Complex analysis exam: Spring 2002
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Notes: Please contact Paul Macklin if there are typos, omissions, or other errors in this document.

1. Let f be a bounded holomorphic function in the upper half plane and continuous on its closure. Suppose also that $f(z)$ is real for real z. Prove that f must be constant.

2. (a) Write the Harnack inequality for positive harmonic functions in the unit disk U.
   (b) Prove that if $f : U \to U \setminus \{0\}$ is analytic and $f(0) = \frac{1}{2}$ then $\left| f\left(\frac{1}{2}\right)\right| \geq \frac{1}{8}$.

3. Evaluate the integral $\int_0^\infty \frac{\sin(x)}{x} dx$.

4. Let $f(z)$ be holomorphic in the unit disk $U = \{|z| < 1\}$ and such that
$$\sup_{0<r<1} \int_0^{2\pi} \left| f(re^{i\theta}) \right| d\theta \leq 2\pi.$$
   Prove that if $f(z) = \sum_{n=1}^\infty a_n z^n$, $|z| < 1$ then $|a_n| \leq 1$, $n \geq 0$.

5. Let $a_n, n \geq 1$ be a bounded sequence and
$$f(z) = \sum_{n=1}^\infty \frac{a_n}{(z-n)^2}.$$
   Prove that f is meromorphic in the complex plane and find all the poles.

6. Define
$$A_n(z) = \sum_{k=1}^n \frac{(-1)^k z^{2k-1}}{2k-1}, n \geq 1, A(z) = \sum_{k=1}^\infty \frac{(-1)^k z^{2k-1}}{2k-1}.$$
   Show that
   (a) for any $0 < r < 1$ there exists $N = N(r)$ such that if $n \geq N$ then $A_n(z)$ has exactly one zero in the disk $U(r) = \{|z| < r\}$;
   (b) $A(\tan z) = z$ in a disk $U(r)$ for some $0 < r < 1$.

7. Let $H = \{z \in \mathbb{C} : \text{Im } z > 0\}$ and $f : H \to H$ be analytic and continuous on $\overline{H} = \{z \in \mathbb{C} : \text{Im } z \geq 0\}$. Assume also that $f(x)$ is real for all real x. Prove the following inequality
$$\left| \frac{f(z) - f(i)}{f(z) - \overline{f(i)}} \right| \leq \left| \frac{z-i}{z+i} \right|, z \in H.$$

8. Prove that there is no a one-to-one holomorphic map from the unit disk $U = \{|z| < 1\}$ onto the punched disk $U \setminus \{0\} = \{0 < |z| < 1\}$. Prove also that there is a conformal (angle preserving) holomorphic mapping of the unit disk $U = \{|z| < 1\}$ onto the punched disk $U \setminus \{0\} = \{0 < |z| < 1\}$.