Complex analysis exam: January 15, 2003
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Notes: Please contact Paul Macklin if there are typos, omissions, or other errors in this document.

Notation: Let Δ be the unit disk in the complex plane $\mathbb{C}$, let $U = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half plane.

Complete Problems 1-8 and either Problem 9 or Problem 10.

1. Does a function f exist which is analytic in the plane with $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^3}$ for all $n = 1, 2, 3, \cdots$? (Justify your answer).

2. Let f be entire such that
$$|f(\sin z)| \le |e^z|, \qquad z \in \mathbb{C}.$$
Show $f(z) \equiv 0$.

3. Show that
$$\frac{1}{\pi} \int_0^\pi \sin^{2n}(\theta) d\theta = \frac{1}{4^n} \binom{2n}{n}.$$

4. Let u be a real-valued harmonic function in an open connected set D. Then u^2 is also harmonic in D if and only if u is a constant.

5. Let $f(z)$ be meromorphic in a neighborhood of the unit disk $|z| \le 1$, with only one singular point z_0, and that a pole of order one on the circle $|z| = 1$. Show that
$$\frac{f^{(n)}(0)}{n!} = \frac{A}{(z_0)^{n+1}} [1 + B_n],$$
where $B_n \to 0$ and $n \to \infty$.

6. Find the radius of convergence R_1 of $\sum_{n=1}^\infty \frac{z^n}{n^2}$ and show the series converges uniformly on $|z| \le R_1$. What is the radius of convergence R_2 of the derivative of this series? Does it converge uniformly on $|z| \le R_2$?

7. Let f and g be analytic on a bounded domain D and continuous on its closure. Show that $|f(z)| + |g(z)|$ attains its maximum on the boundary of D.

8. Find the number of zeros of the function
$$f(z) = 2z^5 + 8z - 1$$
in the annulus $1 < |z| < 2$.

9. Let $\mathcal{F}$ be a family of analytic functions on the unit disk Δ such that for any $f \in \mathcal{F}$, we have $f(0) = 0$. Show that for any compact subset K of Δ with $0 \in K$, there is $C = C(K) \in (0, 1)$ such that
$$\max_{z \in K} |f(z)| \le C \sup_{z \in \Delta} \{|f(z)| : f \in \mathcal{F}\}.$$
(Hint: one may first assume that $\sup_{z\in\Delta} |f(z)| = 1$ for all $f \in \mathcal{F}$.)

10. Let $\{a_n\}$ be a sequence of real numbers monotonically decreasing to zero, and let f be defined by the series
$$f(z) = \sum_{n=0}^\infty a_n z^n.$$

(a) By considering $(1-z)f(z)$ show that the series converges uniformly on any compact subset of $\{z : |z| \leq 1, z \neq 1\}$.

(b) Use part (a) to show that

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$