Smooth 4-manifolds: BIG and small

Reverse Engineering Smooth 4-Manifolds

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September 19, 2008

Joint work with Ron Fintushel

Classify ($\pi_1 = 0$) smooth 4-manifolds ????

A possible scheme

A collection of well-understood 4-manifolds

$\Downarrow$

A collection of surgery operations

$\Downarrow$

All $\pi_1 = 0$ smooth 4-manifolds

$\Uparrow$

Invariants to distinguish up to finite discrepancy

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Seiberg-Witten Invariants

Wild Conjecture

Every topological 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for $n > 4$, every n-manifold has only finitely many distinct smooth n-manifolds which are homeomorphic to it.

Main goal: Discuss recent techniques (reverse engineering) developed to study this conjecture

Basic facts about 4-manifolds

Invariants

- Euler characteristic: $e(X) = \sum_{i=0}^{4}(-1)^j rk(H^j(M;\mathbb{Z}))$
- Intersection form: $H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) \to \mathbb{Z}$;

  $$\alpha \cdot \beta = (\alpha \cup \beta)[X]$$

  is an integral, symmetric, unimodular, bilinear form.

  Signature of X = sign(X) = Signature of intersection form $= b^+ - b^-$

  Type: Even if $\alpha \cdot \alpha$ even for all α; otherwise Odd
- (Freedman, 1980) The intersection form classifies simply connected topological 4-manifolds: There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has $X \times S^1$ smoothable.
- (Donaldson, 1982) Two simply connected *smooth* 4-manifolds are homeomorphic iff they have the same e, sign, and type.

What do we know about smooth 4-manifolds?

Much–but so very little

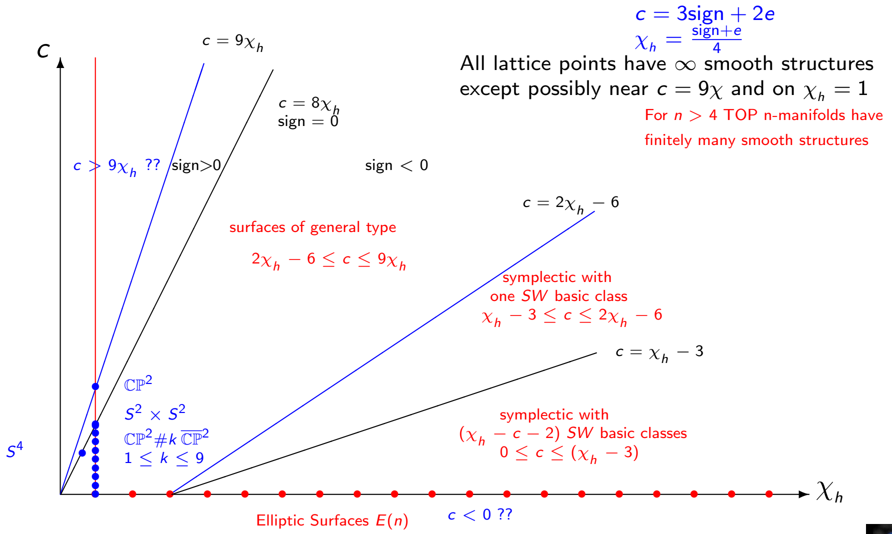
Wild Conjecture

Every 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

- Need more invariants: Donaldson, Seiberg-Witten Invariants
  $$\mathcal{SW} : \{\text{characteristic elements of } H_2(X;\mathbb{Z})\} \to \mathbb{Z}$$
  - $\mathcal{SW}(\beta) \neq 0$ for only finitely many β: called *basic* classes.
  - For each surface $\Sigma \subset X$ with $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma \geq 0$
    $$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|$$
    for every basic class β. (adjunction inequality[Kronheimer-Mrowka])
    Basic classes = smooth analogue of the canonical class of a complex surface
  - $\mathcal{SW}(\kappa) = \pm 1$, κ the first Chern class of a symplectic manifold [Taubes].

