operator space X as follows: for $x \in H \otimes \mathcal{K}_0$, let

$$\|x\|_{X \otimes \mathcal{K}_0} = \sup\left\{\|(u_i a \otimes I_{\mathcal{K}_0})x\|_{E_i \otimes \mathcal{K}_0} \,\middle|\, i \in \mathbb{N},\, a \in \mathcal{A}\right\}. \tag{4.1}$$

X is an operator space, since Ruan's axioms are satisfied. It is easy to see that X is isometric to H (as a Banach space). Moreover, all operators on X, belonging to the class $\mathcal{S}_\mathcal{E}$, are completely bounded:

Lemma 4.3. *If* Y *is an operator space isometric to* ℓ_2*, and* $T : Y \to X$ *belongs to* $\mathcal{S}_{\mathcal{E}}$*, then* $\|T\|_{cb} \leqslant \|T\|_{\mathcal{E}}$*.*

Proof. By the duality between $\mathcal{S}_{\mathcal{E}}$ and $\mathcal{S}_{\mathcal{E}'}$ (see Section III.12 of [17], or Section 1.8 of [45]), $\|u_i aT\|_1 \leqslant \|u_i\|_{\mathcal{E}'}\|a\|\|T\|_{\mathcal{E}} = \|T\|_{\mathcal{E}}$. Thus, by (4.1),

$$\|T\|_{cb} = \sup\{\|u_i aT\|_{cb} \,|\, i \in \mathbb{N}, a \in \mathcal{A}\} \leqslant \sup\{\|u_i aT\|_1 \,|\, i \in \mathbb{N}, a \in \mathcal{A}\} \leqslant \|T\|_{\mathcal{E}}.$$

■

To estimate the c.b. norms of operators from below, we need:

Lemma 4.4. *Suppose* Y *is a subspace of* X*. Consider the operators* $T : Y \to X$*,* $u : X \to \ell_2^n$*, and* $v : \ell_2^n \to Y$*, such that* $\|u\|_{\mathcal{E}'} = \|v\| = 1$*. Let* $C = \sup\{\|uav\|_1 \,|\, a \in \mathcal{A}\}$*. Then* $\|T\|_{cb} \geqslant \|uTv\|_1/(4\max\{C, 1\})$*.*

Proof. Fix $\varepsilon > 0$, and find $i \in \mathbb{N}$ s.t. $n = n_i$, and $\|u - u_i\|_1 < \varepsilon$, and $4^{-i} < \varepsilon$ (we identify ℓ_2^n with E_i). We view u and v as maps from X to E_i and from E_i^* to Y, respectively. By (4.1),

$$\|v\|_{cb} = \sup\{\|u_j av\|_{cb} \,|\, j \in \mathbb{N}, a \in \mathcal{A}\}$$

If $i = j$, then, for any $a \in \mathcal{A}$,

$$\|u_i av\|_{cb} \leqslant \|u_i av\|_1 \leqslant \|uav\|_1 + \|u - u_i\|_1 = C + \varepsilon.$$

If $j \neq i$,

$$\|u_j av\|_{cb} \leqslant \|u_j av\|_2 \leqslant \|u_j\|_2\|a\|\|v\| \leqslant \|u_j\|_{\mathcal{E}'}\|a\|\|v\| = 1.$$

Therefore, $\|v\|_{cb} \leqslant \max\{C + \varepsilon, 1\}$.

By (4.1), $\|u_i\|_{cb} = 1$, hence $\|u\|_{cb} \leqslant \|u_i\|_{cb} + \|u - u_i\|_1 < 1 + \varepsilon$. Therefore,

$$\|T\|_{cb} \geqslant \frac{\|uTv\|_{cb}}{\|u\|_{cb}\|v\|_{cb}} \geqslant \frac{\|uTv\|_1}{(1+\varepsilon)(4+\varepsilon)\max\{C+\varepsilon, 1\}}.$$

However, ε can be chosen to be arbitrarily small. ■

Proof of Theorem 4.1. By (4.1) and the remark following it, a is completely contractive whenever $a \in \mathcal{A}$. By Lemma 4.3, $\|b\|_{cb} \leqslant \|b\|_{\mathcal{E}}$ for any $b \in \mathcal{S}_{\mathcal{E}}$. This proves part (1) of the theorem.

