

Differential Geometry via Harmonic Functions

P. Li*

Abstract

In this talk, I will discuss the use of harmonic functions to study the geometry and topology of complete manifolds. In my previous joint work with Luen-fai Tam, we discovered that the number of infinities of a complete manifold can be estimated by the dimension of a certain space of harmonic functions. Applying this to a complete manifold whose Ricci curvature is almost non-negative, we showed that the manifold must have finitely many ends. In my recent joint works with Jiaping Wang, we successfully applied this general method to two other classes of complete manifolds. The first class are manifolds with the lower bound of the spectrum $\lambda_1(M) > 0$ and whose Ricci curvature is bounded by

$$Ric_M \geq -\frac{m-2}{m-1}\lambda_1(M).$$

The second class are stable minimal hypersurfaces in a complete manifold with non-negative sectional curvature. In both cases we proved some splitting type theorems and also some finiteness theorems.

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1. Introduction

In 1992, the author and Luen-fai Tam [12] discovered a general method to determine if a complete, non-compact, Riemannian manifold have finitely many ends. An end is simply defined to be an unbounded component of the compliment of a compact set in the manifold. If the number of ends is finite, their technique also provides an estimate on the number of ends. In particular, they applied this method to prove that a certain class of manifolds must have finitely many ends.

*Department of Mathematics, University of California, Irvine, CA 92697-3875, USA. E-mail: pli@math.uci.edu

Theorem 1 (Li-Tam). *Let M^m be a complete, non-compact, manifold with*

$$\text{Ric}_M(x) \geq -k(r(x)),$$

where $k(r)$ is a continuous non-increasing function satisfying

$$\int_0^\infty r^{m-1} k(r) dr < \infty.$$

Then there exists a constant $0 < C(m, k) < \infty$ depending only on m and k , such that, M has at most $C(m, k)$ number of ends.

Since a manifold with non-negative Ricci curvature will satisfy the hypothesis, this theorem can be viewed as a perturbed version of the splitting theorem [4] of Cheeger-Gromoll. A weaker version of the above theorem for manifolds with non-negative Ricci curvature outside a compact set was also independently proved by Cai [1].

In some recent work of Jiaping Wang and the author, they successfully applied the general theory of determining the number of ends to other situations. The purpose of this note is to give a quick overview of the theory and its applications to manifolds with positive spectrum and minimal hypersurfaces.

2. General Theory

Throughout this article, we will assume that (M^m, ds_M^2) is an m -dimensional, complete, non-compact Riemannian manifold without boundary. In terms of local coordinates (x_1, x_2, \dots, x_m) , if the metric is given by

$$ds_M^2 = g_{ij} dx_i dx_j,$$

then the Laplacian is defined by

$$\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(g_{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right),$$

where $(g^{ij}) = (g_{ij})^{-1}$ and $g = \det(g_{ij})$. A function is said to be *harmonic* on M if it satisfies the Laplace equation

$$\Delta f(x) = 0$$

for all $x \in M$.

In order to state the general theorem, it is necessary for us to define the following spaces.

Definition 1. *Let*

$$\mathcal{H}_D(M) = \{f \mid \Delta f = 0, \|f\|_\infty < \infty, \int_M |\nabla f|^2 < \infty\}$$

be the space of bounded harmonic functions with finite Dirichlet integral defined on M .

Definition 2. *Let*

$$\mathcal{H}_+(M) = \langle \{f \mid \Delta f = 0, f > 0\} \rangle$$

be the space spanned by the set of positive harmonic functions defined on M .

Definition 3. *Let*

$$\mathcal{H}'(M) = \langle \{f \mid \Delta f = 0, \text{bounded from one side on each end}\} \rangle$$

be the space spanned by the set of harmonic functions defined on M , which has the property that each one is bounded either from above or below on each end.

Observe that these spaces are monotonically contained in each other, i.e.,

$$\mathcal{H}_D(M) \subset \mathcal{H}_+(M) \subset \mathcal{H}'(M).$$

Let us also recalled the following potential theoretic definition.

Definition 4. *An end E of M is non-parabolic if it admits a positive Green's function with Neumann boundary condition on ∂E . Otherwise, E is said to be parabolic.*

It is important to note that if M has at least one non-parabolic end, then M admits a positive Green's function. In this case, we say that M is non-parabolic. The interested reader can refer to [11] for more detail descriptions. Let us now state the general theorem in [12].

Theorem (Li-Tam). *Let M be a complete, non-compact manifold without boundary. Then there exists a subspace $\mathcal{K} \subset \mathcal{H}'(M)$, such that, $\dim \mathcal{K}$ is equal to the number of ends of M .*

Moreover, if M is non-parabolic, then the subspace \mathcal{K} can be taken to be in $\mathcal{H}_+(M)$. Also there exists another subspace $\mathcal{K}_N \subset \mathcal{H}_D(M)$, such that, $\dim \mathcal{K}_N$ is equal to the number of non-parabolic ends of M .