Oriented minimal ($\pi_1 = 0$) 4-manifolds with $\mathcal{SW} \neq 0$

Geography



Every 4-manifold has zero or infinitely many distinct smooth structures

- One way to try to prove this conjecture is to find a "dial" to change the smooth structure at will.
- We will show that an effective dial is Surgery on null-homologous tori

T: any self-intersection 0 torus $\subset X$, Tubular nbd $N_T \cong T^2 \times D^2$.

Surgery on T: $X \smallsetminus N_T \cup_\varphi T^2 \times D^2$, $\varphi : \partial(T^2 \times D^2) \to \partial(X \smallsetminus N_T)$

$$\varphi(\mathrm{pt} \times \partial D^2) = \text{surgery curve}$$

Result determined by $\varphi_*[\mathrm{pt} \times \partial D^2] \in H_1(\partial(X \smallsetminus N_T)) = \mathbb{Z}^3$

Choose basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial N_T)$ where $\{\alpha, \beta\}$ are pushoffs of a basis for $H_1(T)$.

$$\varphi_*[\mathrm{pt} \times \partial D^2] = p\alpha + q\beta + r[\partial D^2]$$

Write $X \smallsetminus N_T \cup_\varphi T^2 \times D^2 = X_T(p, q, r)$

This operation does not change $e(X)$ or $\sigma(X)$

Note: $X_T(0, 0, 1) = X$

Need formula for the Seiberg-Witten invariant of $X_T(p, q, r)$ to determine when the smooth structure changes:

Due to Morgan, Mrowka, and Szabó (1996).

The Morgan, Mrowka, Szabó Formula

$$\sum_i \mathrm{SW}_{X_{T(p,q,r)}}(k + 2i[T_{(p,q,r)}]) = p \sum_i \mathrm{SW}_{X_{T(1,0,0)}}(k' + 2i[T_{(1,0,0)}])$$
$$+q \sum_i \mathrm{SW}_{X_{T(0,1,0)}}(k'' + 2i[T_{(0,1,0)}]) + r \sum_i \mathrm{SW}_{X}(k''' + 2i[T])$$

k characteristic element of $H_2(X_{T(p,q,r)})$

$$\begin{array}{ccccccc} H_2(X_T(p,q,r)) & \to & H_2(X_T(p,q,r), N_{T_{(p,q,r)}}) & & k & \to & \bar{k} \\ & & \downarrow\cong & & & & \downarrow \\ & & H_2(X \setminus N_T, \partial) & & & & \hat{k} = \hat{k}' \\ & & \uparrow\cong & & & & \uparrow \\ H_2(X_T(1,0,0)) & \to & H_2(X_T(1,0,0), N_{T_{(1,0,0)}}) & & k' & \to & \bar{k}' \end{array}$$

- All basic classes of $X_T(p,q,r)$ arise in this way.
- Useful to determine situations when sums collapse to single summand.

Surgery on Tori

Reducing to one summand

$$\mathrm{SW}_{X_{T(p,q,r)}} = p\mathrm{SW}_{X_{T(1,0,0)}} + q\mathrm{SW}_{X_{T(0,1,0)}} + r\mathrm{SW}_X$$

- When torus T is nullhomologous, and
- when a core torus is essential, there is a torus that intersects it algebraically nontrivially.

Some observations about null-homologous tori:

- With null-homologous framing: $H_1(X_{T(p,q,1)}) = H_1(X)$,
  So for an effective dial want, say, $\mathrm{SW}_{X_{T(1,0,0)}} \neq 0$;
- $b_1(X_{T(1,0,0)}) = b_1(X_{T(0,1,0)}) = b_1(X) + 1$.

Dual situations for surgery on tori T

a. T primitive, $\alpha \subset T$ essential in $X \setminus T$.
   $\Rightarrow \quad T_{(1,0,r)}$ nullhomologous in $X_T(1, 0, r)$.

b. T nullhomologous, α bounds in $X \setminus N_T$
   $\Rightarrow \quad (1, 0, 0)$ surgery on T gives (a).

