

**Complex analysis exam:** September 2004

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**Notes:** Please contact Paul Macklin if there are typos, omissions, or other errors in this document.

1. Let  $f_n(z) = \cos(nz)$  Show that

$$\sum_{n=0}^{\infty} \frac{f_n(z)}{n!}$$

converges uniformly on compact subsets of the plane. Find the sum.

2. Evaluate  $\int_0^{2\pi} e^{\cos \theta} d\theta$ .

3. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{z^2 + n^2}$$

is analytic in the complex plane excluding the zeros of  $z^2 + n^2 : n = 1, 2, \dots$ .

4. Let  $\{u_n(z)\}$  be a sequence of real-valued functions harmonic on an open set  $G$  which converge to a function  $u(z)$ , uniformly on each subset of  $G$ . Show that  $u$  is harmonic on  $G$ .

5. Let  $D = \{z : |z| < 1\}$ , and let  $f : D \rightarrow D$  have a zero of order  $n$  at zero. Show that  $|f^{(n)}(z)| \leq |z|^n$  on  $D$  and  $|f^{(n)}(0)| \leq n!$ .

6. Prove that all the zeros of  $f(z) = 4z^3 + 2z^2 + z + 10$  are in the annulus  $\{z : 1 < |z| < 2\}$ .

7. Find the radius of convergence for the series:

$$\sum_{n=1}^{\infty} \frac{z^{2n}}{n!} \text{ and } \sum_{n=1}^{\infty} \frac{z^{n!}}{2n}.$$

8. Let  $f(z)$  be an entire function whose range does not contain any non-negative real numbers. Show that  $f$  is constant.

9. Let  $f$  be analytic in a region  $D$  and define  $f_n(z) = f^{(n)}(z)$  to be the  $n$ -th derivative of  $f$ . Suppose that the sequence  $f_n(z)$  converges uniformly on compacta to a continuous function  $g$ . Show that  $g(z) = ce^z$ .