

Smooth 4-manifolds: **BIG** and small

Ronald J. Stern
University of California, Irvine
August 11, 2008

Joint work with Ron Fintushel

The Pre-4-manifold Kirby

Annulus Conjecture, Torus trick, Hauptvermutung using Engulfing, Surgery theory

- ▶ Phd 1965 – Advisor: Eldon Dyer, University of Chicago
- ▶ Assistant Professor UCLA: 1966-69; Full Professor UCLA: 1969-71; Full Professor UCB: 1971 —
- ▶ 1971 Fifth Veblen Prize: The annulus conjecture is true: a region in n -space bounded by two locally flat $(n - 1)$ -spheres is an annulus ($n > 5$): *Stable Homeomorphisms and the Annulus Conjecture*, Annals of Math 89 (1969), 574–82.
- ▶ With Larry Siebenmann: The Hauptvermutung is false: PL structures (up to isotopy) on a PL manifold M correspond to elements of $H^3(M; \mathbb{Z}_2)$ ($n > 4$)
- ▶ The triangulation conjecture is false: a topological manifold has no PL structure when an obstruction in $H^4(M; \mathbb{Z}_2)$ is non-zero ($n > 4$)
- ▶ Simple homotopy type is a topological invariant ($n > 4$)
Foundational essays on topological manifolds, smoothings, and triangulations, Annals of Mathematics Studies, No. 88. Princeton University Press, Princeton, N.J. 1977. vii+355 pp.
- ▶ Key new idea: Torus Trick; works in dimension > 4 , so led to Rob's 4-dimensional interests and his 4-manifold legacy —
- ▶ 50 PhD students; 82 grandchildren; 16 great grandchildren



Basic facts about 4-manifolds

Invariants

- ▶ Euler characteristic: $e(X) = \sum_{i=0}^4 (-1)^i \text{rk}(H^i(M; \mathbb{Z}))$
- ▶ Intersection form: $H^2(X; \mathbb{Z}) \otimes H^2(X; \mathbb{Z}) \rightarrow \mathbb{Z}$;

$$\alpha \cdot \beta = (\alpha \cup \beta)[X]$$

is an integral, symmetric, unimodular, bilinear form.

Signature of $X = \text{sign}(X) =$ Signature of intersection form
 $= b^+ - b^-$

Type: **Even** if $\alpha \cdot \alpha$ even for all α ; otherwise **Odd**

- ▶ (Freedman, 1980) **The intersection form classifies simply connected topological 4-manifolds:** There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has $X \times S^1$ smoothable.
- ▶ (Donaldson, 1982) Two simply connected *smooth* 4-manifolds are homeomorphic iff they have the same **e**, **sign**, and **type**.



What do we know about smooth 4-manifolds?

Much—but so very little

Wild Conjecture

Every 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for $n > 4$, every n -manifold has only finitely many distinct smooth n -manifolds which are homeomorphic to it.

Main goal: Discuss invariants and techniques developed to study this conjecture

- ▶ Need more invariants: Donaldson, Seiberg-Witten Invariants

$$SW : \{\text{characteristic elements of } H_2(X; \mathbb{Z})\} \rightarrow \mathbb{Z}$$

- ▶ $SW(\beta) \neq 0$ for only finitely many β : called *basic classes*.
- ▶ For each surface $\Sigma \subset X$ with $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma \geq 0$