Old Application: Knot Surgery

K: Knot in S^3, T: square 0 essential torus in X

- $$X_K = X \smallsetminus N_T \cup S^1 \times (S^3 \smallsetminus N_K)$$

Note: $S^1 \times (S^3 \smallsetminus N_K)$ has the homology of $T^2 \times D^2$.

Facts about knot surgery

- If X and $X \smallsetminus T$ both simply connected; so is X_K
  (So X_K homeo to X)
- If K is fibered and X and T both symplectic; so is X_K.
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Conclusions

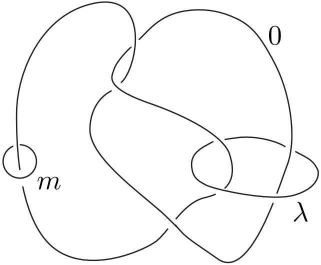
- If X, $X \smallsetminus T$, simply connected and $\mathcal{SW}_X \neq 0$, then there is an infinite family of distinct manifolds all homeomorphic to X.
- If in addition X, T symplectic, K fibered, then there is an infinite family of distinct symplectic manifolds all homeomorphic to X.

e.g. $X = K3$, $\mathcal{SW}_X = 1$, $\mathcal{SW}_{X_K} = \Delta_K(t^2)$

Knot surgery and nullhomologous tori

Knot surgery on torus T in 4-manifold X with knot K:

$X_K = X \#_{T = S^1 \times m} \; S^1 \times$



$\Lambda = S^1 \times \lambda =$ nullhomologous torus — Used to change crossings:

Now apply Morgan-Mrowka-Szabó formula + tricks

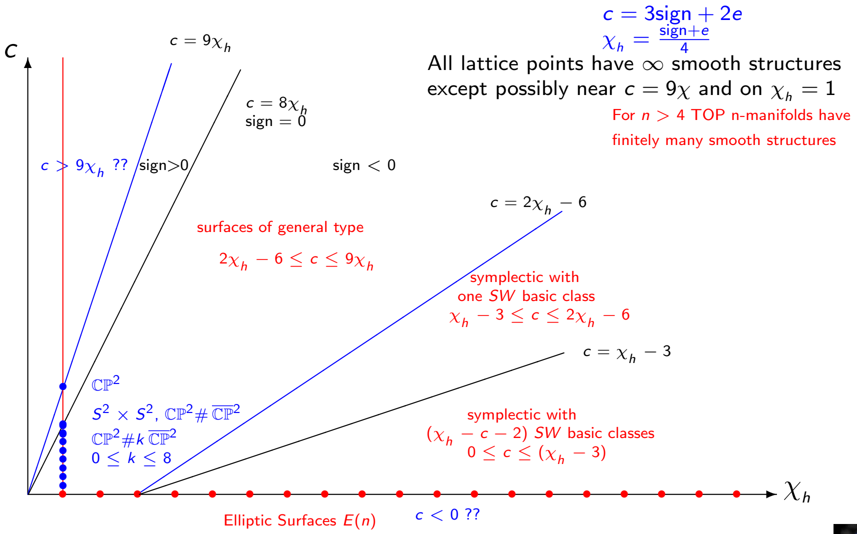
- Weakness of construction: Requires T to be homologically essential

  Open conjecture: If $\chi(X) > 1$, $\mathcal{SW}_X \neq 0$, then X contains a homologically essential torus T with trivial normal bundle.

- If X homeomorphic to $\mathbb{CP}^2$ blown up at 8 or fewer points, then X contains no such torus - so what can we do for these small manifolds?

Oriented minimal $\pi_1 = 0$ 4-manifolds with $\mathcal{SW} \neq 0$

Geography



Reverse Engineering

- Difficult to find useful nullhomologous tori like Λ used in knot surgery.
  - Recall: $SW_{X_{T(p,q,r)}} = pSW_{X_{T(1,0,0)}} + qSW_{X_{T(0,1,0)}} + rSW_X$
  - With null-homologous framing: $H_1(X_{T(p,q,1)}) = H_1(X)$. So want, say, $SW_{X_{T(1,0,0)}} \neq 0$;
  - $b_1(X_{T(1,0,0)}) = b_1(X_{T(0,1,0)}) = b_1(X) + 1$.
    - Recall: Dual situations for surgery on tori T

a. T primitive, $\alpha \subset T$ essential in $X \smallsetminus T$.
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IDEA: First construct $X_{T(1,0,0)}$ so that $SW_{X_{T(1,0,0)}} \neq 0$ and then surger to reduce b_1.

