Complex analysis exam: September 19, 2001
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Notes: Please contact Paul Macklin if there are typos, omissions, or other errors in this document.

Notation: Let Δ be the unit disk in the complex plane $\mathbb{C}$.

1. (a) State the Rouché Theorem.
   (b) Find the number of solutions of the equation $z^{32} + 16z^2 + 16 = 0$ in the left-half plane $D = \{z : \text{Re } z < 0\}$.

2. (a) State the Schwarz reflection principle for holomorphic functions.
   (b) Let $f(z)$ be holomorphic in the $G = \{z \in \mathbb{C} : -1 < \text{Re } z < 1, \text{Im } z > 0\}$ and continuous on $\overline{G} = \{z \in \mathbb{C} : -1 \le \text{Re } z \le 1, \text{Im } z \ge 0\}$. If $f(x) = x^2$ for all $x \in [0,1]$. Find $f(z)$ and show your work.

3. Show that the equation
$$f'(z) = f(z), \qquad f(0) = 1$$
has a **unique** holomorphic solution $f(z) = e^z$.

4. (a) State the Riemman Mapping Theorem.
   (b) Construct a conformal map with maps the unit disk Δ onto D where
$$D = \left\{ z = |z|\, e^{i\theta} \in \mathbb{C} : 0 < \theta < \frac{\pi}{4} \right\}.$$

5. (a) State the Liouville Theorem.
   (b) Find all entire holomorphic functions $f(z)$ on $\mathbb{C}$ satisfying
$$|f(z)| \le |z|\, e^x, \qquad z = x + iy \in \mathbb{C}.$$

6. Let D_1 and D_2 be two domains in $\mathbb{C}$. Let $\{f_n(z)\}$ be a sequence of conformal holomorphic maps (i.e., angle preserving maps) from D_1 into D_2. Suppose that f_n converges to g uniformly on any compact subset of D_1. Show that either $g \equiv$ constant or g is also a conformal map from D_1 into D_2.

7. Evaluate the integral
$$\int_0^\infty \frac{\sin^2 x}{x^2} dx.$$

8. Let
$$f(z) = \sum_{n=1}^{\infty} z^{n!}.$$
   (a) Prove that $f(z)$ is holomorphic in the unit disk $\Delta = \{|z| < 1\}$.
   (b) Prove that $f(z)$ does not have any holomorphic extension, i.e., there is no holomorphic function g on domain U with $\Delta \subset U$ and $U \neq \Delta$ so that $f \equiv g$ on Δ. (Hint: prove $f(e^{i\theta}) = \infty$ for θ being rational number.)

9. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be holomorphic in $\mathbb{C}$.
   (a) Prove
$$\frac{1}{2\pi} \int_0^{2\pi} \left| f(e^{i\theta}) \right|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2.$$

(b) Prove there is a constant C depending only on f such that

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(e^{i\theta})\right|^2 d\theta - \sum_{n=0}^{N} |a_n|^2 \leq Ce^{-N}$$

holds for all positive integer N.