To prove part (2), pick $T \in CB(X)$ with $\|T\|_{cb} \leqslant 1$, and show that, for every $\varepsilon > 0$, there exist $a \in \mathbb{C}\mathcal{A}$ and $b \in \mathcal{S}_{\mathcal{E}}$ satisfying $\rho(a) + \|b\|_{\mathcal{E}} < 4C + \varepsilon$, and $T = a + b$. Indeed, otherwise there exist $\gamma \geqslant 1$, and contractions $u_0, v_0 \in B(H)$, s.t. $\|u_0 av_0\|_{\mathcal{E}} \leqslant \gamma$ for any $a \in \mathcal{A}$, yet $\|u_0 Tv_0\|_{\mathcal{E}} > 4\gamma$. By the duality between $\mathcal{E}$ and $\mathcal{E}'$, there exist $n \in \mathbb{N}$, $v_1 \in B(\ell_2^n, X)$, and $u_1 \in B(X, \ell_2^n)$, s.t. $\|v_1\| = 1 = \|u_1\|_{\mathcal{E}'}$, and $\|u_1 u_0 Tv_0 v_1\|_1 > 4\gamma$. Let $u = u_1 u_0$, and $v = v_0 v_1$. Then $\|u\|_{\mathcal{E}'} \leqslant 1$, $\|v\| \leqslant 1$, and therefore, for any $a \in \mathcal{A}$,

$$\|uav\|_1 = \|u_1 u_0 av_0 v_1\|_1 \leqslant \|u_1\|_{\mathcal{E}'}\|u_0 av_0\|_{\mathcal{E}}\|v_1\| \leqslant \gamma.$$

By Lemma 4.4, $\|T\|_{cb} > 1$, a contradiction. ■

Acknowledgments. We are grateful to D. Larson and E. Ricard for valuable conversations. We thank the organizers of of the workshop in Linear Analysis and Probability in College Station, TX, where part of this work was carried out.

References

[1] A. Abramovich and C. Aliprantis. *An invitation to operator theory*, American Mathematical Society, Providence, RI, 2002.
[2] A. Andrew and W. Green. On James' quasi-reflexive Banach space as a Banach algebra, *Canadian J. Math.*, 32:1080–1101, 1980.
[3] E. Asplund and V. Ptak. A minimax inequality for operators and a related numerical range, *Acta Math.*, 126:53–62, 1971.
[4] W. Arveson. Interpolation problems in nest algebras, *J. Funct. Anal.*, 20:208–233, 1975.
[5] E. Azoff and H. Shehada. On separation by families of linear functionals, *J. Funct. Anal.*, 96:96–116, 1991.
[6] R. Bhatia. *Matrix analysis*, Springer Verlag, New York, 1997.
[7] J. B. Conway. *A course in operator theory*, Amer. Math. Soc., Providence RI, 2000.
[8] E. Christensen. Perturbations of operator algebras II, *Indiana Univ. Math. J.*, 26:891–904, 1977.
[9] E. Christensen. On some problems studied by R. V. Kadison, *Acta Math. Sin.*, 19:523–534, 2003.
[10] K. Davidson. The distance to the analytic Toeplitz operators, *Illinois J. Math.*, 31:265–273, 1987.
[11] K. Davidson. *Nest algebras. Triangular forms for operator algebras on Hilbert space*, Longman Scientific & Technical, Harlow, 1988.
[12] K. Davidson and S. Szarek. Local operator theory, random matrices and Banach spaces, in *Handbook of the geometry of Banach spaces*, Vol. 1, 317–366, North-Holland, Amsterdam, 2001.
[13] A. Defant and K. Floret. *Tensor norms and operator ideals*, North-Holland, Amsterdam, 1993.
[14] J. Diestel, H. Jarchow, and A. Tonge. *Absolutely summing operators*, Cambridge University Press, Cambridge, 1995.
[15] E. Effros and Z.-J. Ruan. *Operator spaces*, Oxford University Press, New York, 2000.
[16] H. Fetter and B. Gamboa de Buen. *The James forest*, Cambridge University Press, New York, 1997.
[17] I. C. Gohberg and M. G. Kreĭn. *Introduction to the theory of linear nonselfadjoint operators.* American Mathematical Society, Providence, RI, 1969.
[18] P. Halmos. Shifts on Hilbert spaces, *J. Reine Angew. Math.*, 208:102–112, 1961.
[19] T. Hoover. Derivations, homomorphisms, and operator ideals, *Proc. Amer. Math. Soc.*, 63:293–298, 1977.
[20] W. Johnson. Banach spaces all of whose subspaces have the approximation property, in *Seminar on Functional Analysis, 1979–1980*, Exp. No. 16, 15–26, École Polytech., Palaiseau, 1980.
[21] W. Johnson and G. Schechtman. Finite dimensional subspaces of L_p, in *Handbook of the geometry of Banach spaces*, Vol. 1, 837–870, North-Holland, Amsterdam, 2001.
[22] D. Larson. On similarity of nests in Hilbert space and in Banach spaces, in *Functional Analysis (proceedings of the Seminar at the University of Texas at Austin, 1986-1987*, Lecture Notes in Mathematics 1332, 179–194, Springer, Berlin, 1988.
[23] D. Lewis. Finite dimensional subspaces of L_p, *Studia Math.*, 63:207–212, 1978.
[24] J. Li and Z. Pan. Reflexivity and hyperreflexivity of operator spaces, *J. Math. Anal. Appl.*, 279:210–215, 2003.
[25] J. Lindenstrauss and H. Rosenthal. The $\mathcal{L}_p$ spaces, *Israel J. Math.*, 7:325–349, 1969.
[26] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces I*, Springer-Verlag, Berlin, 1977.
[27] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces II*, Springer-Verlag, Berlin, 1979.