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- Procedure to insure the existence of effective null-homologous tori

1. Find model manifold M with same Euler number and signature as desired manifold, but with $b_1 \neq 0$ and with $\mathcal{SW} \neq 0$.
2. Find b_1 disjoint essential tori in M containing generators of H_1. Surger to get manifold X with $H_1 = 0$. Want result of each surgery to have $\mathcal{SW} \neq 0$ (except perhaps the very last).
3. X will contain a "useful" nullhomologous torus.

Luttinger Surgery

- For model manifolds with $H_1 \neq 0$: nature hands you symplectic manifolds.
- We seek tori that will kill b_1. Nature hands you Lagrangian tori.

X: symplectic manifold T: Lagrangian torus in X

Preferred framing for T: Lagrangian framing
w.r.t. which all pushoffs of T remain Lagrangian

$(1/n)$-surgeries w.r.t. this framing are again symplectic
(Luttinger; Auroux, Donaldson, Katzarkov)

If $S^1_\beta =$ Lagrangian pushoff, $X_T(0, \pm 1, 0)$: symplectic mfd

$\Longrightarrow X_T(0, \pm 1, 0)$ has $\mathcal{SW} \neq 0$

Recall: $\mathrm{SW}_{X_{T(p,q,r)}} = p\mathrm{SW}_{X_{T(1,0,0)}} + q\mathrm{SW}_{X_{T(0,1,0)}} + r\mathrm{SW}_X$

Then have infinitely many $H_1 = 0$ manifolds - keep fingers crossed $\pi_1 = 0$.

Reverse Engineering in Action

Infinite families of fake $\mathbb{CP}^2 \# k\, \overline{\mathbb{CP}}^2$

Need Model Manifolds for $\mathbb{CP}^2 \# k\, \overline{\mathbb{CP}}^2$

i.e. symplectic manifolds X_k with same e and sign as $\mathbb{CP}^2 \# k\, \overline{\mathbb{CP}}^2$, and $b_1 \geq 1$ disjoint lagrangian tori carrying basis for H_1.

- Surger lagraginan tori to decrease b_1.
- Resulting manifold has $H_1 = 0$ - but with a dial.

- Get infinite family of distinct manifolds all homology equivalent to $\mathbb{CP}^2 \# k\, \overline{\mathbb{CP}}^2$
- Keep fingers crossed that result has $\pi_1 = 0$, so all homeomorphic to $\mathbb{CP}^2 \# k\, \overline{\mathbb{CP}}^2$

Model Manifolds for $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$

Basic Pieces: X_0, X_1, X_2, X_3, X_4

$X_r \#_{\Sigma_2} X_s$ is a model for $\mathbb{CP}^2 \# (r+s+1)\,\overline{\mathbb{CP}}^2$

X_0: $\Sigma_2 \subset T^2 \times \Sigma_2$ representing $(0,1)$

X_1: $\Sigma_2 \subset T^2 \times T^2 \# \overline{\mathbb{CP}}^2$ representing $(2,1) - 2e$

X_2: $\Sigma_2 \subset T^2 \times T^2 \# 2\overline{\mathbb{CP}}^2$ representing $(1,1) - e_1 - e_2$

X_3: $\Sigma_2 \subset S^2 \times T^2 \# 3\overline{\mathbb{CP}}^2$ representing $(1,3) - 2e_1 - e_2 - e_3$

X_4: $\Sigma_2 \subset S^2 \times T^2 \# 4\overline{\mathbb{CP}}^2$ representing $(1,2) - e_1 - e_2 - e_3 - e_4$

Exception: $X_0 \#_{\Sigma_2} X_0 = \Sigma_2 \times \Sigma_2$ is a model for $S^2 \times S^2$