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|$$

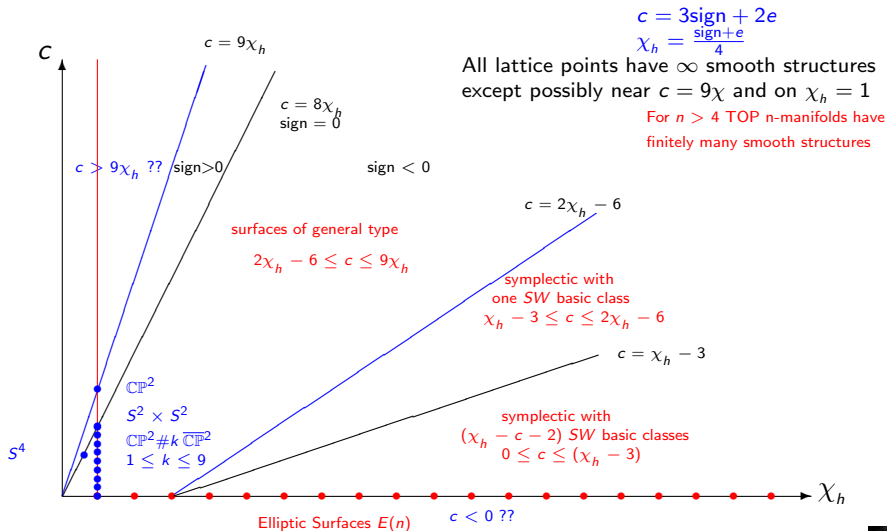
for every basic class β . (adjunction inequality[Kronheimer-Mrowka])

Basic classes = smooth analogue of the canonical class of a complex surface

- ▶ $SW(\kappa) = \pm 1$, κ the first Chern class of a symplectic manifold [Taubes].



Oriented minimal ($\pi_1 = 0$) 4-manifolds with $SW \neq 0$ Geography



Every 4-manifold has zero or infinitely many distinct smooth structures

- ▶ One way to try to prove this conjecture is to find a “dial” to change the smooth structure at will.
- ▶ Since Rob’s last party — An effective dial: **Surgery on null-homologous tori**

T : any self-intersection 0 torus $\subset X$, Tubular nbd $N_T \cong T^2 \times D^2$.

Surgery on T : $X \setminus N_T \cup_{\varphi} T^2 \times D^2$, $\varphi: \partial(T^2 \times D^2) \rightarrow \partial(X \setminus N_T)$
 $\varphi(\text{pt} \times \partial D^2) = \text{surgery curve}$

Result determined by $\varphi_*[\text{pt} \times \partial D^2] \in H_1(\partial(X \setminus N_T)) = \mathbb{Z}^3$

Choose basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial N_T)$ where $\{\alpha, \beta\}$ are pushoffs of a basis for $H_1(T)$.

$$\varphi_*[\text{pt} \times \partial D^2] = p\alpha + q\beta + r[\partial D^2]$$

Write $X \setminus N_T \cup_{\varphi} T^2 \times D^2 = X_T(p, q, r)$

This operation does not change $e(X)$ or $\sigma(X)$

Note: $X_T(0, 0, 1) = X$

Need formula for the Seiberg-Witten invariant of $X_T(p, q, r)$ to determine when the smooth structure changes:

Due to Morgan, Mrowka, and Szabó (1996).



The Morgan, Mrowka, Szabó Formula

$$\sum_i SW_{X_T(p,q,r)}(k + 2i[T_{(p,q,r)}]) = p \sum_i SW_{X_T(1,0,0)}(k' + 2i[T_{(1,0,0)}]) \\ + q \sum_i SW_{X_T(0,1,0)}(k'' + 2i[T_{(0,1,0)}]) + r \sum_i SW_X(k''' + 2i[T])$$

k characteristic element of $H_2(X_{T(p,q,r)})$

$$\begin{array}{ccc} H_2(X_T(p, q, r)) & \rightarrow & H_2(X_T(p, q, r), N_{T(p,q,r)}) & k & \rightarrow & \bar{k} \\ & & \downarrow \cong & & & \downarrow \\ & & H_2(X \setminus N_T, \partial) & & & \hat{k} = \hat{k}' \\ & & \uparrow \cong & & & \uparrow \\ H_2(X_T(1, 0, 0)) & \rightarrow & H_2(X_T(1, 0, 0), N_{T(1,0,0)}) & k' & \rightarrow & \bar{k}' \end{array}$$

- All basic classes of $X_T(p, q, r)$ arise in this way.
- Useful to determine situations when sums collapse to single summand.