At this point, it is important to point out that even though an estimate on the dimension of the spaces $\mathcal{H}'(M)$, $\mathcal{H}_+(M)$, or $\mathcal{H}_D(M)$ will imply an estimate on the number of ends of corresponding type, however, in general, these spaces can be bigger than \mathcal{K} or \mathcal{K}_N . Hence to effectively use the above theorem, one should use the constructive argument in the proof of the theorem to give an estimate on \mathcal{K} and \mathcal{K}_N directly. Indeed, this was the case in the proof of Theorem 1. This is also true for the two applications stated in the subsequence sections.

3. Manifolds with Positive Spectrum

A complete manifold (M, ds_M^2) is *conformally compact* if M is topologically a manifold with boundary given by ∂M . Moreover, there is a background metric ds_0^2 on $(M, \partial M)$ such that

$$ds_M^2 = \rho^{-2} ds_0^2,$$

where ρ is a defining function for ∂M satisfying the conditions

$$\rho = 0 \quad \text{on} \quad \partial M$$

and

$$d\rho \neq 0 \quad \text{on} \quad \partial M.$$

A direct computation reveals that the sectional curvature, K_M , of the complete metric ds^2 has asymptotic value given by

$$K_M \sim -|d\rho|^2,$$

near ∂M . Hence if (M, ds_M^2) is also assumed to be Einstein with

$$\text{Ric}_M = -(m-1),$$

then

$$K_M(x) \sim -1,$$

as $x \rightarrow \infty$.

In 1999, Witten-Yau [19] proved a theorem concerning the AdS/CFT correspondence, which effectively ruled out the existence of worm holes. It is also a very interesting theorem in Riemannian geometry.

Theorem (Witten-Yau). *Let M^m be a conformally compact, Einstein manifold of dimension at least 3. Suppose the boundary ∂M of M has positive Yamabe constant, then*

$$H_{m-1}(M, \mathbb{Z}) = 0.$$

In particular, this implies that ∂M is connected and M must have only 1 end.

Shortly after, Cai-Galloway [2] relaxed the assumption of Witten-Yau by assuming the boundary ∂M has non-negative Yamabe constant. We would also like to point out that by a theorem of Schoen [17], a compact manifold has non-negative Yamabe constant is equivalent to the fact that it is conformally equivalent to a manifold with non-negative scalar curvature.

In his Stanford thesis, X. Wang [18] generalized the Witten-Yau, Cai-Galloway theorem by studying L^2 harmonic 1-forms.

Theorem (Wang). *Let M^m be a conformally compact manifold of dimension at least 3. Suppose the Ricci curvature of M is bounded by*

$$\text{Ric}_M \geq -(m-1)$$

and the lower bound of the spectrum of the Laplacian $\lambda_1(M)$ has a positive lower bound given by

$$\lambda_1(M) \geq (m-2),$$

then either

- (1) M has no non-constant L^2 -harmonic 1-forms, i.e.,

$$H^1(L^2(M)) = 0; \quad \text{or}$$

- (2) $M = \mathbb{R} \times N$ with the warped product metric

$$ds_M^2 = dt^2 + \cosh^2 t ds_N^2,$$

where (N, ds_N^2) is a compact manifold with $Ric_N \geq -(m-2)$. Moreover, $\lambda_1(M) = m-2$.

To see that this is indeed a generalization of the theorems of Witten-Yau and Cai-Galloway, one uses a theorem of Mazzeo [16] asserting that on a conformally compact manifold

$$H^1(L^2(M)) \simeq H^1(M, \partial M).$$

By a standard exact sequence argument, the conclusion that $H^1(L^2(M)) = 0$ implies that M has only 1 end. In addition to this, one also uses a theorem of Lee [10] giving a lower bound on λ_1 for conformally compact, Einstein manifold with non-negative Yamabe constant on ∂M .

Theorem (Lee). *Let M be a conformally compact, Einstein manifold with*

$$Ric_M = -(m-1).$$

Suppose that ∂M has non-negative Yamabe constant, then

$$\lambda_1(M) \geq \frac{(m-1)^2}{4}.$$

Since $\frac{(m-1)^2}{4} \geq m-2$, Wang's theorem implies the theorems of Witten-Yau and Cai-Galloway. Observe that the warped product case in Wang's theorem has negative Yamabe constant on ∂M .

At this point, let us also recall a theorem of Cheng [5] stating that:

Theorem (Cheng). *Let M be a complete manifold with*

$$Ric_M \geq -(m-1),$$

then

$$\lambda_1(M) \leq \frac{(m-1)^2}{4}.$$

Combining the results of Cheng and Lee we conclude that

$$\lambda_1(M) = \frac{(m-1)^2}{4}$$

for conformally compact, Einstein manifolds, whose Ricci curvature is given by

$$Ric_M = -(m-1)$$

and has non-negative Yamabe constant for its boundary.