[28] B. Magajna. On the distance to finite-dimensional subspaces in operator algebras, *J. London Math. Soc.*, 47:516–532, 1993.
[29] B. Mathes and V. Paulsen. Operator ideals and operator spaces, *Proc. Amer. Math. Soc.*, 123:1763–1772, 1995.
[30] V. Müller and M. Ptak. Hyperreflexivity of finite-dimensional subspaces, *J. Funct. Anal.*, 218:395–408, 2005.
[31] T. Oikhberg. Operator spaces with prescribed sets of completely bounded maps, *J. Funct. Anal.*, 224:296–315, 2005.
[32] T. Oikhberg. Operator spaces with complete bases, lacking completely unconditional bases, *Houston J. Math.*, 32:551–561, 2006.
[33] T. Oikhberg. The operator shift space, *Proc. Edinburgh Math. Soc.*, to appear.
[34] T. Oikhberg and E. Ricard. Operator spaces with few completely bounded maps, *Math. Ann.*, 328:229–259, 2004.
[35] J. Partington. *Linear operators and linear systems. An analytical approach to control theory*, Cambridge University Press, 2004.
[36] V. Paulsen. *Completely bounded maps and operator algebras*, Cambridge University Press, 2002.
[37] A. Pietsch. *Operator ideals*, North-Holland, Amsterdam, 1980.
[38] G. Pisier. Holomorphic semigroups and the geometry of Banach spaces, *Ann. of Math.*, 115:375–392, 1982.
[39] G. Pisier. Counterexamples to a conjecture of Grothendieck, *Acta Math.*, 151:181–208, 1983.
[40] G. Pisier. *Factorization of linear operators and geometry of Banach spaces*, American Mathematical Society, Providence, RI, 1997.
[41] G. Pisier. *An introduction to the theory of operator spaces*, Cambridge University Press, 2003.
[42] G. Pisier and Q. Xu. Non-commutative L^p-spaces, in *Handbook of the geometry of Banach spaces*, Vol. 2, 1459–1517, North-Holland, Amsterdam, 2003.
[43] S. Rosenoer. Distance estimates for von Neumann algebras, *Proc. Amer. Math. Soc.*, 86:248–252, 1982.
[44] V. Shulman. On vector functionals and minimax inequalities, in *Spectral and evolution problems, Vol. 11 (Sevastopol, 2000)*, 73–77, Natl. Taurida Univ. Press, Simferopol', 2001.
[45] B. Simon. *Trace ideals and their applications. Second edition*, American Mathematical Society, Providence, RI, 2005.
[46] S. Szarek. On the existence and uniqueness of complex structure and spaces with "few" operators, *Trans. Amer. Math. Soc.*, 293:339–353, 1986.
[47] N. Tomczak-Jaegermann. *Banach-Mazur distances and finite-dimensional operator ideals*, Longman Scientific and Technical, Harlow, 1989.

Department of Mathematics, University of California - Irvine, Irvine CA 92697
E-mail address: toikhber@math.uci.edu