Complex analysis exam: June 21, 2005
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Notes: Please contact Paul Macklin if there are typos, omissions, or other errors in this document.

Notation: Let $D(z_0, r)$ be the disk centered at z_0 and radius r in the complex plane $\mathbb{C}$, let $U = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half plane.

Choose and complete eight problems from the following nine problems.

1. Let u and v be real-valued harmonic functions on the whole complex plane such that
$$u(z) \le v(z), \qquad z \in \mathbb{C}.$$
Find the relation between u and v.

2. If $f(z)$ is continuous in the region $\operatorname{Re} z \ge \sigma$ (σ is a fixed real number) and $\lim_{|z|\to\infty} f(z) = 0$, then for any negative number t
$$\lim_{R\to\infty} \int_{\Gamma_R} e^{tz} f(z)dz = 0,$$
where $\Gamma_R = \{z = x + iy : |z| = R, \text{ and } x \ge \sigma\}$.

3. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function.
   (a) Evaluate the following integral:
   $$\int_{|z|=1} f\left(\sin\left(\frac{1}{z}\right)\right) dz$$
   (b) If $\left|f\left(\sin\left(\frac{1}{z}\right)\right)\right| \le \frac{10}{|z|^2}$ for all $z \in \mathbb{C}\setminus\{0\}$, then f must be a constant.

4. Let f be holomorphic in $D(0,R)$ and continuous on $\overline{D(0,R)}$. Assume that $f(0) \ne 0$ and $f(z) \ne 0$ on $|z| = R$. Let $\{a_j : 1 \le j \le n\}$ be all the zeros of f in $D(0,R)$ counting multiplicity.
   (a) Prove Jensen's formula:
   $$\log|f(0)| + \sum_{j=1}^{n} \log\frac{R}{|a_j|} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\theta}\right)\right| d\theta$$
   (b) If $|f(z)| \le 16$ on $D(0,2)$ and $|f(0)| = 1$. Show that $f(z)$ has at most 4 zeros in the unit disk $D(0,1)$.

5. Let $f(z)$ be holomorphic in the upper half plane $U = \{z = x + iy : y > 0\}$ and continuous on $\overline{U}$. Assume $f(x) = ix^3$ for all $x \in (0, 10)$. Find all such $f(z)$.

6. Show that for any $R > 0$, there is N_R such that when $n \ge N_R$, the function
$$P_n(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} \ne 0, \qquad \text{for all } |z| \le R.$$

7. Complete the following problems:
   (a) State Montel's Theorem for a normal family;
   (b) Let $f_j : D(0,1) \to D(0,1)$ be holomorphic for all $j = 1, 2, 3, \cdots$. Suppose
   $$\lim_{z\to\infty} f_j(0) = 1.$$
   Show that $f_j(z) \to 1$ uniformly on any compact subset of $D(0,1)$.

8. Let f be an entire holomorphic function so that $|f(z)| = 1$ when $|z| = 1$. Show that $f(z)$ is a polynomial.

9. Let $A(r, R) = \{z \in \mathbb{C} : r < |z| < R\}$ with $0 < r < R < \infty$. Prove that there is a number $\epsilon > 0$ so that for any polynomial $p(z)$ we have

$$\sup \left\{ \left| p(z) - \frac{1}{z^2} \right| : z \in A(r, R) \right\} \geq \epsilon.$$