Smooth 4-manifolds: BIG and small

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Joint work with Ron Fintushel

Basic facts about 4-manifolds

Invariants

- Euler characteristic: $e(X) = \sum_{i=0}^{4}(-1)^j rk(H^j(M;\mathbb{Z}))$
- Intersection form: $H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) \to \mathbb{Z}$;

$$\alpha \cdot \beta = (\alpha \cup \beta)[X]$$

  is an integral, symmetric, unimodular, bilinear form.

  Signature of X = $\text{sign}(X)$ = Signature of intersection form $= b^+ - b^-$

  Type: Even if $\alpha \cdot \alpha$ even for all α; otherwise Odd
- (Freedman, 1980) The intersection form classifies simply connected topological 4-manifolds: There is one homeomorphism type if the form is even; there are two if odd — exactly one of which has $X \times S^1$ smoothable.
- (Donaldson, 1982) Two simply connected *smooth* 4-manifolds are homeomorphic iff they have the same e, sign, and type.

What do we know about smooth 4-manifolds?

Wild Conjecture

Every 4-manifold has either zero or infinitely many distinct smooth 4-manifolds which are homeomorphic to it.

In contrast, for $n > 4$, every n-manifold has only finitely many distinct smooth n-manifolds which are homeomorphic to it.

The goal of this lecture —Discuss techniques used to study this conjecture

- Need new invariants: Donaldson, Seiberg-Witten Invariants

$$\mathcal{SW} : \{\text{characteristic elements of } H_2(X;\mathbb{Z})\} \to \mathbb{Z}$$

  - $\mathcal{SW}(\beta) \neq 0$ for only finitely many β: called *basic* classes.
  - For each surface $\Sigma \subset X$ with $g(\Sigma) > 0$ and $\Sigma \cdot \Sigma \geq 0$

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |\Sigma \cdot \beta|$$

  for every basic class β. (adjunction inequality[Kronheimer-Mrowka])

  Basic classes = smooth analogue of the canonical class of a complex surface

  - $\mathcal{SW}(\kappa) = \pm 1$, κ the first Chern class of a symplectic manifold [Taubes].

Oriented minimal ($\pi_1 = 0$) 4-manifolds with $\mathcal{SW} \neq 0$

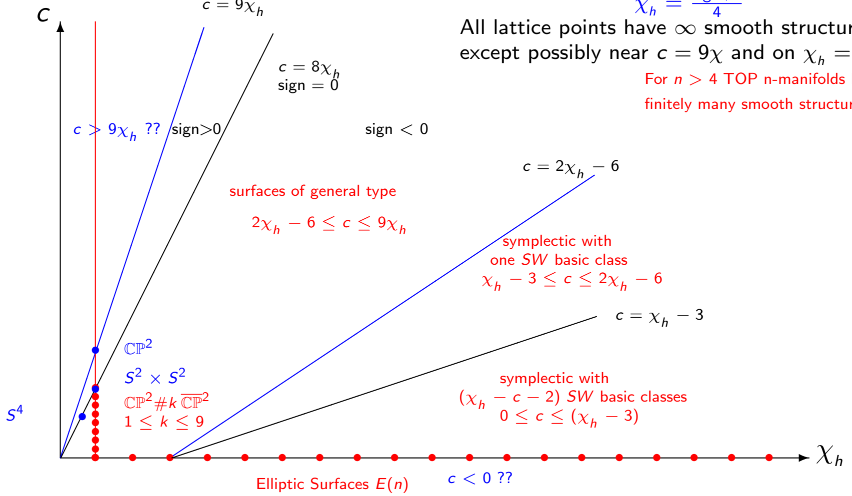
Geography

$c = 3\text{sign} + 2e$

$\chi_h = \frac{\text{sign}+e}{4}$

All lattice points have ∞ smooth structures except possibly near $c = 9\chi$ and on $\chi_h = 1$

For $n > 4$ TOP n-manifolds have finitely many smooth structures



Every 4-manifold has zero or infinitely many distinct smooth structures

Need techniques to prove wild conjectures!

- One way to try to prove this conjecture is to find a "dial" to change the smooth structure at will.
- Goal of lecture – An effective dial: Surgery on null-homologous tori

T: any self-intersection 0 torus $\subset X$, Tubular nbd $N_T \cong T^2 \times D^2$.