In the authors recent joint work with Jiaping Wang [14], they proved this splitting type theorem without assuming the manifold is conformally compact.

Theorem 2 (Li-Wang). *Let M^m be a complete manifold with dimension $m \geq 3$. Suppose the Ricci curvature of M is bounded by*

$$Ric_M \geq -(m-1)$$

and

$$\lambda_1(M) \geq m-2,$$

then either

- (1) M has only 1 end with infinite volume; or
- (2) $M = \mathbb{R} \times N$ with the warped product metric

$$ds_M^2 = dt^2 + \cosh^2 t ds_N^2,$$

where (N, ds_N^2) is compact with $Ric_N \geq -(m-2)$. Moreover, $\lambda_1(M) = m-2$.

It is worth noting that this theorem implies that when the lower bound for $\lambda_1(M)$ of Cheng is achieved, then either

- (1) M has only 1 end with infinite volume, or
- (2) $M = \mathbb{R} \times N$ is the warped product and $m = 3$.

Also, since all the ends of a conformally compact manifold must have infinite volume, Theorem 2 is, in fact, a generalization of the theorems of Witten-Yau, Cai-Galloway, and Wang. It is also interesting to note that without the conformally compactness assumption, it is possible to have finite volume ends as indicated by following example.

Example 1. *Let $M^m = \mathbb{R} \times N^{m-1}$ with the warped product metric*

$$ds_M^2 = dt^2 + \exp(2t) ds_N^2,$$

where N is a compact manifold with

$$Ric_N \geq 0.$$

A direct computation shows that M has Ricci curvature bounded by

$$Ric_M \geq -(m-1)$$

and

$$\lambda_1(M) \geq m-2.$$

In fact, when $m = 3$, $\lambda_1(M) = 1$. Obviously M has two ends. One end E has infinite volume growth with

$$V_E(r) \sim C \exp((m-1)r),$$

while the other end e has finite volume with volume decay given by

$$V_e(\infty) - V_e(r) \sim C \exp(-(m-1)r).$$

We would like to point out that the pair of conditions

$$\left. \begin{array}{l} Ric_M \geq -(m-1) \\ \lambda_1(M) \geq m-2 \end{array} \right\} \quad (1)$$

is equivalent to the pair of conditions

$$\left. \begin{array}{l} Ric_M \geq -\frac{m-1}{m-2} \lambda_1(M) \\ \lambda_1(M) > 0. \end{array} \right\} \quad (2)$$

On the other hand, the pair of conditions

$$\left. \begin{array}{l} Ric_M \geq -\frac{m-1}{m-2} \lambda_1(M) \\ \lambda_1(M) = 0 \end{array} \right\} \quad (3)$$

are equivalent to the single assumption that

$$Ric_M \geq 0,$$

because the condition $\lambda_1(M) = 0$ is a consequence of the curvature assumption.

Taking this point of view, Theorem 2 can be viewed as an analogue to the splitting theorem of Cheeger-Gromoll. Similarly to the fact that Theorem 1 is a perturbed version of the Cheeger-Gromoll splitting theorem, the following theorem in [14] is a perturbed version of Theorem 2.

Theorem 3 (Li-Wang). *Let M^m be a complete manifold with $m \geq 3$. Suppose $B_p(R) \subset M$ is a geodesic ball such that*

$$Ric_M \geq -(m-1) \quad \text{on} \quad M \setminus B_p(R)$$

and the lower bound of the spectrum of the Dirichlet Laplacian on $M \setminus B_p(R)$ is bounded by

$$\lambda_1(M \setminus B_p(R)) \geq m-2 + \epsilon$$

for some $\epsilon > 0$. Then there exists a constant $0 < C(m, R, \alpha, v, \epsilon) < \infty$ depending only on $m, R, \alpha = \inf_{B_p(3R)} Ric_M, v = \inf_{x \in B_p(2R)} V_x(R)$, and ϵ , so that the number of infinite volume ends of M is at most $C(m, R, \alpha, v, \epsilon)$.

In both Theorem 2 and Theorem 3, the authors only managed to estimate the number of infinite volume ends by estimating the number of non-parabolic ends. In fact, when a manifold has positive spectrum, they proved that an end must either be non-parabolic with exponential volume growth, or it must be parabolic and finite volume with exponential volume decay. Moreover, these growth and decay estimates can be localized at each end.

