MATH 280B WINTER 2016 HOMEWORK 5

Due date: Friday March 18 before 12:00noon

Rules: Write as efficiently as possible – and think carefully what to write and what not. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt. I will not grade any text that exceeds the specified length.

1. (1/2 page) let $R(z, x_1, \ldots, x_\ell)$ be a Σ_1-relation. Prove that the relation R' defined by

$$R'(y, x_1, \ldots, x_\ell) \iff (\forall z < y) R(z, x_1, \ldots, x_\ell)$$

is also a Σ_1-relation.

2. (1 page) Prove that all basic functions are Σ_1-definable and the operations composition and the μ-operator, when applied to Σ_1-definable partial functions, give Σ_1-definable partial functions.

3. (5 lines) The Chinese remainder theorem says that if $a_0, \ldots, a_{n-1}$ are positive integers which are relatively prime in pairs, that is, $\gcd(a_i, a_j) = 1$ whenever $i, j < n$ and $i \neq j$, then the function

$$s \mapsto \langle \mathsf{rm}(s, a_0), \ldots, \mathsf{rm}(s, a_{n-1}) \rangle$$

is a bijection of $a_0 \cdot a_1 \cdot \cdots \cdot a_{n-1}$ onto $\prod_{i<n} a_i$. Here $\mathsf{rm}(a, q)$ is the remainder of the division of a by q.

Notice also that if $b \in \mathbb{N}$ then the numbers

$$1 + 1 \cdot b! \quad 1 + 2 \cdot b! \quad \ldots \quad 1 + (b+1)b!$$

are relatively prime in pairs.

The above observations give a tool for coding finite sequences of numbers by numbers which has simpler definition than the coding based on exponentiation.

Show that the function $\beta : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ defined by

$$\beta(s, d, i) = \mathsf{rm}(s, 1 + (1 + i) \cdot d)$$

is Σ_0-definable.

The function β is called the **Gödel β-function**.

4. (1 page) Assume $f(x_1, \ldots, x_\ell)$ and $g(u, v, x_1, \ldots, x_\ell)$ are partial Σ_1-definable functions and $h(y, x_1, \ldots, x_\ell)$ is a partial function constructed from f and g in terms of primitive recursion, that is,

$$\begin{aligned} h(0, x_1, \ldots, x_\ell) &\simeq f(x_1, \ldots, x_\ell) \\ h(y+1, x_1, \ldots, x_\ell) &\simeq g(h(y, x_1, \ldots, x_\ell), y, x_1, \ldots, x_\ell) \end{aligned}$$

Prove that h is Σ_1-definable.

Remark. You will need the above exercises for this task. In particular, use the Gödel β-function to code the sequence of calculations leading to the calculation of $h(y, x_1, \ldots, x_\ell)$.

5. (1/2 page) Prove that there is a primitive recursive relation $\mathsf{FR}(x, y)$ such that for every $s, v \in \mathbb{N}$, if s codes a formula and v codes a variable then

$$\mathsf{FR}(s, v) \iff \text{the variable coded by } v \text{ has a free occurrence in the formula coded by } s.$$

For this, use the primitive recursive functions and relations we developed in the lecture.