Surgery on T: $X \smallsetminus N_T \cup_\varphi T^2 \times D^2$, $\varphi : \partial(T^2 \times D^2) \to \partial(X \smallsetminus N_T)$

$$\varphi(\mathrm{pt} \times \partial D^2) = \text{surgery curve}$$

Result determined by $\varphi_*[\mathrm{pt} \times \partial D^2] \in H_1(\partial(X \smallsetminus N_T)) = \mathbb{Z}^3$

Choose basis $\{\alpha, \beta, [\partial D^2]\}$ for $H_1(\partial N_T)$ where $\{\alpha, \beta\}$ are pushoffs of a basis for $H_1(T)$.

$$\varphi_*[\mathrm{pt} \times \partial D^2] = p\alpha + q\beta + r[\partial D^2]$$

Write $X \smallsetminus N_T \cup_\varphi T^2 \times D^2 = X_T(p, q, r)$

This operation does not change $e(X)$ or $\sigma(X)$

Note: $X_T(0, 0, 1) = X$

Need formula for the Seiberg-Witten invariant of $X_T(p, q, r)$ to determine when the smooth structure changes:

Due to Morgan, Mrowka, and Szabo.

The Morgan, Mrowka, Szabo Formula

$$\sum_i SW_{X_{T(p,q,r)}}(k + 2i[T_{(p,q,r)}]) = p \sum_i SW_{X_{T(1,0,0)}}(k' + 2i[T_{(1,0,0)}])$$
$$+q \sum_i SW_{X_{T(0,1,0)}}(k'' + 2i[T_{(0,1,0)}]) + r \sum_i SW_X(k''' + 2i[T])$$

k characteristic element of $H_2(X_{T(p,q,r)})$

$$\begin{array}{ccc} H_2(X_T(p,q,r)) & \to & H_2(X_T(p,q,r), N_{T_{(p,q,r)}}) \\ & & \downarrow\cong \\ & & H_2(X \setminus N_T, \partial) \\ & & \uparrow\cong \\ H_2(X_T(1,0,0)) & \to & H_2(X_T(1,0,0), N_{T_{(1,0,0)}}) \end{array} \qquad \begin{array}{ccc} k & \to & \bar{k} \\ & & \downarrow \\ & & \hat{k} = \hat{k}' \\ & & \uparrow \\ k' & \to & \bar{k}' \end{array}$$

- All basic classes of $X_T(p,q,r)$ arise in this way.
- Useful to determine situations when sums collapse to single summand.

Surgery on Tori

Reducing to one summand

$$SW_{X_{T(p,q,r)}} = pSW_{X_{T(1,0,0)}} + qSW_{X_{T(0,1,0)}} + rSW_X$$

- When a core torus is nullhomologous.
- When a core torus is essential, but there is a square 0 torus that intersects it algebraically nontrivially.

Dual situations for surgery on T

a. T primitive, $\alpha \subset T$ essential in $X \smallsetminus T$.
$\Rightarrow \quad T_{(1,0,r)}$ nullhomologous in $X_T(1,0,r)$.
(Its meridian is $\alpha + r\mu_T \sim \alpha \not\sim 0$ in $X \smallsetminus N_T$.)
Let $\alpha' =$ surgery curve on $\partial N_{T_{(1,0,r)}} \subset X_T(1,0,r)$ which gives back X
α' bounds in $X_T(1,0,r) \smallsetminus N_{T_{(1,0,r)}} = X \smallsetminus N_T$.

b. T nullhomologous, α bounds in $X \smallsetminus N_T$
$(1,0,0)$ (*i.e.* nullhomologous) surgery on T gives (a).

First Application: Knot Surgery

K: Knot in S^3, T: square 0 essential torus in X

- $$X_K = X \setminus N_T \cup S^1 \times (S^3 \setminus N_K)$$

Note: $S^1 \times (S^3 \setminus N_K)$ has the homology of $T^2 \times D^2$.

Facts about knot surgery

- If X and $X \setminus T$ both simply connected; so is X_K

  (So X_K homeo to X)
- If K is fibered and X and T both symplectic; so is X_K.
- $\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t^2)$

Conclusions

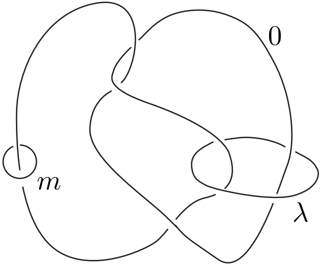
- If X, $X \setminus T$, simply connected and $\mathcal{SW}_X \neq 0$, then there is an infinite family of distinct manifolds all homeomorphic to X.
- If in addition X, T symplectic, K fibered, then there is an infinite family of distinct symplectic manifolds all homeomorphic to X.

e.g. $X = K3, \quad \mathcal{SW}_X = 1, \quad \mathcal{SW}_{X_K} = \Delta_K(t^2)$

Knot surgery and nullhomologous tori

Knot surgery on torus T in 4-manifold X with knot K:

$X_K = X \#_{T = S^1 \times m} \; S^1 \times$



$\Lambda = S^1 \times \lambda =$ nullhomologous torus — Used to change crossings:

Now apply Morgan-Szabo-Taubes formula + tricks

- Weakness of construction: Need T to be homologically essential
- Open conjecture: If $\chi(X) > 1$, then X contains a homologically essential torus T with trivial normal bundle (in the complement of all the basic classes)
- If X homeomorphic to CP^2 blown up at 8 or fewer points, then X contains no such torus - so what can we do here?