Surgery on Tori

Reducing to one summand

$$SW_{X_{T(p,q,r)}} = pSW_{X_{T(1,0,0)}} + qSW_{X_{T(0,1,0)}} + rSW_X$$

- ▶ When torus T is **nullhomologous**, and
- ▶ when a core torus is **essential**, there is a torus that intersects it algebraically nontrivially.

Some observations about null-homologous tori:

- With null-homologous framing: $H_1(X_{T(p,q,1)}) = H_1(X)$,
So for an effective dial want, say, $SW_{X_{T(1,0,0)}} \neq 0$;
- $b_1(X_{T(1,0,0)}) = b_1(X_{T(0,1,0)}) = b_1(X) + 1$.

Dual situations for surgery on tori T

- T primitive, $\alpha \subset T$ essential in $X \setminus T$.
 $\Rightarrow T_{(1,0,r)}$ nullhomologous in $X_T(1,0,r)$.
- T nullhomologous, α bounds in $X \setminus N_T$
 $\Rightarrow (1,0,0)$ surgery on T gives (a).



Old Application: Knot Surgery

K : Knot in S^3 , T : square 0 essential torus in X

$$\blacktriangleright X_K = X \setminus N_T \cup S^1 \times (S^3 \setminus N_K)$$

Note: $S^1 \times (S^3 \setminus N_K)$ has the homology of $T^2 \times D^2$.

Facts about knot surgery

- \blacktriangleright If X and $X \setminus T$ both simply connected; so is X_K
(So X_K homeo to X)
- \blacktriangleright If K is fibered and X and T both symplectic; so is X_K .
- \blacktriangleright $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Conclusions

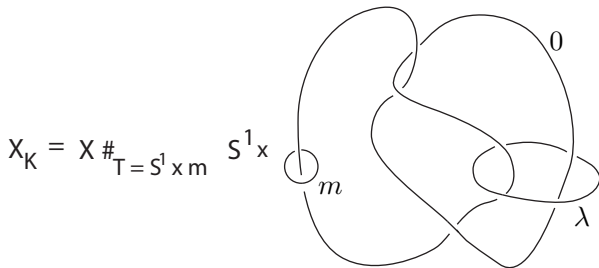
- \blacktriangleright If X , $X \setminus T$, simply connected and $\mathcal{SW}_X \neq 0$, then there is an infinite family of distinct manifolds all homeomorphic to X .
- \blacktriangleright If in addition X , T symplectic, K fibered, then there is an infinite family of distinct symplectic manifolds all homeomorphic to X .

e.g. $X = K3$, $\mathcal{SW}_X = 1$, $\mathcal{SW}_{X_K} = \Delta_K(t^2)$



Knot surgery and nullhomologous tori

Knot surgery on torus T in 4-manifold X with knot K :



$\Lambda = S^1 \times \lambda =$ nullhomologous torus — Used to change crossings:

Now apply Morgan-Mrowka-Szabó formula + tricks

- ▶ Weakness of construction: **Requires T to be homologically essential**

Open conjecture: If $\chi(X) > 1$, $SW_X \neq 0$, then X contains a homologically essential torus T with trivial normal bundle.

- ▶ If X homeomorphic to $\mathbb{C}P^2$ blown up at 8 or fewer points, then X contains no such torus - **so what can we do for these small manifolds?**



Reverse Engineering

- ▶ Difficult to find useful nullhomologous tori like Λ used in knot surgery.
 - ▶ Recall: $SW_{X_{T(p,q,r)}} = pSW_{X_{T(1,0,0)}} + qSW_{X_{T(0,1,0)}} + rSW_X$
 - ▶ With null-homologous framing: $H_1(X_{T(p,q,1)}) = H_1(X)$. So want, say, $SW_{X_{T(1,0,0)}} \neq 0$;
 - ▶ $b_1(X_{T(1,0,0)}) = b_1(X_{T(0,1,0)}) = b_1(X) + 1$.
 - ▶ Recall: Dual situations for surgery on tori T
 - T primitive, $\alpha \subset T$ essential in $X \setminus T$.
 $\Rightarrow T_{(1,0,r)}$ nullhomologous in $X_T(1,0,r)$.
 - T nullhomologous, α bounds in $X \setminus N_T$
 $\Rightarrow (1,0,0)$ surgery on T gives (a).
- IDEA:** First construct $X_{T(1,0,0)}$ so that $SW_{X_{T(1,0,0)}} \neq 0$ and then surger to reduce b_1 .



