

Complex analysis exam: September 2002

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Notes: Please contact Paul Macklin if there are typos, omissions, or other errors in this document.

1. Let $f(z)$ and $g(z)$ be entire analytic functions with $|f(z)| \leq |g(z)|$ for all z . Show that there is a constant c such that $f(z) = cg(z)$.
2. Let $f(z)$ be analytic in the disk $U = \{|z| < 1\}$, with $f(0) = f'(0) = 0$. Show that

$$g(z) = \sum_{n=1}^{\infty} f\left(\frac{z}{n}\right)$$

defines an analytic function on U . Moreover, show that the above function $g(z)$ satisfies

$$g(z) = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) f(z)$$

if and only if $f(z) \equiv cz^2$.

3. (a) State the Rouchet theorem.
(b) Show that $f(z) = \alpha e^z - z$ has only one zero in $U = \{|z| < 1\}$ if $|\alpha| < \frac{1}{3}$ and no zeros if $|\alpha| > 3$.
4. (a) State the Montel Theorem for normal family.
(b) Let \mathcal{F} be a set of holomorphic functions on the unit disk $U = \{|z| < 1\}$ so that

$$\sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})| d\theta < 1.$$

Show that \mathcal{F} is a normal family.

5. Let $\varphi(z, t)$ is continuous as a function of two variables t in $[0, 1]$ and $z \in U = \{|z| < 1\}$. Suppose that for each fixed $z \in U$ fixed $t \in [0, 1]$, $\varphi(z, t)$ is analytic in U . Let

$$f(z) = \int_0^1 \varphi(z, t) dt.$$

- (a) Show that $f(z)$ is analytic in U .
(b) If $\varphi(z, t) = \sin(e^t z)$ find $f'(z)$ explicitly.
6. Prove that there is no entire analytic function so that

$$\bigcup_{n=0}^{\infty} \{z \in \mathbb{C} : f^{(n)}(z) = 0\} = \mathbb{R},$$

where \mathbb{C} is the whole complex plane, and \mathbb{R} is the whole real line.

7. Let $f(z)$ be analytic in the unit disk punched disk $U_0 = \{0 < |z| < 1\}$ such that there is a positive integer n with $|f^{(n)}(z)| \leq |z|^{-n}$ for all $z \in U_0$. Show that $z = 0$ is a removable singularity for f .
8. Let $f(z)$ be analytic on an open set containing $\{|z| \leq 1\}$ and suppose that $|f(z)| = 1$ when $|z| = 1$. Show that there is a finite number of points $\alpha_1, \dots, \alpha_m$ and a constant c , with

$$f(z) = c \prod_{n=1}^m \frac{z - \alpha_n}{1 - \bar{\alpha}_n z}.$$

A corollary is that if $f(z)$ is entire, $f(z) = cz^m$.