Oriented minimal $\pi_1 = 0$ 4-manifolds with $\mathcal{SW} \neq 0$

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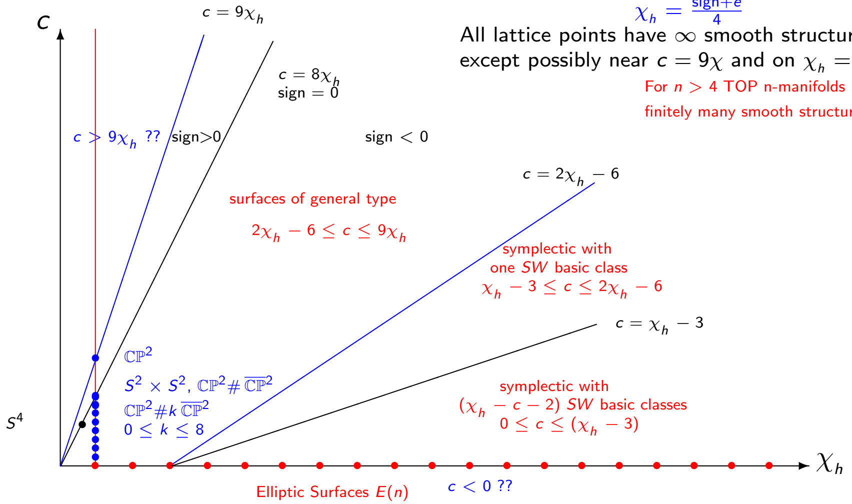

Reverse Engineering

- Difficult to find useful nullhomologous tori like Λ used in knot surgery.

Recall: $SW_{X_{T(p,q,r)}} = pSW_{X_{T(1,0,0)}} + qSW_{X_{T(0,1,0)}} + rSW_X$

Note: $b_1(X_{T(1,0,0)}) = b_1(X_{T(0,1,0)}) = b_1(X) + 1$

IDEA: First construct $X_{T(1,0,0)}$ so that $SW_{X_{T(1,0,0)}} \neq 0$ and then surger to reduce b_1.

Recall: Dual situations for surgery on T

a. T primitive, $\alpha \subset T$ essential in $X \smallsetminus T$.

$\Rightarrow$ $T_{(1,0,r)}$ nullhomologous in $X_T(1,0,r)$.

b. T nullhomologous, α bounds in $X \smallsetminus N_T$

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- Procedure to insure the existence of effective null-homologous tori

1. Find model manifold M with same Euler number and signature as desired manifold, but with $b_1 \neq 0$ and with $\mathcal{SW} \neq 0$.
2. Find b_1 disjoint essential tori in M containing generators of H_1. Surger to get manifold X with $H_1 = 0$. Want result of each surgery to have $\mathcal{SW} \neq 0$ (except perhaps the very last).
3. X will contain a "useful" nullhomologous torus.

Luttinger Surgery

- For our model manifolds: nature hands you symplectic manifolds.
- We seek tori that will kill b_1. Nature hands you Lagrangian tori.

X: symplectic manifold T: Lagrangian torus in X

Preferred framing for T: Lagrangian framing
w.r.t. which all pushoffs of T remain Lagrangian

$(1/n)$-surgeries w.r.t. this framing are again symplectic
(Luttinger; Auroux, Donaldson, Katzarkov)

If $S^1_\beta =$ Lagrangian pushoff, $X_T(0, \pm 1, 0)$: symplectic mfd

$\Longrightarrow$ if $b^+ > 1$, $X_T(0, \pm 1, 0)$ has $\mathcal{SW} \neq 0$

Reverse Engineering in Action

Infinite families of fake $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$

Need Model Manifolds for $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$, $k \geq 2$

i.e. symplectic manifolds X_k with same e and sign as $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$, and $b_1 \geq 1$ disjoint lagrangian tori carrying basis for H_1.

- Surger lagraginan tori to decrease b_1.
- Resulting manifold has $H_1 = 0$ - but with a dial.