Reverse Engineering

- ▶ Difficult to find useful nullhomologous tori like Λ used in knot surgery.
- ▶ Recall: $SW_{X_{T(p,q,r)}} = pSW_{X_{T(1,0,0)}} + qSW_{X_{T(0,1,0)}} + rSW_X$
- ▶ With null-homologous framing: $H_1(X_{T(p,q,1)}) = H_1(X)$. So for effective dial want, say, $SW_{X_{T(1,0,0)}} \neq 0$;
- ▶ $b_1(X_{T(1,0,0)}) = b_1(X_{T(0,1,0)}) = b_1(X) + 1$.
IDEA: First construct $X_{T(1,0,0)}$ so that $SW_{X_{T(1,0,0)}} \neq 0$ and then surger to reduce b_1 .
- ▶ Procedure to insure the existence of effective null-homologous tori
 1. Find model manifold M with same Euler number and signature as desired manifold, but with $b_1 \neq 0$ and with $SW \neq 0$.
 2. Find b_1 disjoint essential tori in M containing generators of H_1 .
Surger to get manifold X with $H_1 = 0$. Want result of each surgery to have $SW \neq 0$ (except perhaps the very last).
 3. X will contain a “useful” nullhomologous torus.



Luttinger Surgery

- ▶ For model manifolds with $H_1 \neq 0$: nature hands you **symplectic** manifolds.
- ▶ We seek tori that will kill b_1 . Nature hands you **Lagrangian** tori.

X : symplectic manifold T : Lagrangian torus in X

Preferred framing for T : **Lagrangian framing**
w.r.t. which all pushoffs of T remain Lagrangian

$(1/n)$ -surgeries w.r.t. this framing are again symplectic
(Luttinger; Auroux, Donaldson, Katzarkov)

If $S_\beta^1 = \text{Lagrangian pushoff}$, $X_T(0, \pm 1, 0)$: symplectic mfd

\implies if $b^+ > 1$, $X_T(0, \pm 1, 0)$ has $\mathcal{SW} \neq 0$

Then have infinitely many $H_1 = 0$ manifolds - keep fingers crossed $\pi_1 = 0$.



Reverse Engineering in Action

Infinite families of fake $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$

Need Model Manifolds for $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$

i.e. symplectic manifolds X_k with same e and sign as $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$,
and $b_1 \geq 1$ disjoint lagrangian tori carrying basis for H_1 .

- ▶ Surger lagrangian tori to decrease b_1 .
- ▶ Resulting manifold has $H_1 = 0$ - but with a **dial**.
- Get infinite family of distinct manifolds all homology equivalent to $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$
- Keep fingers crossed that result has $\pi_1 = 0$, so all homeomorphic to $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$



Model Manifolds for $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$

Basic Pieces: X_0, X_1, X_2, X_3, X_4

$X_r \#_{\Sigma_2} X_s$ is a model for $\mathbb{C}P^2 \# (r + s + 1) \overline{\mathbb{C}P^2}$

X_0 : $\Sigma_2 \subset T^2 \times \Sigma_2$ representing $(0, 1)$

X_1 : $\Sigma_2 \subset T^2 \times T^2 \# \overline{\mathbb{C}P^2}$ representing $(2, 1) - 2e$

X_2 : $\Sigma_2 \subset T^2 \times T^2 \# 2\overline{\mathbb{C}P^2}$ representing $(1, 1) - e_1 - e_2$

X_3 : $\Sigma_2 \subset S^2 \times T^2 \# 3\overline{\mathbb{C}P^2}$ representing $(1, 3) - 2e_1 - e_2 - e_3$

X_4 : $\Sigma_2 \subset S^2 \times T^2 \# 4\overline{\mathbb{C}P^2}$ representing $(1, 2) - e_1 - e_2 - e_3 - e_4$

