Complex analysis exam: September 23, 1999
LaTeX'ed by: Paul Macklin
email: pmacklin@math.uci.edu.NOSPAM
www: http://math.uci.edu/~pmacklin

Notes: Please contact Paul Macklin if there are typos, omissions, or other errors in this document.

1. Let f and g be entire analytic functions in the complex plane $\mathbb{C}$ such that $g(0) \neq 0$ and $|f(z)| \leq |g(z)|$ on $\mathbb{C}$. Show that
$$f(z) = \frac{f(0)}{g(0)} g(z).$$

2. (a) State the Residue theorem.
   (b) Let $f(z)$ be holomorphic in $\mathbb{C}$. Let the zero set of f in $\mathbb{C}$ be $\mathbb{N}$ (where $\mathbb{N}$ denotes the set of all positive integers). Suppose $f'(k) = k(k+1)$ for $k \in \mathbb{N}$. Find Res $\left(\frac{\cos^2(\pi z)}{f(z)}; k \right)$ for any $k \in \mathbb{N}$.
   (c) Evaluate
$$\lim_{n \to \infty} \int_{|z|=n+\frac{1}{2}} \frac{\cos^2(\pi z)}{f(z)} dz.$$

3. Evaluate
$$\int_0^{2\pi} \frac{1}{2 + \sin t} dt.$$

4. Let $f(z)$ be an one-to-one and onto holomorphic map from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C} \setminus \{0\}$. Prove that either $f(z) = az$ or $f(z) = a/z$ for some $z \neq 0$.

5. (a) Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of holomorphic functions on a domain D which converges uniformly on any compact subsets to a holomorphic function f in D. Show that $f_j'(z)$ converges to $f'(z)$ uniformly on any compact subset in D.
   (b) Let $\{p_k(z)\}_{k=1}^{\infty}$ be a sequence of holomorphic polynomials of degree n which converges to a function f uniformly on any compact subset in $\mathbb{C}$. Prove that f is a holomorphic polynomial of degree less than or equal to n.

6. (a) State the uniqueness theorem for holomorphic functions.
   (b) Let f be holomorphic in disk $D(0,2)$. Answer the following two questions:
      i. Prove or disprove that if $|f(z)| = \left|z^5\right|$ on the circle $|z| = 1$ then $f(z) = e^{i\theta} z^5$ for some real number θ.
      ii. Prove or disprove that if $|f(z)| = \left|z^5\right|$ on the annulus $\frac{1}{2} < |z| < 1$ then $f(z) = e^{i\theta} z^5$ for some real number θ.

7. Let $f(z)$ be an entire holomorphic function in the complex plane such that $|f(z)| = 1$ when $|z| = 1$ and $f''(0) = 2$. Prove $f(z) = z^2$.

8. (a) State the Riemann mapping theorem.
   (b) Find a conformal mapping φ which maps the strip $H = \left\{ z \in \mathbb{C} : \frac{1}{3} < \text{Re } z < 1 \right\}$ to the disk $D(0,1)$.

9. (a) State Harnack's inequality on the unit disk.
   (b) Let $f_n : D(0,1) \to D(0,1) \setminus \{0\}$ be holomorphic functions (or maps) with $f_n(0) = \frac{1}{n}$ $(n = 1, 2, 3, \cdots)$ Prove $\sum_{n=1}^{\infty} f_n(z)^3$ converges uniformly and absolutely on any compact subset K in $D(0, 1/2)$.
   (c) Construct a sequence of holomorphic functions $f_n : D(0,1) \to D(0,1) \setminus \{0\}$ such that $f_n(0) = \frac{1}{n}$ but $\sum_{n=1}^{\infty} f_n \left(\frac{1}{2}\right)^3$ diverges.