Complex analysis exam: January 2000
LaTeX'ed by: Paul Macklin
email: pmacklin@math.uci.edu.NOSPAM
www: http://math.uci.edu/~pmacklin

Notes: Please contact Paul Macklin if there are typos, omissions, or other errors in this document.

Notation: Let Δ be the unit disk in $\mathbb{C}$. Let D be a domain–connected, open set.

1. (a) State the Uniqueness Theorem for holomorphic function in a domain D.
   (b) Let $f(z)$ be holomorphic in the unit disc Δ satisfying
   $$f\left(\frac{\sqrt{-1}}{k}\right) = \frac{100}{k^4}, \quad k = 2, 3, 4, \cdots$$
   What is $f(z)$ exactly?
2. (a) State Rouché's theorem.
   (b) Let $f(z) = z^{16} + 16z + 16$. Then find the number of zeros of $f(z)$ in the right half plane $D = \{z \in \mathbb{C} : \text{Re } z > 0\}$.
3. (a) Construct a holomorphic map which is conformal from the unit disc Δ onto the punctured disk $\Delta \setminus \{0\}$.
   (b) Prove that Δ is not conformally equivalent to $\Delta \setminus \{0\}$.
4. Evaluate
   $$\int_0^\infty \frac{\cos(2x)}{1+x^2} dx.$$
5. Let f be a function in a domain D in $\mathbb{C}$ such that $f(z)^3$ is holomorphic in D.
   (a) Is $f(z)$ holomorphic in D? (prove or give a counterexample to your answer)
   (b) If $f \in C^1(D)$, can you conclude f is holomorphic in D (verify your answer.)
6. Let $\text{Aut}(\mathbb{C} \setminus \{0\})$ be the set of all biholomorphic (one-to-one, onto holomorphic) maps from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C} \setminus \{0\}$. Show that $\text{Aut}(\mathbb{C} \setminus \{0\}) = \{az \text{ or } b/z : a, b \in \mathbb{C} \setminus \{0\}\}$.
7. Let Ω be simply connected domain in $\mathbb{C}$ and let $\varphi : \Omega \to \Delta$ be a biholomorphic map. Let $f : \Omega \to \Delta$ be holomorphic with $f(p) = 0$, where $p = \varphi^{-1}(0)$. Prove $|f'(p)| \le |\varphi'(p)|$; and, equality holds if and only if $f(z) \equiv e^{i\theta}\varphi(z)$ for some $\theta \in [0, 2\pi)$.
8. (a) Let $\Delta(a, r)$ be the disk centered at $a \in \mathbb{C}$ with radius $r > 0$, and let $g(z)$ be holomorphic in $\Delta(0, r)$. Then
   $$|g(z)| \le \frac{1}{\pi r^2} \int_{\Delta(a,r)} |g(z)|\, dA(z) = \frac{1}{\pi r^2} \int_{\Delta(a,r)} |g(z)|\, dx\, dy$$
   (b) Let f be holomorphic in $D = \Delta \setminus \{0\} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. If
   $$\int_D |f(z)|^2\, dA(z) = \int_D |f(x+iy)|^2\, dx\, dy < \infty$$
   then $z = 0$ is a removable singularity of f.