Complex analysis exam: September 17, 1997
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Notes: Please contact Paul Macklin if there are typos, omissions, or other errors in this document.

Instructions: Do all problems. Use only one side of each page. Write your name and do at most one problem on each page. Justify your answers fully. Where appropriate, quote without proof theorems and results that you use in your solutions.

Notation: $\mathbb{C}$ is the complex plane, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk.

Problem 1 *Let f be an analytic function on Δ.*

(a) If f is not identically zero, show that the zeros of f are isolated, that is, if $f(a) = 0$ for some $a \in \Delta$, then there exists $\delta > 0$ such that $f(z) \neq 0$ for $0 < |z - a| < \delta$

(b) Show that if $\left|f(n^{-2})\right| \leq 2^{-n}$ for $n \geq 2$, then f is identically zero.

Problem 2 *Let f be a non-constant entire function.*

(a) Prove that if $\lim_{|z|\to+\infty} |f(z)| = +\infty$ then f must be a polynomial.

(b) Prove that the range of f is dense in $\mathbb{C}$.

Problem 3 *True or false. Justify your answers.*

(a) There is an analytic function $f : \Delta \to \Delta \setminus \{0\}$ which is one-to-one and onto.

(b) There is an analytic function $f : \Delta \to \mathbb{C}$ which is one-to-one and onto.

(c) There is an analytic function $f : \{z = x + iy \in \mathbb{C} : |x| > 2, x + y < 4\} \to \Delta$ which is one-to-one and onto.

Problem 4 *Let $f(z) = e^z + 3z^8$.*

(a) How many zeros does f have in Δ?

(b) Are they distinct?

Problem 5 *(a) Let f be an analytic function on Δ satisfying $M := \int_\Delta |f|\, dx\, dy < \infty$. Show that for any $z \in \Delta$, we have*

$$|f(z)| \leq \frac{M}{\pi(1 - |z|)^2}$$

(b) Let f_n be a sequence of analytic functions on Δ such that

$$\lim_{m,n\to\infty} \int_\Delta |f_n - f_m|\, dx\, dy = 0.$$

Prove that there is an <u>analytic</u> function f on Δ such that $\lim_{n\to\infty} f_n(z) = f(z)$ for every $z \in \Delta$.

Problem 6 *Let*

$$f(z) = z^{-1/2}/(z+1) \text{ for } z \neq 0,\ 0 < \arg z < 2\pi.$$

(a) Prove

$$\lim_{\rho\to 0} \int_{|z|=\rho} f(z)\, dz = 0 \text{ and } \lim_{R\to\infty} \int_{|z|=R} f(z)\, dz = 0.$$

(b) Evaluate the integral:

$$\int_0^\infty \frac{r^{-1/2}}{r+1} dr.$$

Problem 7 *Let D be a bounded, connected, open subset of $\mathbb{C}$, and let f be a non-constant continuous function on $\overline{D}$ which is analytic on D. Assume that $|f(z)| = 1$ for z in the boundary of D.*

(a) Show that 0 is in the range of f.

(b) Show that f maps D onto Δ

Problem 8 *Suppose that u is a positive harmonic function on the disk $|z| < r$. Prove that*

$$\frac{1}{3}u(0) \leq u(z) \leq 3u(0) \text{ for } |z| < r/2.$$