Complex analysis exam: September 2000
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Notes: Please contact Paul Macklin if there are typos, omissions, or other errors in this document.

Notation: Let Δ be the unit disk in the complex plane $\mathbb{C}$, let $U = \{z \in \mathbb{C} : \text{Im } z > 0\}$ be the upper half plane, and $i = \sqrt{-1}$.

1. Evaluate the integral for $a > 1$
$$\int_0^{2\pi} \frac{1}{a^2 - 2a\cos\theta + 1} d\theta.$$

2. (a) State Rouche's theorem.
   (b) Find all roots of the equation $2z + \sin z = 0$ in the unit disk Δ.

3. Let $f(z)$ be an entire function having properties: Re $f(z) > -1$ on $\mathbb{C}$ and $f(0) = 2$. Find $f(z)$ exactly (show your work).

4. Let $f(z)$ be an analytic automorphism of bounded domain G (f is one to one, analytic in G and $f(G) = G$). Let $f_1 = f$ and $f_{n+1}(z) = f \circ f_n(z) = f(f_n(z))$, for $z \in G$ and $n \geq 1$.
   (a) There is a subsequence of $\{f_n\}$ converges to an analytic function g uniformly on any compact subset in G. Why?
   (b) If g is not constant, prove g is an automorphism of G;
   (c) If the sequence $\{f_n\}$ itself converges to $g(z)$ then $f(z) \equiv z$.

5. Let $f(z)$ be analytic on upper half plane $U = \{z \in \mathbb{C} : \text{Im } z > 0\}$ and continuous on $\overline{U} = \{z \in \mathbb{C} : \text{Im } z \geq 0\}$. Prove that if $f(z)$ equals zero on any non-empty interval of the real axis then $f(z)$ must be identically zero on $\overline{U}$.

6. (a) State the Riemann mapping theorem.
   (b) Find the Riemann mapping $f(z) : D \to \Delta$ so that $f(1) = 0$ and $f'(1) > 0$. Here
$$D = \left\{ z \in \mathbb{C} : |\arg z| < \frac{\pi}{4} \right\}.$$

7. Let $f(z)$ be an entire function on $\mathbb{C}$ with $|f(z)| = 1$ on the unit circle $|z| = 1$. Find all such $f(z)$.

8. (a) State Riemann Lemma for removable singularity;
   (b) Let f be holomorphic in $D = \Delta \setminus \{0\} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. If
$$\int_D |f(z)|\, dA(z) = \int_D |f(z + iy)|\, dx\, dy < \infty$$
   then $z = 0$ is either a removable singularity or simple pole of f.