MATH 280B WINTER 2016 HOMEWORK 3

Due date: Wednesday, February 17

Rules: Write as efficiently as possible – and think carefully what to write and what not. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt. I will not grade any text that exceeds the specified length.

Problems 1 and 2 build on Problems 4 and 5 in Homework 2 assignment. For the notation refer to those problems. Use whatever tools have been developed and facts have been proved in those exercises. **Do not** re-prove those facts.

1. (1 page) Given uncountable cardinals $\kappa < \theta$, assume κ is regular. Let $\mathcal{H} = (H_\theta, \in, <^*)$ as in Problem 4 in Homework 2 assignment. Assume $S \subseteq \kappa$. Prove that the following are equivalent.

(a) S is stationary.
(b) There is an $\mathcal{M} \prec H_\theta$ such that $\delta_\mathcal{M} \in S$, $S \in M$ and $\delta_\mathcal{M} \subseteq M$.
(c) There is an $\mathcal{M} \prec H_\theta$ such that $\delta_\mathcal{M} \in S$ and $S \in M$.

2. (1/2 page) Here κ and θ are as in Problem 1. Notice that if $X \subseteq H_\theta$ then there is a **smallest** $\mathcal{M} \prec \mathcal{H}$ such that $X \subseteq M$. Here "smallest" means that if $\mathcal{M}' \prec \mathcal{H}$ is such that $X \subseteq M'$ then $M \subseteq M'$ (equivalently $\mathcal{M} \prec \mathcal{M}'$).

Assume $\mathcal{M} \prec \mathcal{H}$ is such that $\kappa \in M$. Let $\mathcal{M}^*$ be the smallest elementary substructure of $\mathcal{H}$ such that $\delta_\mathcal{M} \cup M \subseteq \mathcal{M}^*$. Show that $\delta_\mathcal{M} = \delta_{\mathcal{M}^*}$. That is, no new ordinals from the interval $[\delta_\mathcal{M}, \kappa)$ are added to $\mathcal{M}'$.

3. (1/2 page) Let T be a complete $\mathcal{L}$-theory and $\mathcal{M} \models T$. Show that any n-type of T can be realized in some elementary extension of $\mathcal{M}$.

This is a fact from the beginning of the section on types. Do the proof without using any facts proved about types in the lecture; all you can use are Löwenheim-Skolem theorems.

4. (3/2 page) In the lecture we proved a special case of the Omitting Types Theorem in that we constructed a model which omits one given nonisolated type. Modify the proof to get a proof of the full Omitting Types Theorem, that is, that would give a model which omits a given countable list of nonisolated types.

Try to focus on the new elements in the argument; parts in the argument which go the same way as in the case presented in the lecture should be mentioned as briefly as possible, and without proof, just with the reference that they can be proved the same way as in the lecture.

5. (1 page) Let $\mathcal{L}_{\mathsf{PA}} = \{\dot{0}, \dot{S}, \dot{+}, \dot{\times}, \dot{<}\}$ be the language of arithmetic with the obvious meaning of the symbols. Let $\mathcal{M}, \mathcal{M}'$ be two $\mathcal{L}_{\mathsf{PA}}$-structures. We say that $\mathcal{M}'$ is an **end-extension** of $\mathcal{M}$ iff $\mathcal{M}$ is a substructure of $\mathcal{M}'$ such that for every

$a \in M$ and $b \in M'$ we have $a \dot{<}^{\mathcal{M}'} b$. Let $\mathcal{M}_{\mathsf{std}} = (\mathbb{N}, 0, S, +, \cdot, <)$ be the standard model of Arithmetic.

(a) Given a countable model $\mathcal{M} \models \mathsf{Th}(\mathcal{M}_{\mathsf{std}})$, use the omitting types theorem to construct an elementary end-extension of $\mathcal{M}$.

(b) Using the result in (a), construct an elementary end-extension $\mathcal{M}^*$ of $\mathcal{M}_{\mathsf{std}}$ such that
   - $\mathcal{M}^*$ is of size ω_1, and
   - Every proper initial segment of $\mathcal{M}^*$ is countable.