Enough lagrangian tori to kill H_1; The art is to find tori and show result has $\pi_1 = 0$

- First successful implementation of this strategy for $\mathbb{CP}^2 \# \, 3\overline{\mathbb{CP}}^2$ (i.e. find tori, show surgery on model manifold results in $\pi_1 = 0$) obtained by Baldridge-Kirk; Akhmedov-Park
- Full implementation (i.e. infinite families) for $\mathbb{CP}^2 \# \, 3\overline{\mathbb{CP}}^2$: Fintushel-Park-Stern using the 2-fold symmetric product $Sym^2(\Sigma_3)$ as model.
- Full implementation (i.e. infinite families) for $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$, $k \geq 4$ by Baldridge-Kirk, Akhmedov-Park, Fintushel-Stern, Akhmedov-Baykur-Baldridge-Kirk-Park, Ahkmedov-Baykur-Park.

Oriented minimal $\pi_1 = 0$ 4-manifolds with $\mathcal{SW} \neq 0$

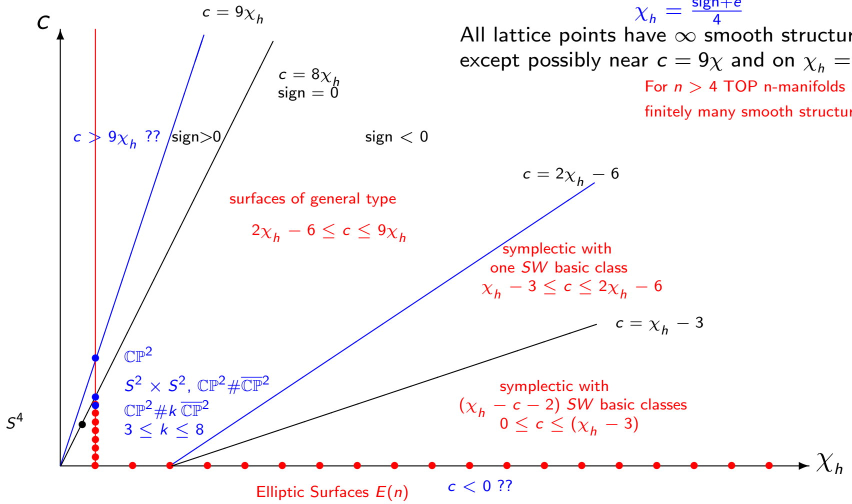
Geography

$c = 3\text{sign} + 2e$

$\chi_h = \frac{\text{sign}+e}{4}$

All lattice points have ∞ smooth structures except possibly near $c = 9\chi$ and on $\chi_h = 1$

For $n > 4$ TOP n-manifolds have finitely many smooth structures



Next Challenges

- Model for $\mathbb{CP}^2$; topological construction of the Mumford plane.
- What about $S^2 \times S^2$; $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$; $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$? ($\pi_1$ issues)
- Are the fake $\mathbb{CP}^2 \# k\, \overline{\mathbb{CP}}^2$ obtained by surgery on null-homologous torus in the standard $\mathbb{CP}^2 \# k\, \overline{\mathbb{CP}}^2$?

  (see Fintushel-Stern: *Surgery on nullhomologous tori and simply connected 4-manifolds with* $b^+ = 1$, Journal of Topology 1 (2008), 1-15, for first attempts)
- More generally are all 4- manifolds obtained from either $\ell\, \mathbb{CP}^2 \# k\, \overline{\mathbb{CP}}^2$ or $nE(2) \# m\, (S^2 \times S^2)$ via a sequence of surgeries on null-homologous tori?

  Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on (null-homologous) tori.

$$S^4 \#_k \mathbb{CP}^2 \#_\ell \overline{\mathbb{CP}}^2 \#_s (S^2 \times S^2) \#_t K3$$

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Surgery on (null-homologous) tori

$\Downarrow$

All $\pi_1 = 0$ smooth 4-manifolds

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Seiberg-Witten Invariants

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Wormholes!