MATH 280B WINTER 2016 HOMEWORK 1

Due date: Wednesday, January 20

Rules: Write as efficiently as possible – and think carefully what to write and what not. Each problem indicates maximum allowed length; this is usually much more than needed. If you type, do not use font smaller than 10pt. I will not grade any text that exceeds the specified length.

1. (6 lines) In the lecture I defined the notion of embedding/elementary embedding of and $\mathcal{L}$-structure $\mathcal{M}$ into an $\mathcal{L}$-structure $\mathcal{N}$ as a map $\pi : \mathcal{M} \to \mathcal{N}$ such that for every quantifier-free formula/every formula $\varphi(v_1, \dots, v_\ell)$ and every tuple $a_1, \dots, a_\ell$ of elements of $\mathcal{M}$ we have

$$\mathcal{M} \models \varphi(a_1, \dots, a_\ell) \implies \mathcal{N} \models \varphi(\pi(a_1), \dots, \pi(a_\ell)).$$

Prove that in either case we actually have $\iff$ in place of $\implies$.

2. (3 lines) Show that if $\pi : \mathcal{M} \to \mathcal{N}$ is an embedding of two $\mathcal{L}$-structures then π is injective.

3. (1 page) Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures with domains M, N and $\pi : M \to N$ be a map such that

(a) $\pi(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for every constant symbol of $\mathcal{L}$,
(b) $\pi(f^{\mathcal{M}}(a_1, \dots, a_\ell)) = f^{\mathcal{N}}(\pi(a_1), \dots, \pi(a_\ell))$ for every ℓ-place function symbol of $\mathcal{L}$ and every tuple $a_1, \dots, a_\ell \in M$, and
(c) $(a_1, \dots, a_\ell) \in R^{\mathcal{M}}$ iff $(\pi(a_1), \dots, \pi(a_\ell)) \in R^{\mathcal{N}}$ for every ℓ-place relation symbol of $\mathcal{L}$ (including the equality symbol) and every tuple $a_1, \dots, a_\ell \in M$.

Prove that π is an embedding of $\mathcal{M}$ into $\mathcal{N}$.

Hint. Do the induction on complexity of quantifier-free formulae.

4. (1 page) Let $\mathcal{L}$ be a language, c be a constant symbol of $\mathcal{L}$ and $\varphi(v)$ be an $\mathcal{L}$-formula with a single free variable v and no occurrence of c. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $a = c^{\mathcal{M}}$. Prove:

$$\mathcal{M} \models \varphi(a) \quad \iff \quad \mathcal{M} \models \varphi(v/c)$$

where $\varphi(v/c)$ is the $\mathcal{L}$-formula obtained from $\varphi(v)$ by replacing all free occurrences of v in $\varphi(v)$ with c.

Hint. Use the induction on complexity of φ.

5. (1 page) Let $\mathcal{L} = \{\dot{0}, \dot{S}, \dot{+}, \dot{\times}, \dot{<}\}$ be the language of arithmetic, that is, $\dot{0}$ is the constant symbol denoting 0, $\dot{S}$ is the unary function symbol denoting the successor function, $\dot{+}$ and $\dot{\times}$ are binary function symbols denoting addition and multiplication, and $\dot{<}$ is a binary relation symbol denoting the ordering of natural numbers. Let $\mathcal{N} = \{\mathbb{N}, 0, S, +, \cdot, <\}$ be the standard model of arithmetic with standard interpretations of the above symbols.

Construct an elementary exetension $\mathcal{N}'$ of $\mathcal{N}$ such that there is an order-preserving map $\sigma : (\mathbb{R}, <_{\mathbb{R}}) \to (N', \dot{<}^{\mathcal{N}'})$ where $\mathbb{R}$ is the set of real numbers and $<_{\mathbb{R}}$ is the natural ordering of real numbers.

Hint. Use the compactness theorem and any of the propositions 1.11 and 1.12 or Corollary 1.13 from the lecture. Do **not** reproduce proofs of these propositions/corollary, just apply them where appropriate.