Exception: $X_0 \#_{\Sigma_2} X_0 = \Sigma_2 \times \Sigma_2$ is a model for $S^2 \times S^2$

Enough lagrangian tori to kill H_1 ; The art is to find tori and show result has $\pi_1 = 0$

- First successful implementation of this strategy for $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ (i.e. find tori, show surgery on model manifold results in $\pi_1 = 0$) obtained by Baldridge-Kirk; Akhmedov-Park
- Full implementation (i.e. infinite families) for $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$: Fintushel-Park-Stern using the 2-fold symmetric product $Sym^2(\Sigma_3)$ as model.
- Full implementation (i.e. infinite families) for $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$, $k \geq 4$ by Baldridge-Kirk, Akhmedov-Park, Fintushel-Park-Stern, Akhmedov-Baykur-Baldridge-Kirk-Park, Ahkmedov-Baykur-Park.



Alternate approaches successful for $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$, $k \geq 5$

Rational Blowdown

1989: First fake $\mathbb{C}P^2 \# 8 \overline{\mathbb{C}P^2}$; D. Kotschick showed Barlow surface exotic.

2004: First fake $\mathbb{C}P^2 \# 7 \overline{\mathbb{C}P^2}$; Jongil Park used rational blowdown (gave others courage to pursue small 4-manifolds).

2004: Related ideas used by Stipsicz-Szabó to produce fake $\mathbb{C}P^2 \# 6 \overline{\mathbb{C}P^2}$

2005: Fintushel-Stern introduced *double-node surgery* to produce infinitely many fake such structures

2005: Two hours later Park-Stipsicz-Szabó used double-node surgery to produce infinitely many exotic $\mathbb{C}P^2 \# 5 \overline{\mathbb{C}P^2}$.

Jongil Park led effort to use rational blowdown techniques to find exotic **complex structures** on $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$, $k \geq 5$

2006: Reverse Engineering introduced

2007: $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$, $k \geq 3$, Baldrige-Kirk, Ahkmedov-Park, Fintushel-Park-Stern, Ahkmedov-Bakyur-Baldrige-Kirk-Park, Ahkmedov-Bakyur-Park.



Next Challenges

- Model for $\mathbb{C}P^2$; topological construction of the Mumford plane.
- What about $S^2 \times S^2$; $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$; $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$? (π_1 issues)
- Are the fake $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$ obtained by surgery on null-homologous torus in the standard $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$?
(see Fintushel-Stern: *Surgery on nullhomologous tori and simply connected 4-manifolds with $b^+ = 1$* , Journal of Topology 1 (2008), 1-15, for first attempts)
- More generally are all 4- manifolds obtained from either $\ell \mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$ or $nE(2) \# m(S^2 \times S^2)$ via a sequence of surgeries on null-homologous tori?

Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on (null-homologous) tori.



Focus Problem

Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on (null-homologous) tori.

Aside: When is log transform on an essential torus a sequence of log transforms on null-homologous tori?

- ▶ Then, euler characteristic, signature, and type will classify smooth 4-manifolds up to surgery on (null-homologous) tori.
- ▶ In other words, algebraic topology will classify smooth 4-manifolds up to



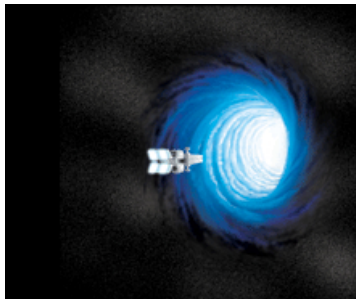
Focus Problem

Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on (null-homologous) tori.

Aside: When is log transform on an essential torus a sequence of log transforms on null-homologous tori?

- ▶ Then, euler characteristic, signature, and type will classify smooth 4-manifolds up to surgery on (null-homologous) tori.
- ▶ In other words, algebraic topology will classify smooth 4-manifolds up to

Wormholes!



An Even Bigger Challenge

Talking at Rob's 80th Birthday Fest.

HAPPY 70th BIRTHDAY, ROB