- Get infinite family of distinct manifolds all homology equivalent to $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$
- Keep fingers crossed that result has $\pi_1 = 0$, so all homeomprphic to $\mathbb{CP}^2 \# k\,\overline{\mathbb{CP}}^2$

Model Manifolds

Basic Pieces: $X_0, X_1, X_2, X_3 X_4 X_5$

X_0: $\Sigma_2 \subset T^2 \times \Sigma_2$ representing $(0,1)$

X_1: $\Sigma_2 \subset T^2 \times T^2 \# \overline{\mathbb{CP}}^2$ representing $(2,1) - 2e$

X_2: $\Sigma_2 \subset T^2 \times T^2 \# 2\overline{\mathbb{CP}}^2$ representing $(1,1) - e_1 - e_2$

X_3: $\Sigma_2 \subset S^2 \times T^2 \# 3\overline{\mathbb{CP}}^2$ representing $(1,3) - 2e_1 - e_2 - e_3$

X_4: $\Sigma_2 \subset S^2 \times T^2 \# 4\overline{\mathbb{CP}}^2$ representing $(1,2) - e_1 - e_2 - e_3 - e_4$

- For a symplectic 4-manifold, X, $c_1^2(X) = \frac{1}{4}(e(X) + \text{sign}(X))$; $\chi(X) = 3\,\text{sign} + 2\,e(X)$
- (Fiber Sums) If X', X'' are symplectic with symplectic submanifolds Σ', Σ'' of square 0 and same genus g, the fiber sum $X = X' \#_{\Sigma' = \Sigma''} X''$ is again symplectic, and $c_1^2(X) = c_1^2(X') + c_1^2(X'') + 8(g-1)$; $\chi(X) = \chi(X') + \chi(X'') + (g-1)$

$X_r \#_{\Sigma_2} X_s$ is a model for $\mathbb{CP}^2 \# (r+s+1)\,\overline{\mathbb{CP}}^2$

Except $X_0 \#_{\Sigma_2} X_0 = \Sigma_2 \times \Sigma_2$ is a model for $S^2 \times S^2$

All have enough lagrangian tori to kill H_1 (π_1???)

- First successful implementation of this strategy for $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ (i.e. show surgery on model manifold results in $\pi_1 = 0$) obtained by Baldridge-Kirk and Akhmedov-Park
- First full implementation (i.e. infinite families) for $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$: Fintushel-Park-Stern using the 2-fold symmetric product $Y = Sym^2(\Sigma_3)$ as model.
- Ahkmedov-Park have paper to implement strategy for $\mathbb{CP}^2 \# 2\,\overline{\mathbb{CP}}^2$ (i.e. show surgery on model manifold results in $\pi_1 = 0$)

WHAT ABOUT $\mathbb{CP}^2$?? will discuss this tomorrow

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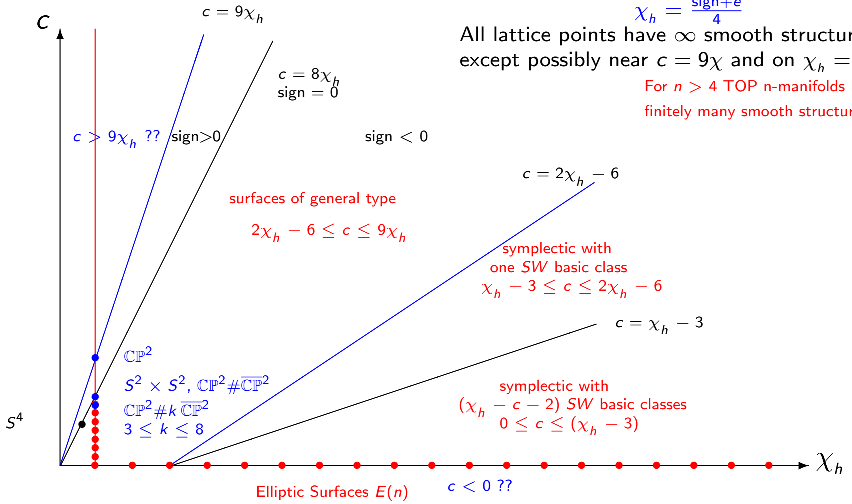

Next Challenge

Two homeomorphic smooth 4-manifolds are related by a sequence of logarithmic transforms on null-homologous tori.

- Then, euler characteristic. signature, and type will classify smooth 4-manifolds up to surgery on null-homologous tori.
- In other words, algebraic topology will classify smooth 4-manifolds up to

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Wormholes!