Complex analysis exam: June 2001
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Notation: Let $\Delta(z_0, r)$ be the disk centered at z_0 and radius r in the complex plane $\mathbb{C}$, let $U = \{z \in \mathbb{C} : \text{Im } z > 0\}$ be the upper half plane.

1. (a) State Rouché's theorem.
   (b) Find the number of solutions of the equation $2z^8 + 15 + 16z = 0$ in the left-half plane $D = \{z : \text{Re } z < 0\}$.

2. (a) State the Schwarz reflection principle for holomorphic functions on the unit disk.
   (b) Let $f(z)$ be holomorphic in the unit disc Δ and continuous on the closed disc $\overline{\Delta}$. If $f(e^{i\theta}) = e^{ie^{i\theta}}$ for $0 < \theta < \frac{\pi}{4}$. Prove $f(z) \equiv e^{iz}$ on Δ.

3. (a) State the Residue theorem.
   (b) Evaluate the integral
   $$\int_0^\infty \frac{1}{(1+x^2)x^{1/2}} dx$$

4. Let D_j $(j = 1, 2)$ are two bounded domains in $\mathbb{C}$. Complete the following problems.
   (a) We say that $f : D_1 \to D_2$ is a proper map if $f^{-1}(K)$ is relatively compact in D_1 for any compact subset K in D_2. If $f : D_1 \to D_2$ is proper then
   $$\lim_{z \to \partial D_1} \text{dist}(f(z), \partial D_2) = 0.$$
   (b) If $D_j = \Delta(0; r_j, R_j)$ be annulus $(j = 1, 2)$. If $f : D_1 \to D_2$ is proper holomorphic map. Prove either $|f(z)| = r_2$ when $|z| = r_1$ and $|f(z)| = R_2$ when $|z| = R_1$ OR $|f(z)| = R_2$ when $|z| = r_1$ and $|f(z)| = r_2$ when $|z| = R_1$.
   (c) Find $\text{Aut}(\Delta(0; 1, 2))$ where $\Delta(0; 1, 2) = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

5. (a) State the Riemann mapping theorem.
   (b) Construct a conformal map which maps D to the unit disc, where
   $$D = \{z \in \mathbb{C} : \text{Re } z > 0 \text{ and } |z - 1| > 1\} = \{z \in \mathbb{C} : \text{Re } z > 0\} \setminus \overline{\Delta(1, 1)}.$$

6. Let $f(z)$ be an entire holomorphic function on $\mathbb{C}$. Prove the following two statements.
   (a) If $|f(z)| \le |z|^{11/5}$ when $|z| \ge 20$ then $f(z)$ is a polynomial of degree at most 2.
   (b) If $|f(z)| \le |e^z|$ when $|z| \ge 20$ then $f(z) = ce^z$ for some constant c satisfying $|c| \le 1$.

7. (a) Find all entire functions $f(z)$ on $\mathbb{C}$ such that $|f(z)| = 1$ when $|z| = 1$ and $f'(0) = 1$.
   (b) Let $f(z)$ be meromorphic in $\mathbb{C}$ so that $|f(z)| = 1$ when $|z| = 1$. Prove that $f(z)$ must be rational.

8. (a) State the Montel theorem for normal families.
   (b) Let $\mathcal{F}$ be a family of normal functions on the unit disc $\Delta(0, 1)$ satisfying that for any $f \in \mathcal{F}$ we have
   $$|f'(z)| (1 - |z|^2) + |f(0)| \le 1, \qquad z \in \Delta(0, 1).$$
   Show that $\mathcal{F}$ is a normal family on $\Delta(0, 1)$.