Theorem 4 (Li-Wang). *Let M be a complete, non-compact, Riemannian manifold. Suppose E is an end of M given by a unbounded component of $M \setminus B_p(R)$, where $B_p(R)$ is a geodesic ball of radius R centered at some fixed point $p \in M$. Assume that the lower bound of the spectrum $\lambda_1(E)$ of the Dirichlet Laplacian on E is positive. Then as $r \rightarrow \infty$, either*

- (1) E is non-parabolic and has volume growth given by

$$V_E(r) \geq C_1 \exp(2\sqrt{\lambda_1(E)} r)$$

for some constant $C_1 > 0$; or

- (2) E is parabolic and has finite volume with volume decay given by

$$V(E) - V_E(r) \leq C_2 \exp(-2\sqrt{\lambda_1(E)} r)$$

for some constant $C_2 > 0$.

In particular, if $\lambda_1(M) > 0$, then M must have exponential volume growth given by

$$V_p(r) \geq C_1 \exp(2\sqrt{\lambda_1(M)} r).$$

Both the volume growth and the volume decay estimates are sharp. For example, the growth estimate is achieved by the hyperbolic m -space, \mathbb{H}^m . Also, in Example 1 when dimension $m = 3$, the infinite volume end achieves the sharp volume growth estimate and the finite volume end achieves the sharp volume decay estimate. It is also interesting to point out that the sharp volume growth estimate is previously not known for manifolds with $\lambda_1(M) > 0$.

4. Minimal Hypersurfaces

Let us recall that the well-known Bernstein's theorem (Bernstein, Fleming, Almgren, DeGiorgi, Simons) asserts that an entire minimal graph $M^m \subset \mathbb{R}^{m+1}$ must be linear if $m \leq 7$. Moreover, the dimension restriction is necessary as indicated by the examples of Bombieri, DeGiorgi, and Guisti. Since minimal graphs are necessarily area minimizing and hence stable (second variation of the area functional is non-negative), Fischer-Colbrie and Schoen [8] considered a generalization of Bernstein's theorem in this category. They proved that a complete, oriented, immersed, stable minimal surface in a complete manifold with non-negative scalar curvature must be conformally equivalent to either \mathbb{C} or $\mathbb{R} \times \mathbb{S}^1$. Moreover, if the ambient manifold is \mathbb{R}^3 then the minimal surface must be planar. This special case was independently proved by do Carmo and Peng [6].

Later, Fischer-Colbrie [7] studied the structure of minimal surfaces with finite index. Recall that a minimal surface has finite index means that there are only a finite dimension of variations such that the second variations of the area functional is negative. In this case, Fischer-Colbrie proved that a complete, oriented, immersed, minimal surface with finite index in a complete manifold with non-negative

scalar curvature must be conformally equivalent to a compact Riemann surface with finitely many punctures. In particular, M must have finitely many ends. The special case when $N = \mathbb{R}^3$ was also independently proved by Gulliver [9]. It is in the spirit of the number of ends that Cao, Shen and Zhu [3] found a higher dimensional statement for stable minimal hypersurfaces in \mathbb{R}^{m+1} .

Theorem (Cao-Shen-Zhu). *Let $M^m \subset \mathbb{R}^{m+1}$ be a complete, oriented, immersed, stable minimal hypersurface in \mathbb{R}^{m+1} , then M must have only 1 end.*

This theorem is recently generalized to minimal hypersurfaces with finite index by the author and Jiaping Wang [13].

Theorem 5 (Li-Wang). *Let $M^m \subset \mathbb{R}^{m+1}$ be a complete, oriented, immersed, minimal hypersurface with finite index in \mathbb{R}^{m+1} , then M must have finitely many ends.*

In another paper [15], they also considered complete, properly immersed, stable (or with finite index) minimal hypersurfaces in a complete, non-negatively curved manifold.

Theorem 6 (Li-Wang). *Let $M^m \subset N^{m+1}$ be a complete, oriented, properly immersed, stable, minimal hypersurface. Suppose N is a complete manifold with non-negative sectional curvature. Then either*

- (1) M has only 1 end; or
- (2) $M = \mathbb{R} \times S$ with the product metric, where S is a compact manifold with non-negative sectional curvature. Moreover, M is totally geodesic in N .

Theorem 7 (Li-Wang). *Let $M^m \subset N^{m+1}$ be a complete, oriented, properly immersed, minimal hypersurface with finite index. Suppose N is a complete manifold with non-negative sectional curvature. Then M must have finitely many ends.*

It is interesting to point out that in the case when $M = \mathbb{R} \times S$, the manifold is parabolic. In this case, it is necessary to estimate the space \mathcal{K} rather than \mathcal{K}' . Again, the crucial point is to follow the construction of \mathcal{K} and obtain sufficient estimates on the functions in \mathcal{K} so that analytic techniques can be applied. In the case of Theorem 5, since the ambient manifold is \mathbb{R}^{m+1} and hence the ends of M must all be non-parabolic, it is sufficient to estimate the space \mathcal{K}' as stated in Theorem 2.

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