Complex analysis exam: January 9, 1997
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Notes: Please contact Paul Macklin if there are typos, omissions, or other errors in this document.

Instructions: Do all problems. Use only one side of each page. Write your name and do at most one problem on each page. Justify your answers fully. Where appropriate, quote without proof theorems and results that you use in your solutions. If you cannot prove a part (a) or (b) of a particular problem, you are still free to use it in the rest of the problem.

Notation: $\mathbb{C}$ is the complex plane, $\mathbb{R}$ is the real number system, $\Delta(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$ and $\Delta = \Delta(0,1)$ is the open unit disk.

Problem 1 *(a) Let f be analytic on the unit disk Δ. If $f(i/n) = n^{-4}$ for all positive integers n, what is f?*

(b) Find all entire functions f such that $f(x) = x^2 - 2x + 10$ for all $x \in \mathbb{R}$.

Problem 2 *Find a one-to-one analytic function which maps G onto the unit disk Δ, where*

$$G = \left\{ z = x + iy : |z| < 1 \text{ and } y > -1/\sqrt{2} \right\}.$$

Problem 3 *(a) Show that every one-to-one analytic function mapping the unit disk Δ onto itself has the form*

$$f(z) = e^{i\theta} \frac{a - z}{1 - \overline{a}z}$$

for some $\theta \in \mathbb{R}$ and $|a| < 1$.

(b) Let $f : \overline{\Delta} \to \overline{\Delta}$ be a continuous function such that $f|_\Delta$ is analytic. Suppose that $|f(z)| = 1$ whenever $|z| = 1$. Express f in terms of its zeros in Δ.

Problem 4 *Let f be analytic on Δ.*

(a) Let $a \in \Delta$. Show that for any $0 < r < 1 - |a|$, we have

$$f(a) = \frac{1}{\pi r^2} \int_{\Delta(a,r)} f(z) dx\, dy$$

(b) Show that if $M = \int_\Delta |f(z)|\, dx\, dy$, then

$$|f(a)| \leq \frac{M}{\pi(1 - |a|)^2}, \quad a \in \Delta.$$

(c) Let $\mathcal{F}$ be a family of analytic functions on Δ for which there exists $M > 0$ such that

$$\int_\Delta |f(z)|\, dx\, dy \leq M \text{ for all } f \in \mathcal{F}.$$

Show that $\mathcal{F}$ is a normal family.

Problem 5 *Evaluate the following integrals:*

(a) $\int_{\partial\Delta(0,r)} \frac{dz}{(z-b)(z-a)^m}$, where $|a| < r < |b|$ and m is an arbitrary integer.

(b) $\int_{|z|=2} \frac{4z^7 - 1}{z^8 - 2z + 1} dz$

Problem 6 *Let f be an entire function.*

(a) *Let $Z = \{z \in \mathbb{C} : f(z) = 0\}$ be the set of zeros of f. If f is not identically zero, show that Z is at most countable.*

(b) *Suppose that for each complex number a, the series expansion*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \quad z \in \mathbb{C},$$

has at least one coefficient $c_n = 0$. Show that f is a polynomial.

Problem 7 *Prove or disprove:*

(a) *If f is a function on Δ with $f(z)^2$ analytic on Δ, then f itself is analytic.*

(b) *If f is a continuously differentiable function on Δ, and if $f(z)^2$ is analytic on Δ, then f itself is analytic.*

Problem 8 *We say a funtion f defined on $\mathbb{C}$ is analytic (resp. has a pole) at ∞ if $f(1/z)$ is analytic in a neighborhood of $z = 0$ with a removable singularity (resp. pole) at $z = 0$.*

(a) *If f is analytic in the extended plane $\mathbb{C} \cup \{\infty\}$, what can you say about f?*

(b) *If f is an entire function and has a pole at infinity, what can you say about f?*

(c) *If f is an entire function satisfying the estimate*

$$|f(z)| \leq 1 + |z|^{99/2}, \quad z \in \mathbb{C},$$

show that f is a polynomial and determine the best upper bound for the degree of f.

Problem 9 *Let u be a harmonic function on Δ which is not a constant. Show that u has no maximum or minimum on Δ.*